How to determine the masses of the lightest quarks

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Outline

Introduction
- QCD
- Chiral perturbation theory

How to determine $m_u - m_d$

$\eta \to 3\pi$ and Q
- The $\eta \to 3\pi$ amplitude

Summary

Work in collaboration with S. Lanz, H. Leutwyler and E. Passemar
PRL 118 (17) 022001 and in progress

Maxwell's theory of electromagnetism

Lagrangian leading to Maxwell equations

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu = (\phi, \vec{A}), \;\; j_\mu = (\rho, \vec{A})$$

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At the microscopic level, the current is made out of fermions, described by Dirac's theory:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu \left[\sum_i e_i \bar{\psi}_i \gamma_\mu \psi_i \right] + \sum_i \bar{\psi}_i \left[i\partial_\mu \gamma^\mu - m_i \right] \psi_i$$

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$$\mathcal{L}_{\rm QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{\psi}_i \left[i D_\mu \gamma^\mu - m_i \right] \psi_i$$

QED is a gauge theory=theory defined by local invariance

$D_\mu = \partial_\mu + ieA_\mu$ the relevant covariant derivative

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Theory of strong interactions: QCD

Strong interactions are also described by a gauge theory

the local invariance group is larger than that of QED:

$$\mathrm{QED}: U(1) \quad \longrightarrow \quad \mathrm{QCD}: SU(3)$$

and the covariant derivative changes accordingly

$$D_\mu = \partial_\mu + ieA_\mu \quad \longrightarrow \quad D_\mu = \partial_\mu + igA_\mu^a \lambda_a \qquad (a = 1, \ldots 8)$$

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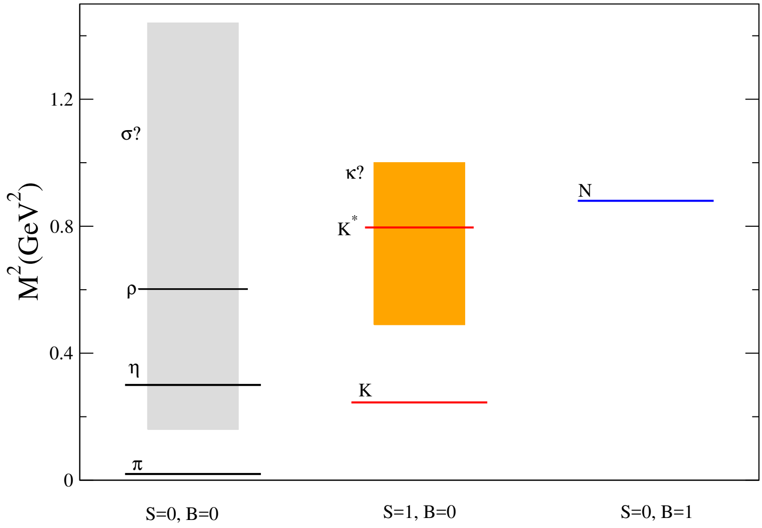
The behaviour of QED and QCD is very different

Determining the fermion masses m_i in the two theories is a completely different matter

The QCD spectrum



Quark masses

QCD Lagrangian:

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + \sum_i \bar{q}_i(i\not{D} - m_{q_i})q_i + \sum_j \bar{Q}_j(i\not{D} - m_{Q_j})Q_j$$

- In the limit $m_{q_i} \to 0$ and $m_{Q_j} \to \infty$: $M_{\rm hadrons} \propto \Lambda$
- Observe that $m_{q_i} \ll \Lambda$ while $m_{Q_j} \gg \Lambda$ [$\Lambda \sim M_N$]
- Quarks do not propagate:
  quark masses are coupling constants! (not observables)

  they depend on the renormalization scale μ (like α_s)
  for light quarks by convention: $\mu = 2$ GeV

The QCD spectrum

- the lowest-lying particles in the spectra are well understood: they would become exactly massless in the chiral limit of QCD (Goldstone bosons)

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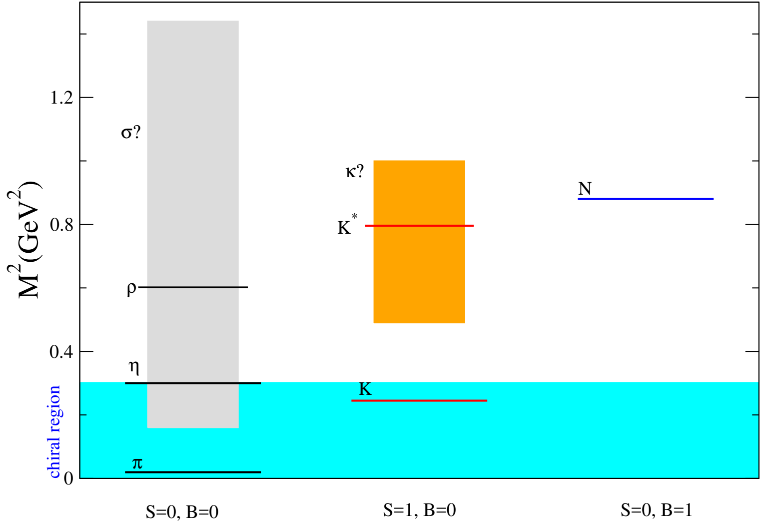

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- the positions of the low-lying resonances is more difficult to determine and understand
- they set the limit of validity of the chiral expansion

Systems with spontaneous symmetry breaking

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- effective Lagrangian: systematic method to construct this expansion, respecting symmetry and all the general principles of quantum field theory Weinberg (79)
- The method leads to predictions – even very sharp ones

Quantum Chromodynamics in the chiral limit

$$\mathcal{L}^{(0)}_{\mathrm{QCD}} = -\frac{1}{4}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Large global symmetry group:

$$SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

1. $U(1)_V \Rightarrow$ baryonic number
2. $U(1)_A$ is anomalous
3. $$SU(3)_L \times SU(3)_R \Rightarrow SU(3)_V$$

   $\Rightarrow$ Goldstone bosons with the quantum numbers of pseudoscalar mesons will be generated

Quark masses, chiral expansion

In the real world quarks are not massless:

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^{(0)} + \mathcal{L}_m, \qquad \mathcal{L}_m := -\bar{q}\mathcal{M}q$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

the mass term $\mathcal{L}_m$ can be considered as a small perturbation $\Rightarrow$
Expand around $\mathcal{L}_{QCD}^{(0)}$ $\equiv$ Expand in powers of m_q

