Introduction to chiral perturbation theory

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0. Motivation and literature

1. QCD and chiral symmetry

2. Spontaneous symmetry breaking

3. Chiral perturbation theory for mesons

4. Chiral perturbation theory for baryons

5. Including resonances, etc.

Motivation of Keywords

- Chiral perturbation theory (ChPT) is the effective field theory (EFT) of the Standard Model/strong interactions at low energies.

- EFTs are low-energy approximations to (more) fundamental theories.

- Instead of solving the underlying theory, low-energy physics is described with a set of variables (effective degrees of freedom) that is suited for the particular energy region you are interested in.

- In our case: Pions and nucleons instead of the more fundamental quarks and gluons of QCD.

- Calculate physical quantities in terms of an expansion in p/Λ, where p stands for momenta or masses that are smaller than a certain momentum scale Λ.

- There exists a regime where both fundamental and effective theories yield the same results.

- EFTs are based on the most general Lagrangian, which includes all terms that are compatible with the symmetries of the underlying theory. $\Rightarrow$ Infinite number of terms. Each term is accompanied by a low-energy coupling constant (LEC).

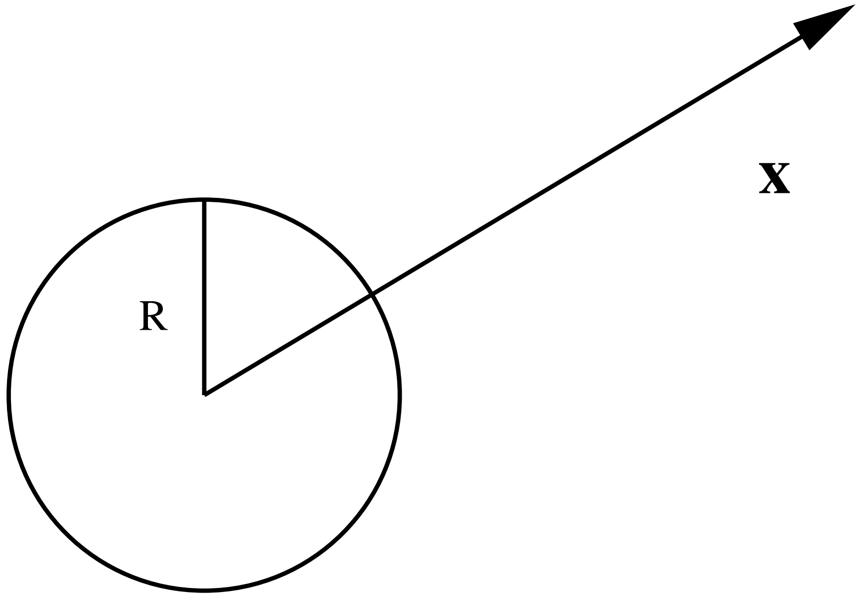
- One needs a method that allows one to decide which terms contribute in a calculation up to a certain accuracy: Weinberg's power counting.

- In actual calculations only a finite number of terms in the expansion in p/Λ has to be considered. $\Rightarrow$ Predictive power.

- Effective field theories are non-renormalizable in the traditional sense. However, as long as one considers all terms that are allowed by the symmetries, divergences that occur in calculations up to any given order of p/Λ can be renormalized by redefining fields and parameters of the Lagrangian of the effective field theory. The so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.

- **First example from electrostatics illustrating the idea of a scale (here distance scale).**

  Consider charge distribution $\rho(\vec{x}\,')$ which is localized inside a sphere of radius R.



  Potential from solution to Poisson equation,

$$\Delta\phi = -\rho,$$

reads

$$\phi(\vec{x}) = \frac{1}{4\pi} \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'.$$

Make use of

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

$\Rightarrow$

– Solution for $|\vec{x}| \lesssim R$ complicated

– Solution for $|\vec{x}| \gg R$ simple, because

$$\phi(\vec{x}) = \sum_{l,m} \underbrace{\left[\int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}\,') d^3x'\right]}_{\text{multipole moment } q_{lm}} \frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}.$$

– Far away, only the leading-order terms contribute.

– Systematic improvement possible.

– For smaller r, higher multipoles become more important.

– q_{lm} parameterize short-distance physics.

– $\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ determine the long-distance effects of short-distance physics.

– (Simplified) analogies[1]

Multipole expansion	EFT
q_{lm}	LECs
$\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$	Structures of most general $\mathcal{L}_{\mathrm{EFT}}$

– Here: Simple separation of scales (R).

– ChPT: Scales depend on underlying dynamics and masses of the participating particles.

[1] Observables will be calculated in perturbation theory using $\mathcal{L}_{\mathrm{EFT}}$.

- **Second example: "Integrating out" a heavy degree of freedom in a toy model ($m \ll M$):**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2) + \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) - \frac{\lambda}{2} \Phi \varphi^2.$$

Equations of motion:

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} - \frac{\partial \mathcal{L}}{\partial \Phi} = \Box \Phi + M^2 \Phi + \frac{\lambda}{2} \varphi^2 &= 0, \qquad (*) \\ \Box \varphi + m^2 \varphi + \lambda \varphi \Phi &= 0. \qquad (**) \end{aligned}$$

Formally solve $(*)$:

$$\Phi = -\frac{\lambda}{2M^2} \frac{1}{1 + \frac{\Box}{M^2}} \varphi^2.$$

Insert solution into $(**)$. $\Rightarrow$

$$\Box \varphi + m^2 \varphi - \frac{\lambda^2}{2M^2} \varphi \frac{1}{1 + \frac{\Box}{M^2}} \varphi^2 = 0$$

and expand to leading order in $1/M^2$:

$$\Box\varphi + m^2\varphi - \frac{\lambda^2}{2M^2}\varphi^3 = 0.$$

What is the corresponding effective Lagrangian to this order?