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Chiral perturbation theory, the low-energy effective theory of QCD, is a simultaneous expansion in powers of momenta and quark masses

Quark mass expansion of meson masses

General quark mass expansion for the P particle:

$$M_P^2 = M_0^2 + \langle P|\bar{q}\mathcal{M}q|P\rangle + O(m_q^2)$$

For the pion $M_0^2 = 0$:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2)$$

where we have used a Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

$\langle 0|\bar{q}q|0\rangle$ is an order parameter for the chiral spontaneous symmetry breaking

Gell-Mann, Oakes and Renner (68)

Quark mass expansion of meson masses

Consider the whole pseudoscalar octet:

$$
\begin{aligned}
M_\pi^2 &= (m_u + m_d)B_0 + O(m_q^2) \\
M_{K^+}^2 &= (m_u + m_s)B_0 + O(m_q^2) \\
M_{K^0}^2 &= (m_d + m_s)B_0 + O(m_q^2) \\
M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + O(m_q^2)
\end{aligned}
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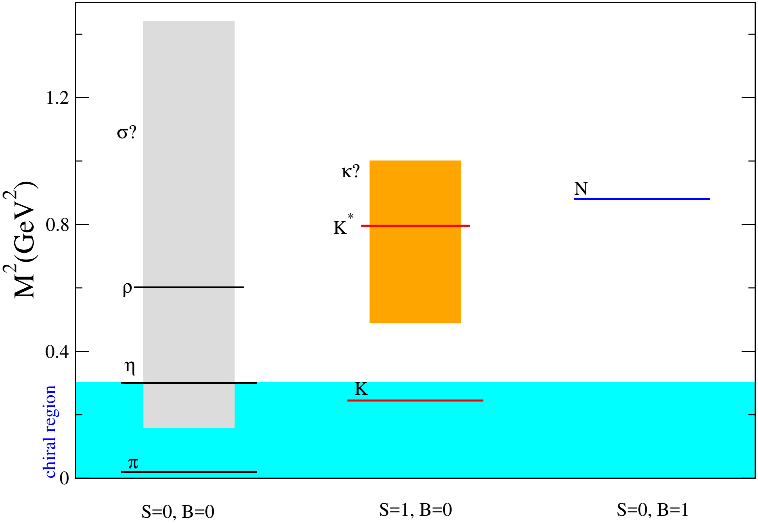
Consequences: $(\hat{m} = (m_u + m_d)/2)$

$$
\begin{aligned}
M_K^2/M_\pi^2 &= (m_s + \hat{m})/2\hat{m} & \Rightarrow m_s/\hat{m} = 25.9 \\
M_\eta^2/M_\pi^2 &= (2m_s + \hat{m})/3\hat{m} & \Rightarrow m_s/\hat{m} = 24.3 \\
3M_\eta^2 &= 4M_K^2 - M_\pi^2 & \text{Gell-Mann–Okubo (62)} \\
(0.899 &= 0.960)\ \text{GeV}^2
\end{aligned}
$$

Quark mass expansion of meson masses



How to determine quark masses

- From their influence on the spectrum — χPT, lattice
  - $m_Q \gg \Lambda$

    $$M_{\bar{Q}q_i} = m_Q + \mathcal{O}(\Lambda)$$
  - $m_q \ll \Lambda$

    $$M_{\bar{q}_i q_j} = M_{0\,ij} + \mathcal{O}(m_{q_i}, m_{q_j}) \qquad M_{0\,ij} = \mathcal{O}(\Lambda)$$

  In both cases need to understand the $\mathcal{O}(\Lambda)$ term
- From their influence on any other observable — χPT, sum rules

  Quark masses are coupling constants
  $\Rightarrow$ exploit the sensitivity to them of any observable
  [e.g. η decays, spectral functions from τ decays, etc.]

Isospin symmetry

Originally introduced as symmetry between proton and neutron

(Heisenberg 1932)

Symmetry of the QCD Lagrangian if $m_u = m_d$

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^{(0)}_{\rm QCD} - \hat{m}(\bar{u}u + \bar{d}d)$$

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Symmetry of the QCD Lagrangian if $m_u = m_d$ $\quad q = \begin{pmatrix} u \\ d \end{pmatrix}$

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Broken by:

$m_u \neq m_d$ and $Q_u \neq Q_d$

strong and electromagnetic interactions

$m_d + m_u$ is easier to get than $m_d - m_u$

$$m_d,\ m_u \ll \Lambda \quad \Rightarrow \quad \mathcal{L}_m = -m_u \bar{u}u - m_d \bar{d}d = \text{small perturbation}$$

However:

$$\begin{aligned}\mathcal{L}_m &= -\frac{m_d + m_u}{2}(\bar{u}u + \bar{d}d) + (m_d - m_u)\frac{\bar{u}u - \bar{d}d}{2} \\ &= -\hat{m}\underbrace{\bar{q}q}_{\mathcal{O}_{I=0}} + (m_d - m_u)\underbrace{\bar{q}\tau_3 q}_{\mathcal{O}_{I=1}}\end{aligned}$$

selection rules make the effect of $\mathcal{O}_{I=1}$ well hidden

$\Rightarrow$ $\hat{m}$ responsible for the mass of pions
but $(m_d - m_u)$ only contributes at $\mathcal{O}(p^4)$ (a tiny δM_{π^0})

better sensitivity in K masses
but the em interaction competes as a source of isospin breaking

Outline

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Summary

First estimates

Leading-order masses of π and K:

$$M_\pi^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s)$$

Quark mass ratios:

$$\frac{m_u}{m_d} \simeq \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \simeq 0.67$$

$$\frac{m_s}{m_d} \simeq \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20$$

$m_{ud} \equiv (m_u + m_d)/2 \simeq 5.4$ MeV SU(6) relation, Leutwyler (74)

From an analysis of the p-n mass difference: Gasser & Leutwyler (75)

$$m_u \simeq 4 \text{ MeV} \qquad m_d \simeq 6 \text{ MeV} \qquad m_s \simeq 135 \text{ MeV}$$

Electromagnetic corrections to the masses

According to Dashen's theorem

Dashen (69)

$$\begin{aligned} M^2_{\pi^0} &= B_0(m_u + m_d) \\ M^2_{\pi^+} &= B_0(m_u + m_d) + \Delta_{\text{em}} \\ M^2_{K^0} &= B_0(m_d + m_s) \\ M^2_{K^+} &= B_0(m_u + m_s) + \Delta_{\text{em}} \end{aligned}$$

Extracting the quark mass ratios gives

Weinberg (77)

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} &= 0.56 \\ \frac{m_s}{m_d} &= \frac{M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} &= 20.1 \end{aligned}$$

Higher order chiral corrections

Mass formulae to second order

Gasser-Leutwyler (85)

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$$

The same $\mathcal{O}(m)$ correction appears in both ratios

$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ corrections

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2}\left[1 + \mathcal{O}(m^2)\right]$$

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The same $\mathcal{O}(m)$ correction appears in both ratios
$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ and em corrections

$$Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)} = 24.3$$

Violation of Dashen's theorem

In pure QCD ($\hat{M}_P \equiv M_{P|_{\alpha_{em}=0}}$)

$$\begin{aligned}\hat{M}_{K^+} &= B_0(m_s + m_u) + \mathcal{O}(m_q^2)\\ \hat{M}_{K^0} &= B_0(m_s + m_d) + \mathcal{O}(m_q^2)\end{aligned}$$

$$\Rightarrow \quad \hat{M}_{K^+} - \hat{M}_{K^0} = B_0(m_u - m_d) + \mathcal{O}(m_q^2)$$

Define em contributions to masses

$$M_P^\gamma \equiv M_P - \hat{M}_P, \;\; \Delta_P^\gamma \equiv M_P^2 - \hat{M}_P^2$$

Dashen's theorem: $\Delta_{K^+}^\gamma = \Delta_{\pi^+}^\gamma$

and its violation $[\Delta_\pi \equiv M_{\pi^+}^2 - M_{\pi^0}^2]$

$$\Delta_{K^+}^\gamma - \Delta_{K^0}^\gamma - \Delta_{\pi^+}^\gamma + \Delta_{\pi^0}^\gamma = \epsilon \Delta_\pi$$

Estimates of the size of Dashen's theorem violation

χPT + model-based calculations:

$$\epsilon = \begin{cases} 0.8 & \text{Bijnens-Prades (97)} & Q = 22 \text{ (ENJL model)} \\ 1.0 & \text{Donoghue-Perez (97)} & Q = 21.5 \text{ (VMD)} \\ 1.5 & \text{Anant-Moussallam (04)} & Q = 20.7 \text{ (Sum rules)} \end{cases}$$

Lattice-based calculations (the value of Q is calculated in χPT at NLO)

$$\epsilon = \begin{cases} 0.50(8) & \text{Duncan et al. (96)} & Q = 22.9 \\ 0.5(1) & \text{RBC (07)} & Q = 22.9 \\ 0.78(6)(2)(9)(2) & \text{BMW (11)} & Q = 22.1 \\ 0.65(7)(14)(10) & \text{MILC (13)} & Q = 22.6 \\ 0.79(18)(18) & \text{RM123 (13)} & Q = 22.1 \\ 0.73(2)(5)(17) & \text{BMW (16)} & Q = 22.2 \\ 0.73(3)(13)(5) & \text{MILC (16)} & Q = 22.2 \end{cases}$$

Value quoted in FLAG-3: $\epsilon = 0.7(3)$

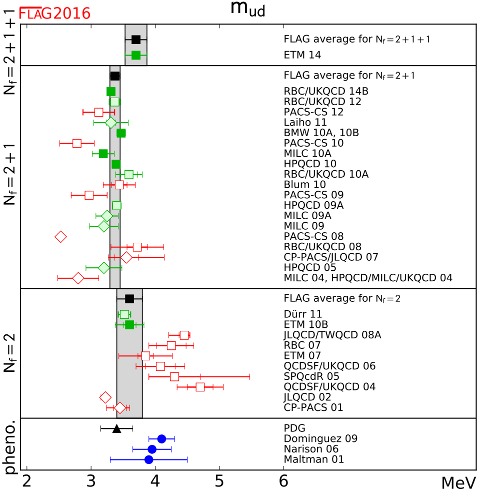
FLAG-3 summary of the quark masses

all masses in MeV

N_F	m_{ud}	m_s	m_s/m_{ud}
2+1+1	3.70(17)	93.9(1.1)	27.30(34)
2+1	3.373(80)	92.0(2.1)	27.43(31)
2	3.6(2)	101(3)	27.3(9)

N_F	m_u	m_d	m_u/m_d	R	Q
2+1+1	2.36(24)	5.03(26)	0.470(56)	35.6(5.1)	22.2 (1.6)
2+1	2.16(9)(7)	4.68(14)(7)	0.46(2)(2)	35.0(1.9)(1.8)	22.5(6)(6)
2	2.40(23)	4.80(23)	0.50(4)	40.7(3.7)(2.2)	24.3(1.4)(0.6)

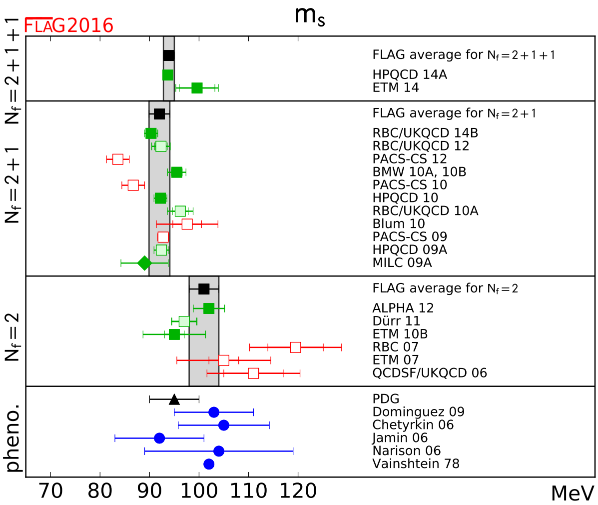
Light quark masses



$$N_f = 2+1: \qquad m_{ud}^{\overline{\rm MS}}(2\,\mathrm{GeV}) = 3.37(8)\,\mathrm{MeV} \qquad (\sim 2.4\%)$$

more precise than PDG

Light quark masses



$N_f = 2+1$: $\quad m_s^{\overline{\rm MS}}(2\,\text{GeV}) = 92.0(2.1)\,\text{MeV} \quad (\sim 2.3\%)$