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) + \frac{\lambda^2}{8M^2}\varphi^4.$$

Compare with original Lagrangian:

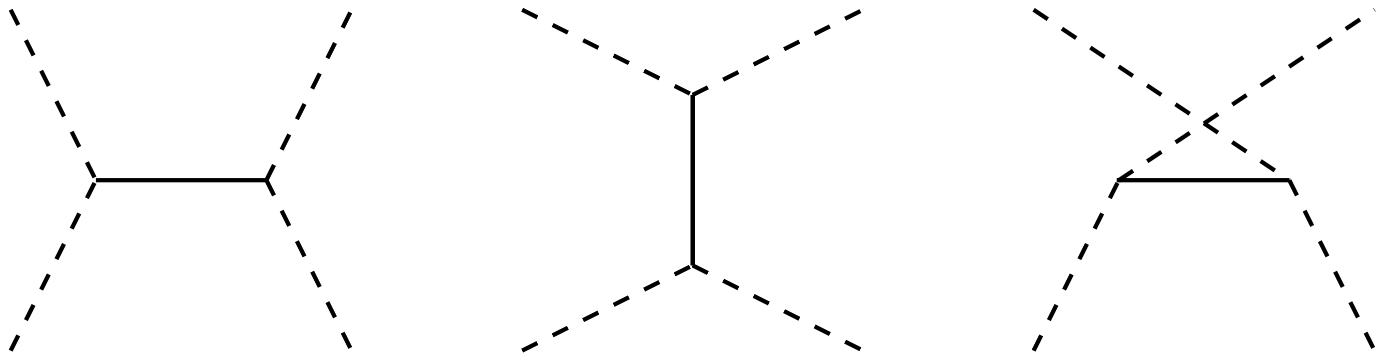
– Heavy degree of freedom is gone.

– A different interaction term has appeared.

Do the two Lagrangians produce the same low-energy scattering amplitude for $\varphi(p_1) + \varphi(p_2) \to \varphi(p_3) + \varphi(p_4)$?

Calculation with original Lagrangian.

Dotted line: Light particle; solid line: Heavy particle.



Mandelstam variables

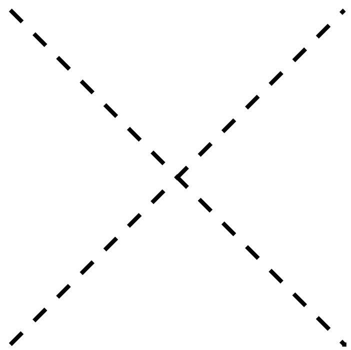
$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\
t &= (p_1 - p_3)^2 = (p_4 - p_2)^2, \\
u &= (p_1 - p_4)^2 = (p_3 - p_2)^2, \\
s + t + u &= 4m^2.
\end{aligned}
$$

Condition: $\{s, |t|, |u|\} \ll M^2 = \Lambda^2. \qquad (*)$

Result:

$$\mathcal{M}_{\text{fund}} = (-i\lambda)^2 \left(\frac{i}{s - M^2 + i0^+} + \frac{i}{t - M^2 + i0^+} + \frac{i}{u - M^2 + i0^+} \right)$$
$$\stackrel{(*)}{=} \frac{3i\lambda^2}{M^2} \left[1 + \mathcal{O}\left(\frac{\{s, t, u\}}{M^2} \right) \right].$$

Effective theory: Description in terms of contact interaction



$$\mathcal{M}_{\text{eff}} = \frac{i\lambda^2 4!}{8M^2} = \frac{3i\lambda^2}{M^2}.$$

Both calculations yield the same result!

EFT calculation simpler.

- Weinberg's effective field theory program:[2]

> ... if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S−matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. ...

Explanation of terms:

- Symmetries: Poincaré invariance, discrete symmetries C, P, T, but also internal symmetries such as isospin symmetry, chiral symmetry including the possibility of a spontaneous symmetry breakdown.

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[2] S. Weinberg, Physica A 96, 327 (1979)

- Analyticity $\leftrightarrow$ Causality.

- Unitarity: The sum over the propabilities of the final states must yield exactly 1:

$$\sum_f |\langle f|S|i\rangle|^2 = 1.$$

- Cluster decomposition:[3] Loosely speaking, distant experiments must yield uncorrelated results:

$$S_{\gamma+\delta\leftarrow\alpha+\beta} \longrightarrow S_{\delta\leftarrow\beta} S_{\gamma\leftarrow\alpha}.$$

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[3] S. Weinberg, The Quantum Theory Of Fields. Vol. 1: Foundations (Cambridge University Press, Cambridge, 1995), chapter 4

- **Aim of these lectures:**

  Most general description of the strong interactions at low energies: $\pi\pi$, πN, etc.

- **Challenge:**

  We need the

  1. **the most general Lagrangian;**

  2. **a consistent power counting scheme to perform perturbative calculations.**

Some original literature

Classical papers

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- H. Leutwyler, in Hadron Physics 94: Topics on the Structure and Interaction of Hadronic Systems, Proceedings, Workshop, Gramado, Brasil, edited by V. E. Herscovitz (World Scientific, Singapore, 1995), hep-ph/9406283

- V. Bernard, N. Kaiser, and U.-G. Meißner, Int. J. Mod. Phys. E 4, 193 (1995), hep-ph/9501384

- A. Pich, Rept. Prog. Phys. 58, 563 (1995), hep-ph/9502366
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QCD and Chiral Symmetry

Some Remarks on SU(3)

SU(3) in the context of the strong interactions

1. Gauge group of quantum chromodynamics (QCD)

2. Flavor SU(3)

3. Chiral symmetry for massless u, d, and s quarks
   $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$

Definition

SU(3) $\equiv$ set of all unitary, unimodular, 3×3 matrices U:

$$U^\dagger U = 1, \ \det(U) = 1.$$

Exponential representation

$$U(\Theta) = \exp\left(-i\sum_{a=1}^{8}\Theta_a\frac{\lambda_a}{2}\right), \quad \Theta_a \text{ real numbers.}$$

Gell-Mann matrices

$$\begin{aligned}
\frac{\lambda_a}{2} &= i\frac{\partial U}{\partial \Theta_a}(0,\cdots,0), \\
\lambda_a &= \lambda_a^\dagger, \\
\mathrm{Tr}(\lambda_a\lambda_b) &= 2\delta_{ab}, \\
\mathrm{Tr}(\lambda_a) &= 0.
\end{aligned}$$

Explicit representation

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\boldsymbol{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Set $\{i\lambda_a\}$: Basis of the Lie algebra su(3).