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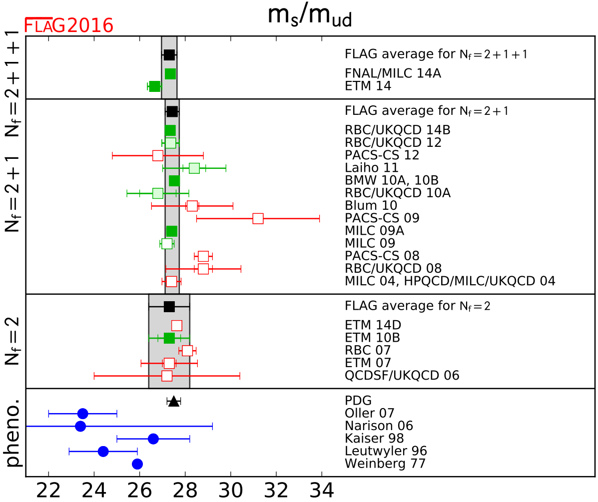

$$N_f = 2+1: \qquad m_s/m_{ud} = 27.43(31) \quad (\sim 1.1\%)$$

more precise than PDG

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$\eta \to 3\pi$ and χPT

The decay $\eta \to 3\pi$ is purely isospin violating:

in an isospin symmetric world it cannot happen

and its mostly due to strong isospin breaking
(electromagnetic contributions are suppressed) Sutherland 66

$\eta \to 3\pi$ and χPT

Lowest order chiral amplitude: Sutherland (66), Osborn, Wallace (70)

$$\mathcal{M}(\eta \to \pi^+\pi^-\pi^0) =: A(s,t,u) \qquad s = (p_{\pi^+} + p_{\pi^-})^2, \ldots$$

$$A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left[1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2} + O(p^2)\right] + O(e^2 m)$$

Relate $m_u - m_d$ to meson masses Dashen (69)

$$B_0(m_u - m_d) = (\hat{M}_{K^+}^2 - \hat{M}_{K^0}^2) + \mathcal{O}(m^2)$$

LO chiral prediction

$\Gamma(\eta \to \pi^+\pi^-\pi^0) \sim 70$ eV $\ll$ $\Gamma_{\text{exp}} = 299 \pm 11$ eV

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$$\Gamma(\eta \to \pi^+\pi^-\pi^0) \sim 70 \text{ eV} \to 160 \pm 50 \text{ eV} \quad \ll \quad \Gamma_{\text{exp}} = 299 \pm 11 \text{ eV}$$

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Large rescattering effects or violations to Dashen's theorem?

Amplitudes $\eta \to 3\pi$ beyond χPT

Decay amplitudes

$$A(\eta \to \pi^+\pi^-\pi^0) \equiv A_c(s,t,u)$$
$$A(\eta \to 3\pi^0) \equiv A_n(s,t,u)$$

Both vanish in the isospin limit and do not receive $\mathcal{O}(\alpha_{\rm em})$ contributions

Sutherland (66)

$$A_c(s,t,u) = -N(\hat{M}^2_{K^0} - \hat{M}^2_{K^+})\,[M(s,t,u) + \mathcal{O}(\delta)]$$
$$A_n(s,t,u) = -N(\hat{M}^2_{K^0} - \hat{M}^2_{K^+})\,[M(s,t,u) + M(t,u,s) + M(u,s,t) + \mathcal{O}(\delta)]$$

$$N := (3\sqrt{3}F_\pi^2)^{-1} \qquad \delta \sim (m_u - m_d, \alpha_{\rm em})$$

Amplitudes $\eta \to 3\pi$ beyond χPT

Decay amplitudes

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$$N := (3\sqrt{3}F_\pi^2)^{-1} \qquad \delta \sim (m_u - m_d, \alpha_{\text{em}})$$

- Leading contribution $M(s,t,u)$ isospin-symmetric: one amplitude describes both decay channels
- $\mathcal{O}(\delta)$-piece different for the two channels

Amplitudes $\eta \to 3\pi$ beyond χPT

Decay amplitudes

$$\begin{aligned} A(\eta \to \pi^+\pi^-\pi^0) &\equiv A_c(s,t,u) \\ A(\eta \to 3\pi^0) &\equiv A_n(s,t,u) \end{aligned}$$

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$$N := (3\sqrt{3}F_\pi^2)^{-1} \qquad \delta \sim (m_u - m_d, \alpha_{\rm em})$$

- Leading contribution $M(s,t,u)$ isospin-symmetric: amenable to a dispersive treatment
- $\mathcal{O}(\delta)$-piece modifies the phase space and the vars. s, t, u

Q from the decay $\eta \to 3\pi$

Steps leading to a determination of Q from $\eta \to 3\pi$

1. estimate the corrections $\mathcal{O}(\delta)$ (χPT)
2. evaluate the isospin-symmetric amplitude $M(s,t,u)$ (χPT or DR)
3. compare the evaluated total width (including $\mathcal{O}(\delta)$ corr.) with the measured one and determine: $-(\hat{M}^2_{K^0} - \hat{M}^2_{K^+})$
4. invoke the low-energy theorem

$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = \frac{M_K^2(M_K^2 - M_\pi^2)}{Q^2 M_\pi^2}$$

to obtain a value for Q

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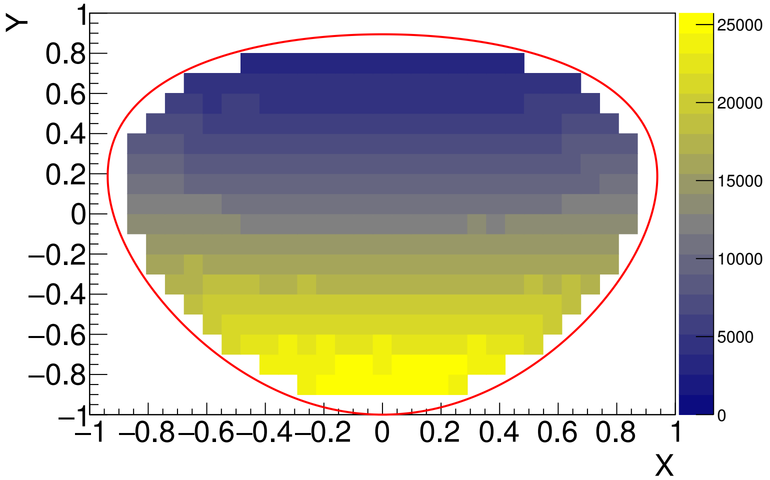
   to obtain a value for Q

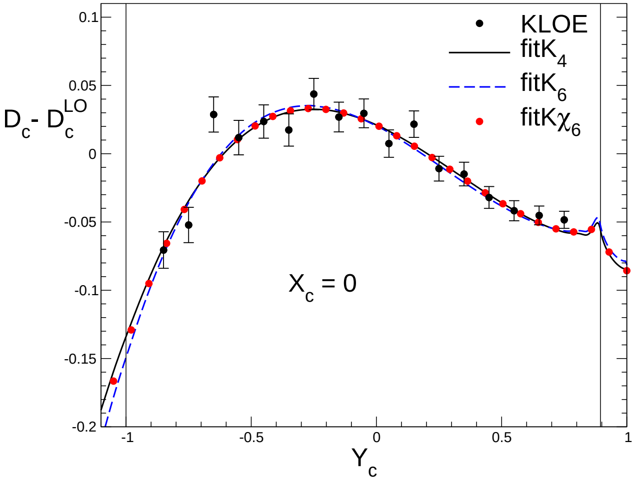
Data on the Dalitz plot $M(s,t,u)/M(s_0,s_0,s_0)$ put constraints on the dynamical calculation for $M(s,t,u)$
but the essential, purely theory input is: $M(s_0,s_0,s_0)$

KLOE data



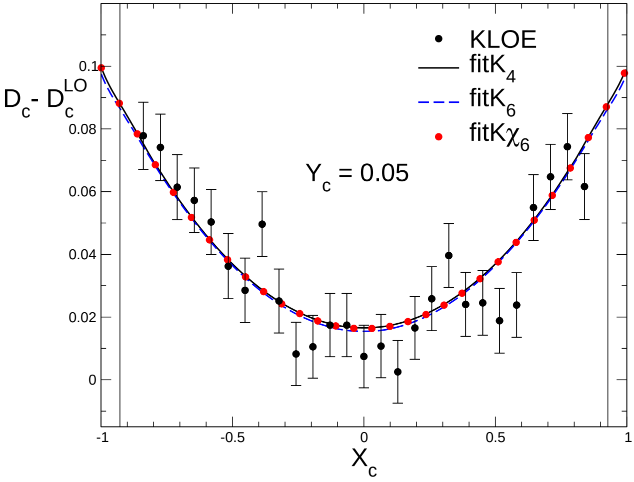
371 bins = data points KLOE collab. JHEP 2016

Momentum dependence



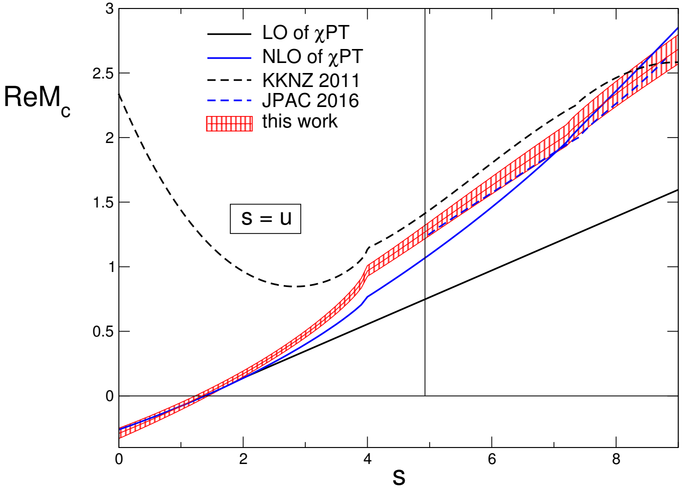
GC, Lanz, Leutwyler, Passemar in prep.

Momentum dependence



GC, Lanz, Leutwyler, Passemar in prep.

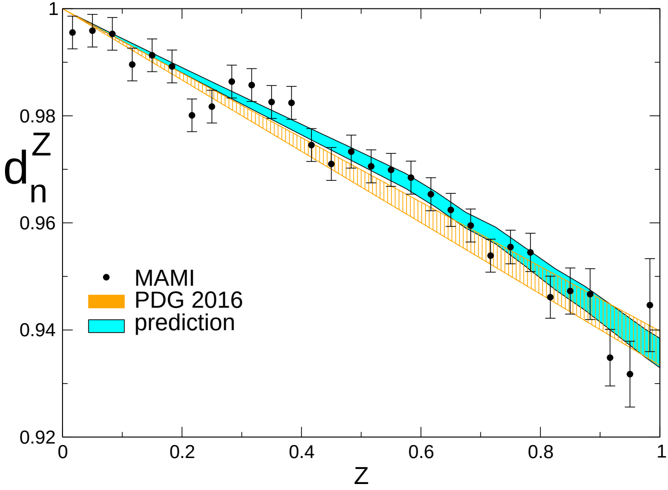
Momentum dependence



GC, Lanz, Leutwyler, Passemar in prep.

Dalitz plot in the neutral channel

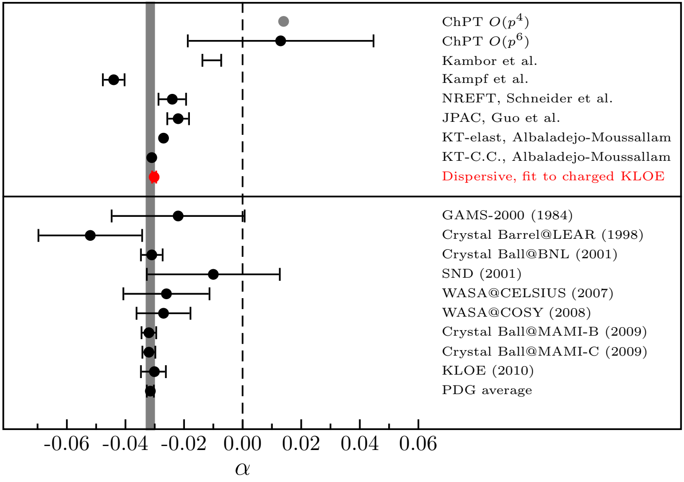
Fit to the charged channel ⇒ prediction for the neutral channel which agrees perfectly with MAMI data GC, Lanz, Leutwyler, Passemar in prep.



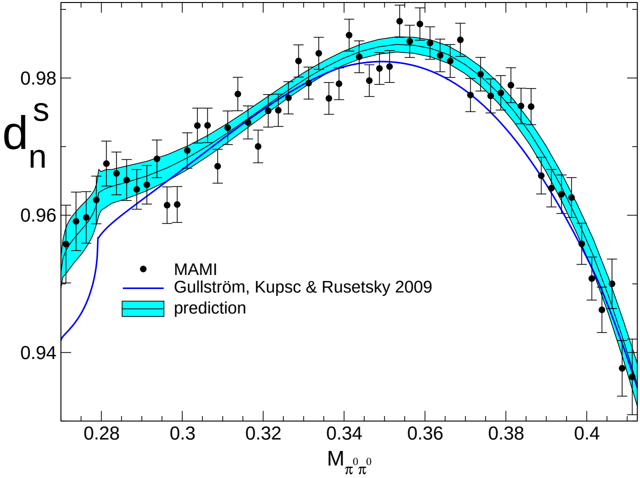
Dalitz plot in the neutral channel: value of α

$\alpha \equiv$ slope at $z = 0$

Comparison with other determinations:



Dalitz plot in the neutral channel



GC, Lanz, Leutwyler, Passemar, in prep.