Commutation relations

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

Totally antisymmetric real structure constants

Exercise: $$f_{abc} = \frac{1}{4i} \mathrm{Tr}([\lambda_a, \lambda_b]\lambda_c)$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$

Anti-commutation relations

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2 d_{abc}\lambda_c$$

Totally symmetric d_{abc}

Exercise: $$d_{abc} = \frac{1}{4} \mathrm{Tr}(\{\lambda_a, \lambda_b\}\lambda_c)$$

Introduce

$$\boldsymbol{\lambda}_0 = \sqrt{2/3}\,\textbf{diag}(1,1,1)$$

Arbitrary 3×3 matrix M can be written as

$$M = \sum_{a=0}^{8} M_a \boldsymbol{\lambda}_a$$

M_a complex numbers

$$M_a = \frac{1}{2}\textbf{Tr}(\boldsymbol{\lambda}_a M)$$

The QCD Lagrangian

Quantum chromodynamics (QCD) is the gauge theory of the strong interactions with color SU(3) as the underlying gauge group

- Ingredients

The matter fields of QCD are the so-called quarks which are spin-1/2 fermions, with six different flavors in addition to their three possible colors

flavor	u	d	s
charge [e]	$2/3$	$-1/3$	$-1/3$
mass [MeV]	5.1 ± 0.9	9.3 ± 1.4	175 ± 25
flavor	c	b	t
charge [e]	$2/3$	$-1/3$	$2/3$
mass [GeV]	$1.15 - 1.35$	$4.0 - 4.4$	$174.3 \pm 3.2 \pm 4.0$

Quark field components

$$q_{f,A,\alpha}$$

$f = 1,2,3,4,5,6$: flavor index (u,d,s,c,b,t)
$A = 1,2,3$: color index (red,green,blue)
$\alpha = 1,2,3,4$: Dirac spinor index

QCD Lagrangian (apply the gauge principle with respect to the group SU(3))

Long version

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f,f'=1}^{6} \sum_{A,A'=1}^{3} \sum_{\alpha,\alpha'=1}^{4} \boxed{\bar{q}_{f,A,\alpha}} \Big[(\gamma^{\mu}_{\alpha\alpha'} i\partial_\mu - m_f \delta_{\alpha\alpha'}) \delta_{AA'} \underbrace{+ g \sum_{a=1}^{8} \mathcal{A}_{\mu,a} \frac{\lambda_{a,AA'}}{2} \gamma^{\mu}_{\alpha\alpha'}}_{\text{from gauge principle}} \Big] \delta_{ff'} \boxed{q_{f',A',\alpha'}} - \sum_{a=1}^{8} \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_{a}$$

Short version

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=\substack{u,d,s,\\c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}_a^{\mu\nu}.$$

Extremely short version

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - \mathcal{M})q - \frac{1}{2}\mathrm{Tr}_c(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu})$$

Gauge transformation of the quark fields

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} \mapsto q'_f = \exp\left[-i\sum_{a=1}^{8}\Theta_a(x)\frac{\lambda_a}{2}\right] q_f = U[\Theta(x)]q_f$$

Transformation behavior of the gauge fields

$$\mathcal{A}_\mu \equiv \mathcal{A}_{\mu,a}\frac{\lambda_a}{2} \mapsto U\mathcal{A}_\mu U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger$$

Covariant derivative of the quark fields

$$D_\mu q_f = (\partial_\mu - ig\mathcal{A}_\mu)q_f \overset{\text{Exercise:}}{\mapsto} (D_\mu q_f)' = D'_\mu q'_f = U D_\mu q_f$$

transforms as quark fields

Field strengths transform as

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu,a}\frac{\lambda_a}{2} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + ig[\mathcal{A}_\mu, \mathcal{A}_\nu] \overset{\text{Exercise:}}{\mapsto} U\mathcal{G}_{\mu\nu}U^\dagger$$

Equivalent (Gell-Mann matrices!)

$$\mathcal{G}_{\mu\nu,a} = \partial_\mu\mathcal{A}_{\nu,a} - \partial_\nu\mathcal{A}_{\mu,a} + gf_{abc}\mathcal{A}_{\mu,b}\mathcal{A}_{\nu,c}$$

Exercise: $\mathcal{L}_{\mathrm{QCD}}$ invariant under local SU(3)

Gauge invariance also allows for

$$\begin{aligned}\mathcal{L}_\theta &= \frac{g^2\bar{\theta}}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{8}\mathcal{G}^a_{\mu\nu}\mathcal{G}^a_{\rho\sigma}\\ &= \frac{g^2\bar{\theta}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}_c(\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma})\end{aligned}$$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ even permutation of } \{0,1,2,3\}\\ -1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ odd permutation of } \{0,1,2,3\}\\ 0 & \text{otherwise}\end{cases}$$

So-called θ term implies explicit P and CP violation in the strong interactions

$\rightsquigarrow$, e.g., electric dipole moment of the neutron

empirical information: very small

Accidental, Global Symmetries of $\mathcal{L}_{\mathrm{QCD}}$

The pion is special!

quark content	mesons
$u\bar{d}$	π^+, ρ^+
$(u\bar{u} - d\bar{d})/\sqrt{2}$	π^0, ρ^0
$d\bar{u}$	π^-, ρ^-

$$\begin{aligned} M_{\pi^+} = 140\,\text{MeV} &\ll M_\rho = 776\,\text{MeV}, \\ M_\pi &\ll m_p = 938\,\text{MeV}. \end{aligned}$$

$$\begin{aligned} M_{\pi^+} &< M_{K^+} = 494\,\text{MeV}, \\ M_{\pi^+} &\ll M_{D^+} = 1869\,\text{MeV}. \end{aligned}$$

$$\left(\begin{array}{c} m_u = 0.005\,\textsf{GeV} \\ m_d = 0.009\,\textsf{GeV} \\ m_s = 0.175\,\textsf{GeV} \end{array}\right) \ll \Lambda_\chi \approx 1\,\textsf{GeV} \leq \left(\begin{array}{r} m_c = (1.15 - 1.35)\,\textsf{GeV} \\ m_b = (4.0 - 4.4)\,\textsf{GeV} \\ m_t = 174\,\textsf{GeV} \end{array}\right)$$

Motivation

$$m_p \gg 2m_u + m_d$$

Consider the light-flavor quarks in the so-called chiral limit $m_u, m_d, m_s \to 0$ as starting point in the discussion of low-energy QCD:

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a$$

What are the global symmetries of $\mathcal{L}^0_{\mathrm{QCD}}$?