Ratio of decay rates

The ratio of decay rates for the two channels can also be calculated and with remarkable accuracy Gasser-Leutwyler (85)

The normalization H_0 also drops out in this ratio

As it turns out most uncertainties cancel out, giving:

$$B \equiv \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)} = 1.44(4)$$

which agrees perfectly with the measured value

$$B_{\mathrm{PDG}}(\text{our fit}) = 1.426(26), \qquad B_{\mathrm{PDG}}(\text{our average}) = 1.48(5)$$

Determination of Q

GC, Lanz, Leutwyler, Passemar (16)

$H_0^{\mathrm{NLO}} = 1.176(53)$ and $\Gamma(\eta \to \pi^+\pi^-\pi^0) = (299 \pm 11)$ eV yield:

$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = 6.27(38) 10^{-3}\,\mathrm{GeV}^2$$

which implies:

$$(M^2_{K^0} - M^2_{K^+})_{\mathrm{QED}} = -2.38(38) 10^{-3}\,\mathrm{GeV}^2$$

This corresponds to

$$\epsilon = 0.9(3)$$

in agreement with recent lattice determinations:

$$\epsilon = \begin{cases} 0.74(18) & \mathrm{BMW} \\ 0.73(14) & \mathrm{MILC} \\ 0.50(6) & \mathrm{QCDSF/UKQCD} \\ 0.801(110) & \mathrm{RM123} \end{cases}$$

Determination of Q

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$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = 6.27(38)10^{-3}\,\text{GeV}^2$$

which implies: (upon use of)

$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = \frac{M_K^2(M_K^2 - M_\pi^2)}{Q^2 M_\pi^2}\Big(1 + \mathcal{O}(m^2)\Big)$$

$$Q = \begin{cases} 21.99(70) & \eta \to \pi^+\pi^-\pi^0 \\ 22.04(70) & \eta \to 3\pi^0 \end{cases} \qquad \Rightarrow \qquad Q = 22.0(7)$$

somewhat lower than recent lattice direct determinations

$$Q = \begin{cases} 23.40(64) & \mathrm{BMW} \\ 23.8(1.1) & \mathrm{RM123} \end{cases}$$

Unexpectedly large $\mathcal{O}(m^2)$ effects?

Single vs Double Quark Mass Ratios

Isospin-limit ratio S:

$$S \equiv \frac{m_s}{m_{ud}} = 27.43(31)$$

FLAG (17)

$$\frac{\bar{M}_K^2}{\bar{M}_\pi^2} = (S+1)(1+\Delta_S) \qquad \Delta_S = -0.055$$

Isospin-breaking ratio R:

$$R \equiv \frac{m_s - m_{ud}}{m_d - m_u} \qquad \frac{\bar{M}_K^2 - \bar{M}_\pi^2}{\hat{M}_{K^0}^2 - \hat{M}_{K^+}^2} = R(1+\Delta R)$$

Relation among R, S and Q: $\Delta_S, \Delta_R \sim \mathcal{O}(m), \ \ \Delta_Q \sim \mathcal{O}(m^2)$

$$Q^2 = \frac{1}{2} R(S+1) \qquad \Rightarrow \qquad (1+\Delta_Q) = (1+\Delta_S)(1+\Delta_R)$$

Single vs Double Quark Mass Ratios

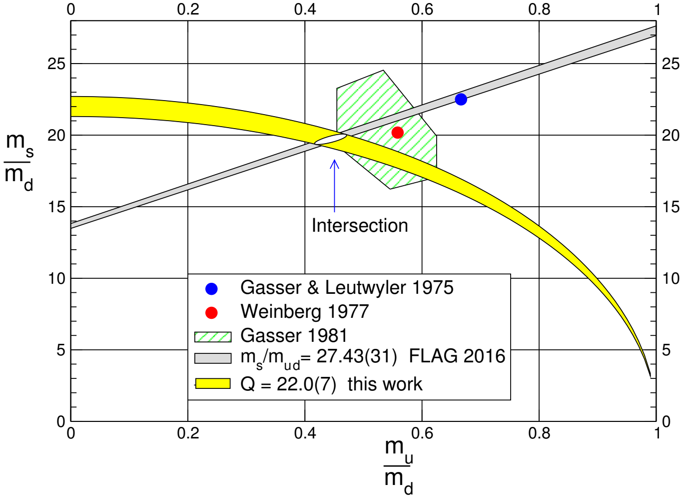
Relation among R, S and Q: $\Delta_S, \Delta_R \sim \mathcal{O}(m)$, $\Delta_Q \sim \mathcal{O}(m^2)$

$$Q^2 = \frac{1}{2}R(S+1) \qquad \Rightarrow \qquad (1+\Delta_Q) = (1+\Delta_S)(1+\Delta_R)$$

	Q	Δ_S	Δ_R	Δ_Q
BMW	23.4(0.4)(0.3)(0.4)	-0.063	-0.028	-0.089
RM123	23.8(1.1)	-0.042	-0.060	-0.099
this work*	22.0(7)	-0.055(11)	0.061(13)	0

* GC, Lanz, Leutwyler, Passemar, in prep.