Chirality matrix

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^2 = 1$$

Projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5) = P_L^\dagger, \quad P_R = \frac{1}{2}(1 + \gamma_5) = P_R^\dagger$$

Exercise: Properties

$$P_L + P_R = 1$$

$$P_L^2 = P_L, \quad P_R^2 = P_R$$

$$P_L P_R = P_R P_L = 0$$

Left- and right-handed quark fields q_L and q_R

$$q_L = P_L q, \quad q_R = P_R q$$

Exercise:

$$\bar{q}\Gamma_i q = \begin{cases} \bar{q}_L \Gamma_1 q_L + \bar{q}_R \Gamma_1 q_R & \text{for} \quad \Gamma_1 \in \{\gamma^\mu, \gamma^\mu\gamma_5\} \\ \bar{q}_R \Gamma_2 q_L + \bar{q}_L \Gamma_2 q_R & \text{for} \quad \Gamma_2 \in \{1, \gamma_5, \sigma^{\mu\nu}\} \end{cases}$$

($\bar{q}_L = \bar{q}P_R$ and $\bar{q}_R = \bar{q}P_L$)

QCD Lagrangian in the chiral limit

$$\mathcal{L}^0_{\rm QCD} = \sum_{l=u,d,s} (\bar{q}_{L,l} i \not{D}\, q_{L,l} + \bar{q}_{R,l} i \not{D}\, q_{R,l}) - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a$$

invariant under (covariant derivative flavor independent!)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^L_a \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^R_a \frac{\lambda_a}{2}\right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

$\mathcal{L}^0_{\rm QCD}$ **has a classical global U(3)$_L$ $\times$ U(3)$_R$ symmetry**

Applying Noether's theorem from such an invariance one would expect a total of $2 \times (8+1) = 18$ conserved currents

Noether's Theorem

Continuous symmetries $\leftrightarrow$ conserved quantities

Lagrangian $\mathcal{L}$ depending on n independent fields Φ_i

$$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

n equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \qquad i = 1, \cdots, n$$

Consider transformations which depend on r real local parameters $\epsilon_a(x)$ (method of Gell-Mann and Lévy)

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

Variation of the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}(\Phi_i', \partial_\mu\Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu\Phi_i) = \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i$$

$$\partial_\mu\delta\Phi_i = -i[\partial_\mu\epsilon_a(x)]F_i^a - i\epsilon_a(x)\partial_\mu F_i^a$$

$$\begin{aligned} \cdots &= \epsilon_a(x)\left(-i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a\right) + \partial_\mu\epsilon_a(x)\left(-i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a\right) \\ &\equiv \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x)J^{\mu,a} \end{aligned}$$

Consistency for solutions to the EOM

$$\partial_\mu J^{\mu,a} = -i\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\right)F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a = -i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a$$

Currents and divergences of currents from variation

$$J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon_a}$$
$$\partial_\mu J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \epsilon_a}$$

Assume Lagrangian to be invariant under a global transformation:

$$\delta \mathcal{L} = 0 \quad \wedge \quad \partial_\mu \epsilon_a(x) J^{\mu,a} = 0$$

$\Rightarrow$ Current $J^{\mu,a}$ is conserved

$$\partial_\mu J^{\mu,a} = 0$$

Charge

$$Q^a(t) = \int d^3x J_0^a(t, \vec{x})$$

is time independent, i.e., a constant of the motion

Classical conservation laws (taken from M. Mojžiš, Bosen lectures 2006)

$$\varphi(x) \to \varphi(x) + \epsilon\delta\varphi(x)$$

symmetry	conservation law	current or charge
$\delta\mathcal{L} = 0$	$\partial_\mu J^\mu(x) = 0$	$J^\mu = \delta\varphi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$
$\delta\mathcal{L} = \epsilon\partial_\mu\mathcal{J}^\mu$	$\partial_\mu J^\mu(x) = 0$	$J^\mu = \delta\varphi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \mathcal{J}^\mu$
$\delta L = 0$	$d_t Q(t) = 0$	$Q = \int d^3x \delta\varphi \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}$
$\delta L = \epsilon d_t \mathcal{Q}(t)$	$d_t Q(t) = 0$	$Q = \int d^3x \delta\varphi \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)} - \mathcal{Q}$
$\delta S = 0$	$\partial_\mu I^\mu = 0$	I^μ not explicitly known

Transition to a Quantum (Field) Theory

Analogy: Point mass m in a central potential $V(\vec{r}) = V(r)$.
Lagrange and Hamilton functions are rotationally invariant.

Consequence: Angular momentum $\vec{l} = \vec{r} \times \vec{p}$ is a constant of the motion.

Transition to quantum mechanics. $\Rightarrow$ Operators

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0.$$

Components of the angular momentum operator

$$\hat{l}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k = -i\hat{p}_j(-i\epsilon_{ijk})\hat{x}_k$$

satisfy the commutation relations

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk}\hat{l}_k$$

i.e., they cannot simultaneously be diagonalized.

Angular momentum operators are generators of rotations:

$$|\Psi'\rangle = \exp(-i\alpha_i \hat{l}_i)|\Psi\rangle.$$

Rotational invariance of the quantum system

$$[\hat{H}, \hat{l}_i] = 0$$

i.e., $\hat{l}_i$ are still constants of the motion.

Simultaneously diagonalize $\hat{H}$, $\hat{\vec{l}}^2$, and $\hat{l}_3$.

Multiplets with eigenvalues $l(l+1)$ and $m = -l, \cdots, l$ $(l = 0, 1, 2, \cdots)$.

Energy eigenvalues depend on V (dynamics).

Classify operators according to transformation behavior. Example: Components $\hat{A}_i$ of a vector operator

$$[\hat{l}_i, \hat{A}_j] = i\epsilon_{ijk}\hat{A}_k.$$

Use Wigner-Eckart theorem to calculate matrix elements.

Analogous case in quantum field theory

Canonical quantization: Fields Φ_i and their conjugate momenta $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i) \Rightarrow$ operators.

$$\begin{aligned}
[\Phi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ij} \quad &\leftrightarrow \quad [\hat{x}_i,\hat{p}_j] = i\delta_{ij} \\
[\Phi_i(t,\vec{x}),\Phi_j(t,\vec{y})] = 0 \quad &\leftrightarrow \quad [\hat{x}_i,\hat{x}_j] = 0 \\
[\Pi_i(t,\vec{x}),\Pi_j(t,\vec{y})] = 0 \quad &\leftrightarrow \quad [\hat{p}_i,\hat{p}_j] = 0
\end{aligned}$$

Consider infinitesimal transformations which are linear in the fields,

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t^a_{ij}\Phi_j(x) \quad \leftrightarrow \quad \hat{x}_i \mapsto \hat{x}_i - i\epsilon_k(-i\epsilon_{kij})\hat{x}_j$$

t^a_{ij} are constants generating a mixing of the fields

$$J^{\mu,a}(x) = -it^a_{ij}\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j$$

$$Q^a(t) = -i\int d^3x\,\Pi_i(x)t^a_{ij}\Phi_j(x) \;\leftrightarrow\; \hat{l}_k = -i\hat{p}_i(-i\epsilon_{kij})\hat{x}_j = \epsilon_{kij}\hat{x}_i\hat{p}_j$$

where $J^{\mu,a}(x)$ and $Q^a(t)$ are now operators

Transformation behavior of field operators

$$\begin{aligned}[Q^a(t),\Phi_k(t,\vec{y})] &= -it^a_{ij}\int d^3x\,[\Pi_i(t,\vec{x})\Phi_j(t,\vec{x}),\Phi_k(t,\vec{y})] \\ &= -t^a_{kj}\Phi_j(t,\vec{y}) \\ &\leftrightarrow [\hat{l}_k,\hat{x}_i] = i\epsilon_{kij}\hat{x}_j\end{aligned}$$

Q^a are generators of the transformations acting on the states of Hilbert space

Global Symmetry Currents of the Light Quark Sector

Consider infinitesimal, local tranformations

$$
\begin{aligned}
q_L &\mapsto \left(1 - i \sum_{a=1}^{8} \epsilon_a^L \frac{\lambda_a}{2} - i\epsilon^L\right) q_L \\
q_R &\mapsto \cdots
\end{aligned}
$$

Variation

$$
\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_L \left(\sum_{a=1}^{8} \partial_\mu \epsilon_a^L \frac{\lambda_a}{2} + \partial_\mu \epsilon^L\right) \gamma^\mu q_L + (L \to R)
$$

Currents

$$
\begin{aligned}
L^{\mu,a} &= \frac{\partial \delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial \partial_\mu \epsilon_a^L} = \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L, \quad \partial_\mu L^{\mu,a} = \frac{\partial \delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial \epsilon_a^L} = 0 \\
L^\mu &= \frac{\partial \delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial \partial_\mu \epsilon^L} = \bar{q}_L \gamma^\mu q_L, \quad \partial_\mu L^\mu = \frac{\partial \delta\mathcal{L}^0_{\mathrm{QCD}}}{\partial \epsilon^L} = 0
\end{aligned}
$$

+ analogous expressions for $R^{\mu,a}$ and R^μ

Linear combinations

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$
$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q$$

Vector and axial-vector current densities, respectively,

$$P : V^{\mu,a}(t,\vec{x}) \mapsto V_\mu^a(t,-\vec{x})$$
$$P : A^{\mu,a}(t,\vec{x}) \mapsto -A_\mu^a(t,-\vec{x})$$

Conserved singlet vector current

$$V^\mu = R^\mu + L^\mu = \bar{q}\gamma^\mu q, \quad \partial_\mu V^\mu = 0$$

Singlet axial-vector current

$$A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu \gamma_5 q$$

This symmetry is not preserved by quantization and there will be extra terms, referred to as anomalies, resulting in

$$\partial_\mu A^\mu = \frac{3g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1$$

The Chiral Algebra

Define "charge operators" as the space integrals of the charge densities

$$\begin{aligned} Q_L^a(t) &= \int d^3x\, q_L^\dagger(t,\vec{x})\frac{\lambda^a}{2}q_L(t,\vec{x}), \qquad a=1,\cdots,8 \\ Q_R^a(t) &= \int d^3x\, q_R^\dagger(t,\vec{x})\frac{\lambda^a}{2}q_R(t,\vec{x}), \qquad a=1,\cdots,8 \\ Q_V(t) &= \int d^3x\, \left[q_L^\dagger(t,\vec{x})q_L(t,\vec{x})+q_R^\dagger(t,\vec{x})q_R(t,\vec{x})\right] \end{aligned}$$

$H^0_{\rm QCD}$ **exhibits a global SU(3)$_L\times$SU(3)$_R\times$U(1)$_V$ symmetry**

$$[Q_L^a, H^0_{\rm QCD}] = [Q_R^a, H^0_{\rm QCD}] = [Q_V, H^0_{\rm QCD}] = 0$$

Lie algebra of SU(3)$_L$ $\times$ SU(3)$_R$ $\times$ U(1)$_V$

$$\begin{aligned} [Q_L^a, Q_L^b] &= i f_{abc} Q_L^c \\ [Q_R^a, Q_R^b] &= i f_{abc} Q_R^c \\ [Q_L^a, Q_R^b] &= 0 \\ [Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0 \end{aligned}$$

How does one verify the commutation relations of the charge operators?

1. **Anti-commutation relations of Fermi fields**

$$\begin{aligned}
\{q_{f,A,\alpha}(t,\vec{x}), q^{\dagger}_{f',A',\alpha'}(t,\vec{y})\} &= \delta^3(\vec{x}-\vec{y})\delta_{ff'}\delta_{AA'}\delta_{\alpha\alpha'} \\
\{q_{f,A,\alpha}(t,\vec{x}), q_{f',A',\alpha'}(t,\vec{y})\} &= 0 \\
\{q^{\dagger}_{f,A,\alpha}(t,\vec{x}), q^{\dagger}_{f',A',\alpha'}(t,\vec{y})\} &= 0
\end{aligned}$$

2. **Exercise:** $[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b$

3. **Let F_i, C_i, and Γ_i be 3×3 flavor matrices, 3×3 color matrices, 4×4 Dirac matrices, respectively**

$$\begin{aligned}
[q^{\dagger}(t,\vec{x})F_1C_1\Gamma_1 q(t,\vec{x}), q^{\dagger}(t,\vec{y})F_2C_2\Gamma_2 q(t,\vec{y})] &= \\
&\delta^3(\vec{x}-\vec{y})q^{\dagger}(t,\vec{x})F_1F_2C_1C_2\Gamma_1\Gamma_2 q(t,\vec{y}) \\
&-\delta^3(\vec{x}-\vec{y})q^{\dagger}(t,\vec{y})F_2F_1C_2C_1\Gamma_2\Gamma_1 q(t,\vec{x})
\end{aligned}$$

4. Insert appropriate projection operators

5. Integrate with respect to $\vec{x}$ and $\vec{y}$

Example (recall $P_L^\dagger = P_L$ and $P_L^2 = P_L$)