Quark mass ratios: results



$$\frac{m_u}{m_d} = 0.44(3)$$

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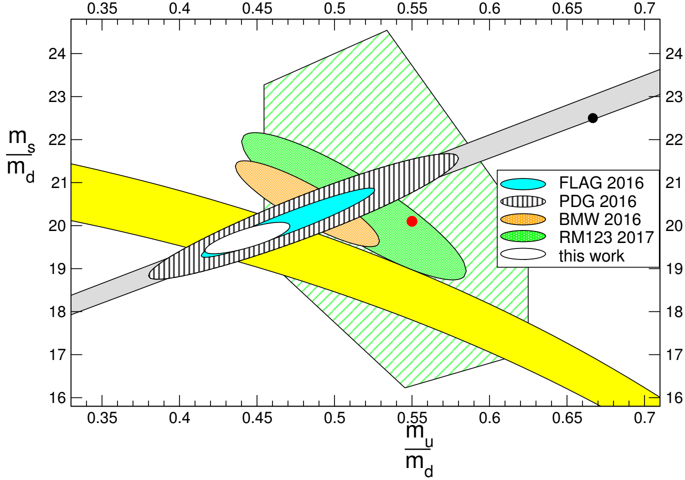

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Outline

Introduction
QCD
Chiral perturbation theory

How to determine $m_u - m_d$

$\eta \to 3\pi$ and Q
The $\eta \to 3\pi$ amplitude

Summary

Summary

- Quark masses are fundamental and yet unexplained parameters of the standard model
- I have discussed methods to determine the values of the light quark masses
- I have reviewed determinations of $m_d - m_u$ based on lattice and χPT + model estimates
- I have discussed the extraction of the quark mass ratio Q from $\eta \to 3\pi$ decays based on dispersion relations

  work in collab. with S. Lanz, H. Leutwyler and E. Passemar (16) and in progress

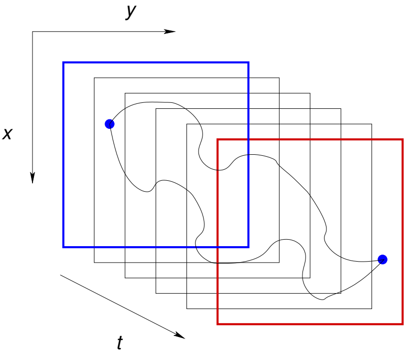
- lattice calculations of Q somewhat higher and show a weird pattern in the chiral series of quark mass ratios: lattice calculations in pure QCD could resolve this puzzle

Backup Slides

The spectrum in lattice QCD

$$C(t) = \int d^3x \, \langle [\bar{q}\gamma_5 q(x)] \, [\bar{q}\gamma_5 q(0)] \rangle \, e^{i\mathbf{px}} \quad \stackrel{t\to\infty}{\longrightarrow} \quad \sum_{n=0}^{\infty} c_n \, e^{-E_n t}$$

$$M_\pi = \lim_{\substack{L\to\infty \\ a\to 0}} M_\pi(L,a) \qquad M_\pi(L,a) = E_n(\mathbf{p}=0)$$



- $\bar{q}\,\gamma_5\, q, \;\; \bar{q}\,\Gamma\, q$

fermion propagator

Dispersive approach

GC, Lanz, Leutwyler, Passemar, (16)

Isospin decomp. of $M(s,t,u)$ Stern, Sazdjian, Fuchs (93), Anisovich, Leutwyler (96)

$$M(s,t,u) = M_0(s)+(s-u)M_1(t)+(s-t)M_1(u)+M_2(t)+M_2(u)-\frac{2}{3}M_2(s)$$

assumes only: $\mathrm{disc}[t_\ell^I(s)] = 0 \qquad \forall \ell \geq 2$ in all channels

Analytic properties of the $M_I(s)$ functions: $[s > 4M_\pi^2]$

$$\mathrm{disc}[M_I(s)] = \mathrm{disc}[t_\ell^I(s)] = t_\ell^I(s) e^{i\delta_\ell^I(s)} \sin \delta_\ell^I(s)$$

$t_\ell^I(s) = M_I(s) + \hat{M}_I(s) =$ part. wave, $I =$ isospin, $\ell =$ ang. momentum

Dispersion relation for M_0 Anisovich, Leutwyler (96), GC, Lanz, Leutwyler, Passemar, (16)

$$M_0(s) = \Omega_0(s)\left\{\alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\hat{M}_0(s')\sin\delta_0^0(s')}{|\Omega_0(s')|s'^2(s'-s-i\epsilon)}\right\}$$

Dispersive approach

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Dispersion relation for M_1 Anisovich, Leutwyler (96), GC, Lanz, Leutwyler, Passemar, (16)

$$M_1(s) = \Omega_1(s)\left\{\beta_1 s + \gamma_1 s^2 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta^1_1(s')\hat{M}_1(s')}{|\Omega_1(s')|s'(s'-s-i\epsilon)}\right\}$$

Dispersive approach

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Analytic properties of the $M_I(s)$ functions: $[s > 4M_\pi^2]$

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Dispersion relation for M_2 Anisovich, Leutwyler (96), GC, Lanz, Leutwyler, Passemar, (16)

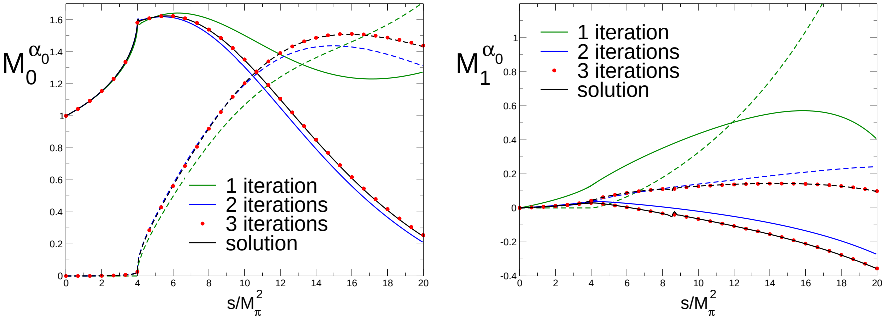
$$M_2(s) = \Omega_2(s)\,\frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \,\frac{\sin\delta_0^2(s')\hat{M}_2(s')}{|\Omega_2(s')|s'^2(s'-s-i\epsilon)}$$

Solution of the dispersion relations

GC, Lanz, Leutwyler, Passemar, (16)

Solutions of the dispersion relations are linear in the subtraction constants α_0, β_0, ..., γ_1:

$$M_I(s) = \alpha_0 M_I^{\alpha_0}(s) + \beta_0 M_I^{\beta_0}(s) + \cdots + \gamma_1 M_I^{\gamma_1}(s)$$



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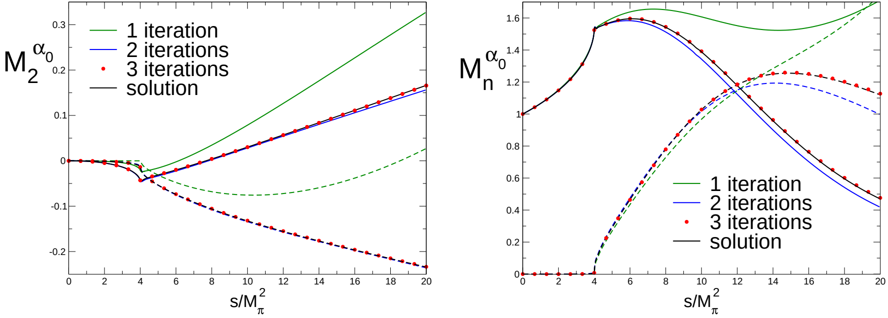