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= \int d^3x d^3y [q^\dagger(t,\vec{x}) P_L^\dagger \frac{\lambda_a}{2} P_L q(t,\vec{x}), q^\dagger(t,\vec{y}) P_L^\dagger \frac{\lambda_b}{2} P_L q(t,\vec{y})] \\
&= \int d^3x d^3y \delta^3(\vec{x}-\vec{y}) q^\dagger(t,\vec{x}) \underbrace{P_L^\dagger P_L P_L^\dagger P_L}_{P_L} \frac{\lambda_a}{2}\frac{\lambda_b}{2} q(t,\vec{y}) \\
&\quad - \int d^3x d^3y \delta^3(\vec{x}-\vec{y}) q^\dagger(t,\vec{y}) P_L \frac{\lambda_b}{2}\frac{\lambda_a}{2} q(t,\vec{x}) \\
&= i f_{abc} \int d^3x q^\dagger(t,\vec{x}) \frac{\lambda_c}{2} P_L q(t,\vec{x}) = i f_{abc} Q_L^c
\end{aligned}
$$

Chiral Symmetry Breaking Due To Quark Masses

$$\mathcal{M} = \textbf{diag}(m_u, m_d, m_s)$$

Quark-mass term mixes left- and right-handed fields

$$\mathcal{L}_{\mathcal{M}} = -\bar{q}\mathcal{M}q = -(\bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}q_R)$$

Transformation of left-handed fields

$$q_L \mapsto \left(1 - i\sum_{a=1}^{8}\epsilon_a^L\frac{\lambda_a}{2} - i\epsilon^L\right)q_L$$

Variation $\delta\mathcal{L}_{\mathcal{M}}$

$$\begin{aligned}
\delta\mathcal{L}_{\mathcal{M}} &= -\left[-i\bar{q}_R\mathcal{M}\left(\sum_{a=1}^{8}\epsilon_a^L\frac{\lambda_a}{2} + \epsilon^L\right)q_L + i\bar{q}_L\left(\sum_{a=1}^{8}\epsilon_a^L\frac{\lambda_a}{2} + \epsilon^L\right)\mathcal{M}q_R\right] \\
&= -i\left[\sum_{a=1}^{8}\epsilon_a^L\left(\bar{q}_L\frac{\lambda_a}{2}\mathcal{M}q_R - \bar{q}_R\mathcal{M}\frac{\lambda_a}{2}q_L\right) + \epsilon^L\left(\bar{q}_L\mathcal{M}q_R - \bar{q}_R\mathcal{M}q_L\right)\right]
\end{aligned}$$

Divergences

$$\partial_\mu L^{\mu,a} = \frac{\partial \delta \mathcal{L}_{\mathcal{M}}}{\partial \epsilon_a^L} = -i\left(\bar{q}_L \frac{\lambda_a}{2}\mathcal{M} q_R - \bar{q}_R \mathcal{M}\frac{\lambda_a}{2} q_L\right)$$

$$\partial_\mu L^{\mu} = \frac{\partial \delta \mathcal{L}_{\mathcal{M}}}{\partial \epsilon^L} = -i\left(\bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L\right)$$

+ analogous expressions for $\partial_\mu R^{\mu,a}$ and $\partial_\mu R^\mu$ $(R \leftrightarrow L)$

More common (linear combinations)

$$\begin{aligned}
\partial_\mu V^{\mu,a} &= i\bar{q}[\mathcal{M}, \frac{\lambda_a}{2}]q \\
\partial_\mu A^{\mu,a} &= i\bar{q}\{\frac{\lambda_a}{2}, \mathcal{M}\}\gamma_5 q \\
\partial_\mu V^{\mu} &= 0 \\
\partial_\mu A^{\mu} &= 2i\bar{q}\mathcal{M}\gamma_5 q + \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}_a^{\mu\nu}\mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1
\end{aligned}$$

Summary

- Massless quarks: 16 conserved currents $L^{\mu,a}$ and $R^{\mu,a}$ ($V^{\mu,a}$ and $A^{\mu,a}$) + 1 conserved singlet vector current V^{μ}. Singlet axial-vector current A^{μ} has an anomaly.

- For any value of quark masses: flavor currents $\bar{u}\gamma^{\mu}u$, $\bar{d}\gamma^{\mu}d$, and $\bar{s}\gamma^{\mu}s$ are always conserved.

- Equal quark masses $m_u = m_d = m_s$:

  8 conserved vector currents $V^{\mu,a}$ ($[\lambda_a, 1] = 0$).
  SU(3) flavor symmetry.

  8 axial-vector currents $A^{\mu,a}$ are not conserved.
  Microscopic origin of the PCAC relation (partially conserved axial-vector current).

- $m_u = m_d$: isospin symmetry.

Green Functions and Chiral Ward Identities

Symmetry currents

$$\begin{aligned} V^{\mu,a} &= R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q \\ V^\mu &= \bar{q}\gamma^\mu q \\ A^{\mu,a} &= R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q \end{aligned}$$

$+$ scalar and pseudoscalar densities (see divergences of currents)

$$\begin{aligned} S_a(x) &= \bar{q}(x)\lambda_a q(x) \\ P_a(x) &= i\bar{q}(x)\gamma_5 \lambda_a q(x) \end{aligned}$$

Green functions: Matrix elements of time-ordered products
Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism: Relation to scattering amplitudes

Examples

"Vacuum" sector

$$\langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \quad \text{pion decay}$$
$$\langle 0|T[P_a(x)J^\mu(y)P_c(z)]|0\rangle \quad \text{pion electromagnetic form factor}$$
$$\langle 0|T[P_a(w)P_b(x)P_c(y)P_d(z)]|0\rangle \quad \text{pion-pion scattering}$$

One-nucleon sector

$$\langle N|J^\mu(x)|N\rangle \quad \text{nucleon electromagnetic form factors}$$
$$\langle N|A_a^\mu(x)|N\rangle \quad \text{axial form factor + ...}$$
$$\langle N|T[J^\mu(x)J^\nu(y)]|N\rangle \quad \text{Compton scattering}$$
$$\langle N|T[J^\mu(x)P_a(y)]|N\rangle \quad \text{pion electroproduction}$$

A chiral Ward identity relates the divergence of a Green function containing at least one factor of $V^{\mu,a}$ or $A^{\mu,a}$ to some linear combination of other Green functions.

Q: Why chiral?
A: $V^{\mu,a}$ and $A^{\mu,a}$ contain $L^{\mu,a}$ and $R^{\mu,a}$

Simple example

$$
\begin{aligned}
G^{\mu,ab}_{AP}(x,y) &= \langle 0|T[A^\mu_a(x)P_b(y)]|0\rangle \\
&= \Theta(x_0-y_0)\langle 0|A^\mu_a(x)P_b(y)|0\rangle \\
&\quad +\Theta(y_0-x_0)\langle 0|P_b(y)A^\mu_a(x)|0\rangle
\end{aligned}
$$

Divergence

$$
\begin{aligned}
&\partial^x_\mu G^{\mu,ab}_{AP}(x,y) \\
&= \partial^x_\mu[\Theta(x_0-y_0)\langle 0|A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A^\mu_a(x)|0\rangle] \\
&= \delta(x_0-y_0)\langle 0|A^0_a(x)P_b(y)|0\rangle - \delta(x_0-y_0)\langle 0|P_b(y)A^0_a(x)|0\rangle \\
&\quad +\Theta(x_0-y_0)\langle 0|\partial^x_\mu A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)\partial^x_\mu A^\mu_a(x)|0\rangle \\
&= \delta(x_0-y_0)\langle 0|[A^0_a(x),P_b(y)]|0\rangle + \langle 0|T[\partial^x_\mu A^\mu_a(x)P_b(y)]|0\rangle
\end{aligned}
$$

We made use of

$$
\partial^x_\mu\Theta(x_0-y_0) = \delta(x_0-y_0)g_{0\mu} = -\partial^x_\mu\Theta(y_0-x_0)
$$

Main features of (chiral) Ward identities:

1. Differentiation of the theta functions $\Rightarrow$ Equal-time commutators between a charge density and the remaining quadratic forms $\Rightarrow$ Reflection of underlying symmetry

2. Divergence of the current operator in question.

   - Perfect symmetry $\Rightarrow$ such terms vanish
     Example: Electromagnetic case with its U(1) symmetry.

   - Approximate symmetry $\Rightarrow$ additional term involving the symmetry breaking appears
     For a soft breaking such a divergence can be treated as a perturbation.

Via induction, the generalization of the above simple example to an $(n+1)$-point Green function is symbolically of the form

$$\begin{aligned}
&\partial_\mu^x \langle 0|T\{J^\mu(x)A_1(x_1)\cdots A_n(x_n)\}|0\rangle = \\
&\quad \langle 0|T\{[\partial_\mu^x J^\mu(x)]A_1(x_1)\cdots A_n(x_n)\}|0\rangle \\
&\quad +\delta(x^0-x_1^0)\langle 0|T\{[J_0(x),A_1(x_1)]A_2(x_2)\cdots A_n(x_n)\}|0\rangle \\
&\quad +\delta(x^0-x_2^0)\langle 0|T\{A_1(x_1)[J_0(x),A_2(x_2)]\cdots A_n(x_n)\}|0\rangle \\
&\quad +\cdots+\delta(x^0-x_n^0)\langle 0|T\{A_1(x_1)\cdots[J_0(x),A_n(x_n)]\}|0\rangle,
\end{aligned}$$

where J^μ stands generically for any of the Noether currents.

The Algebra of Currents

$$
\begin{aligned}
[V_0^a(t,\vec{x}), V_b^\mu(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(t,\vec{x}), \\
[V_0^a(t,\vec{x}), V^\mu(t,\vec{y})] &= 0, \\
[V_0^a(t,\vec{x}), A_b^\mu(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(t,\vec{x}), \\
[V_0^a(t,\vec{x}), S_b(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(t,\vec{x}), \qquad b=1,\cdots,8, \\
[V_0^a(t,\vec{x}), S_0(t,\vec{y})] &= 0, \\
[V_0^a(t,\vec{x}), P_b(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(t,\vec{x}), \qquad b=1,\cdots,8, \\
[V_0^a(t,\vec{x}), P_0(t,\vec{y})] &= 0, \\
[A_0^a(t,\vec{x}), V_b^\mu(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(t,\vec{x}), \\
[A_0^a(t,\vec{x}), V^\mu(t,\vec{y})] &= 0, \\
[A_0^a(t,\vec{x}), A_b^\mu(t,\vec{y})] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(t,\vec{x}), \\
[A_0^a(t,\vec{x}), S_b(t,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}) \left[\sqrt{\frac{2}{3}} \delta_{ab} P_0(t,\vec{x}) + d_{abc} P_c(t,\vec{x}) \right], \\
& \quad b=1,\cdots,8, \\
[A_0^a(t,\vec{x}), S_0(t,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}) \sqrt{\frac{2}{3}} P_a(t,\vec{x}),
\end{aligned}
$$

$$
\begin{aligned}
[A_0^a(t,\vec{x}), P_b(t,\vec{y})] &= -i\delta^3(\vec{x}-\vec{y})\left[\sqrt{\frac{2}{3}}\delta_{ab}S_0(t,\vec{x}) + d_{abc}S_c(t,\vec{x})\right], \\
&\quad b = 1,\cdots,8, \\
[A_0^a(t,\vec{x}), P_0(t,\vec{y})] &= -i\delta^3(\vec{x}-\vec{y})\sqrt{\frac{2}{3}}S_a(t,\vec{x}).
\end{aligned}
$$

For example,

$$
\begin{aligned}
&[V_a^0(t,\vec{x}), V_b^\mu(t,\vec{y})] \\
&= [q^\dagger(t,\vec{x})1\frac{\lambda_a}{2}q(t,\vec{x}), q^\dagger(t,\vec{y})\gamma_0\gamma^\mu\frac{\lambda_b}{2}q(t,\vec{y})] \\
&= \delta^3(\vec{x}-\vec{y})\left[q^\dagger(t,\vec{x})\gamma_0\gamma^\mu\frac{\lambda_a}{2}\frac{\lambda_b}{2}q(t,\vec{y}) - q^\dagger(t,\vec{y})\gamma_0\gamma^\mu\frac{\lambda_b}{2}\frac{\lambda_a}{2}q(t,\vec{x})\right] \\
&= \delta^3(\vec{x}-\vec{y})if_{abc}V_c^\mu(t,\vec{x}).
\end{aligned}
$$

Caveats

- **Schwinger terms**
- **Covariant time-ordered product**
- **Feynman's conjecture**

Naive application of equal-time commutation relations may lead to erroneous results.

Illustration (due to Schwinger)

$$\begin{aligned}[J_0(t,\vec{x}), J_i(t,\vec{y})] &= [\Psi^\dagger(t,\vec{x})\Psi(t,\vec{x}), \Psi^\dagger(t,\vec{y})\gamma_0\gamma_i\Psi(t,\vec{y})] \\ &= \delta^3(\vec{x}-\vec{y})\Psi^\dagger(t,\vec{x})[1,\gamma_0\gamma_i]\Psi(t,\vec{x}) = 0 \qquad (*)\end{aligned}$$

Schwinger: Result cannot be true.