Isospin breaking

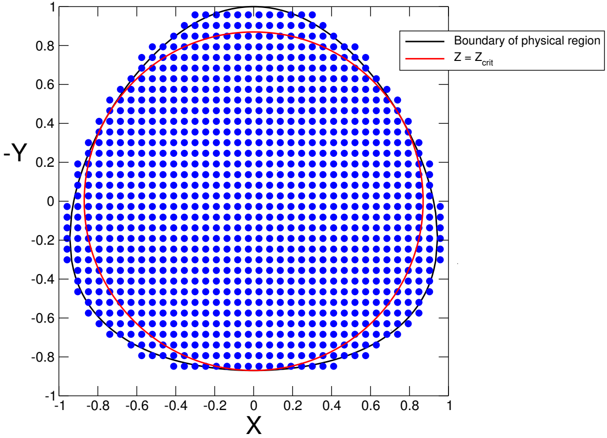
Dispersive calculation performed in the isospin limit:

$$M_\pi = M_{\pi^+} \qquad e = 0$$

- we correct for $M_{\pi^0} \neq M_{\pi^+}$ by "stretching" $s, t, u \Rightarrow$ boundaries of isospin-symmetric phase space = boundaries of physical phase space
- analysis of Ditsche, Kubis, Meissner (09) used as guidance and check. Same for Gullström, Kupsc and Rusetsky (09)
- $e \neq 0$ effects partly corrected for in the data analysis for the rest we rely on one-loop ChPT – formulae given by Ditsche, Kubis, Meissner (09)

Isospin breaking I: boundary preserving map

Phase space boundary in the limit $M_{\pi^0} = M_{\pi^+}$: $z = z_{crit}$



Isospin breaking I: boundary preserving map

Mandelstam variables in the isospin limit ($M_{\pi^i} = M_\pi \equiv$ isoB) used in our dispersive treatment: s, t, u

$$s + t + u = M_\eta^2 + 3M_\pi^2$$

Mandelstam variables in the physical channels:

$$s_c + t_c + u_c = M_\eta + 2M_\pi^2 + M_{\pi^0}^2 \qquad s_n + t_n + u_n = M_\eta + 3M_{\pi^0}^2$$

Define a mapping $(X_i, Y_i) \to (X_i^{\mathrm{bpm}}, Y_i^{\mathrm{bpm}})$, for $i = c, n$ such that boundary/center of phys. reg. $\to$ boundary/center of isoB phase sp.

$$M^{\mathrm{bpm}}(X_c, Y_c) \equiv M^{\mathrm{disp}}(X_c^{\mathrm{bpm}}(X_c, Y_c), Y_c^{\mathrm{bpm}}(Y_c))$$

is used to fit the data in the charged channel

Isospin breaking II: em corrections

We rely on the one-loop ChPT calculation of Ditsche, Kubis, Meissner (09) in the following way:

- we remove the corrections due to real photons and to the Coulomb pole
- we calculate the ratios $N_{n,c}$ and $p_{n,c}(X_{n,c}, Y_{n,c})$

$$N_n \equiv \frac{|M_n^{\mathrm{DKM}}(0,0)|^2}{|M_n^{\mathrm{DKM,bpm}}(0,0)|^2}\ ,\ N_c \equiv \frac{|M_c^{\mathrm{DKM}}(0,0)|^2}{|M_c^{\mathrm{DKM,bpm}}(0,0)|^2}$$

$$\begin{aligned} p_n(X_n, Y_n) &\equiv \frac{1}{N_n}\frac{|M_n^{\mathrm{DKM}}(X_n, Y_n)|^2}{|M_n^{\mathrm{DKM,bpm}}(X_n, Y_n)|^2} \\ p_c(X_c, Y_c) &\equiv \frac{1}{N_c}\frac{|M_c^{\mathrm{mDKM}}(X_c, Y_c)|^2}{|M_c^{\mathrm{mDKM,bpm}}(X_c, Y_c)|^2} \end{aligned}$$

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- we remove the corrections due to real photons and to the Coulomb pole
- we calculate the ratios $N_{n,c}$ and $p_{n,c}(X_{n,c}, Y_{n,c})$
- we fit the data with

$$
\begin{aligned}
|M_n(X_n, Y_n)|^2 &= |M^{\rm bpm}(X_n, Y_n)|^2 N_n p_n(X_n, Y_n) \\
|M_c(X_c, Y_c)|^2 &= |M^{\rm bpm}(X_c, Y_c)|^2 N_c p_c(X_c, Y_c)
\end{aligned}
$$

How we fit the data

GC, Lanz, Leutwyler, Passemar (16)

- our dispersive amplitude, (corrected for isospin breaking) and linear in the subtraction constants
- four up to six subtraction constants: α_0, β_0, γ_0, δ_0, β_1, γ_1
- the normalization of the Dalitz plot is arbitrary, so α_0 ($\rightarrow H_0$) is not fitted
- available data are from:
  - KLOE (2016)
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Fit results

GC, Lanz, Leutwyler, Passemar in prep.

	β_0	γ_0	δ_0	β_1	γ_1	χ^2_{KLOE}	χ^2_{th}
fitχ_4	16.9	-29.4	–	6.6	–	940	0
fitK$_4$	17.3(7)	-34.8(7.1)	–	6.4(8)	–	385	0.61
fitKχ_4	17.2(6)	-34.7(7.1)	–	6.4(7)	–	385	0.60
fitK$_5$	13.6(2.0)	13.5(24.5)	-120(59.5)	12.6(3.3)	–	376	29
fitKχ_5	16.3(8)	-21.2(9.5)	-34.7(18.5)	8.1(1.1)	–	380	0.99
fitK$_6$	-17.9(10.3)	-38.8(76.8)	-61.9(77.6)	72.6(19.2)	-290(89)	369	898
fitKχ_6	16.2(1.2)	-21.3(10.3)	-34.5(18.6)	8.2(2.2)	-1(10.8)	380.0	1.02

fitK$(\chi)_n$=fit to KLOE with n subtr. const.; χ= with chiral constraints