Consider

$$[J_0(t,\vec{x}), \vec{\nabla}_y \cdot \vec{J}(t,\vec{y})]$$

Current conservation: $\partial_\mu J^\mu = 0$. $\Rightarrow$

$$[J_0(t,\vec{x}), \vec{\nabla}_y \cdot \vec{J}(t,\vec{y})] = -[J_0(t,\vec{x}), \partial_t J_0(t,\vec{y})]$$

Assumption: $(*)$ true. $\Rightarrow$

$$0 = [J_0(t,\vec{x}), \partial_t J_0(t,\vec{y})]$$

Evaluate for $\vec{x} = \vec{y}$ between the ground state:

$$0 = \langle 0 | [J_0(t,\vec{x}), \partial_t J_0(t,\vec{x})] | 0 \rangle$$

Insert complete set of states

$$= \sum_n \left(\langle 0 | J_0(t,\vec{x}) | n \rangle \langle n | \partial_t J_0(t,\vec{x}) | 0 \rangle - \langle 0 | \partial_t J_0(t,\vec{x}) | n \rangle \langle n | J_0(t,\vec{x}) | 0 \rangle \right)$$

Make use of

$$\partial_t J_0(t,\vec{x}) = i[H, J_0(t,\vec{x})].$$

Thus

$$0 = 2i \sum_n \underbrace{(E_n - E_0) |\langle 0 | J_0(t,\vec{x}) | n \rangle|^2}_{\geq 0}$$

Conclusion

$$\langle 0 | J_0(t,\vec{x}) | n \rangle = 0 \quad \forall n \neq 0$$

unphysical $\Rightarrow$ $(*)$ not correct

Corrected version picks up an additional, so-called Schwinger term containing a derivative of the delta function.

Jackiw: Equal-time commutation relation between a charge density and a current density can be determined up to one derivative of the δ function

$$[J_0^a(0,\vec{x}), J_i^b(0,\vec{y})] = iC_{abc}J_i^c(0,\vec{x})\delta^3(\vec{x}-\vec{y}) + S_{ij}^{ab}(0,\vec{y})\partial^j\delta^3(\vec{x}-\vec{y}),$$

Schwinger term possesses the symmetry

$$S_{ij}^{ab}(0,\vec{y}) = S_{ji}^{ba}(0,\vec{y}),$$

C_{abc}: Structure constants of the group in question.

Q: Do the chiral Ward identities still work?

A: Yes and no

Above derivation made use of the naive time-ordered product (T).

Covariant time-ordered product (T^*) typically differs by another non-covariant term (so-called seagull).

Feynman's conjecture: Cancellation between Schwinger terms and seagull terms such that a Ward identity obtained by using the naive T product and by simultaneously omitting Schwinger terms ultimately yields the correct result to be satisfied by the Green function (involving the covariant T^* product).

But: Not true in general.

Sufficient condition: Time component algebra of the full theory remains the same as the one derived canonically and does not possess a Schwinger term.

QCD in the Presence of External Fields and the Generating Functional

So far: Explicitly work out the chiral Ward identity you are interested in.

Q: Is it possible to somehow obtain all chiral Ward identities from a single expression?

A: Yes (without proof)

- **Introduce into the Lagrangian of QCD the couplings of the**

  1. **nine vector currents**

  2. **eight axial-vector currents**

3. nine scalar quark densities

4. nine pseudoscalar quark densities

to external c-number fields $v^\mu(x)$, $v^\mu_{(s)}$, $a^\mu(x)$, $s(x)$, and $p(x)$:

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \mathcal{L}_{\rm ext}$$

$$\mathcal{L}_{\rm ext} = \bar{q}\gamma_\mu\left(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu\right)q - \bar{q}(s - i\gamma_5 p)q.$$

Parameterization

$$v^\mu = \sum_{a=1}^{8}\frac{\lambda_a}{2}v^\mu_a, \quad a^\mu = \sum_{a=1}^{8}\frac{\lambda_a}{2}a^\mu_a, \quad s = \sum_{a=0}^{8}\lambda_a s_a, \quad p = \sum_{a=0}^{8}\lambda_a p_a.$$

- Combine all Green functions in a generating functional

$$\exp(iZ[v,a,s,p]) = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{\rm ext}(x)\right]|0\rangle$$

- **Obtain Green function through a functional derivative with respect to the external fields**

Pedagogical illustration

$$\mathcal{L} = \mathcal{L}_0(\phi) + \mathcal{L}_{\text{ext}},$$
$$\mathcal{L}_{\text{ext}} = j(x)\phi(x)$$

Generating functional for Green functions of the type

$$G(x_1, \cdots, x_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle$$

$$\exp(iZ[j]) = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{\text{ext}}(x)\right]|0\rangle$$

$$= 1 + i\int d^4xj(x)\langle 0|\phi(x)|0\rangle$$
$$+\sum_{k=2}\frac{i^k}{k!}\int d^4x_1\cdots d^4x_kj(x_1)\cdots j(x_k)\langle 0|T[\phi(x_1)\cdots\phi(x_k)]|0\rangle$$
$$= \cdots + \frac{i^2}{2}\int d^4x_1d^4x_2j(x_1)j(x_2)\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle + \cdots$$

E.g.

$$\begin{aligned} G(x_1,x_2) &= \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle \\ &= (-i)^2 \frac{\delta^2 Z[j]}{\delta j(x_1)\delta j(x_2)}\bigg|_{j=0} \end{aligned}$$

Powers and sort of functional derivatives must match:

$$1, \qquad i\int d^4x j(x)\langle 0|\phi(x)|0\rangle : \qquad \text{too few terms}$$

$$\frac{i^k}{k!}\int d^4x_1\cdots d^4x_k j(x_1)\cdots j(x_k)\langle 0|\phi(x_1)\cdots\phi(x_k)|0\rangle, k\geq 3 :$$

too many terms, because j is set equal to 0 at the end

Exercise: Make use of

$$\frac{\delta j(x)}{\delta j(y)} = \delta^4(x-y)$$

$$\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}\frac{1}{2}\int d^4x d^4y j(x)j(y)\langle 0|T[\phi(x)\phi(y)]|0\rangle = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$$

- Examples

  - Scalar quark condensate in the chiral limit, $\langle 0|\bar{u}u|0\rangle_0$,

    $$\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)}+\frac{\delta}{\delta s_3(x)}+\frac{1}{\sqrt{3}}\frac{\delta}{\delta s_8(x)}\right]\exp(iZ[v,a,s,p])\Bigg|_{v=a=s=p=0}$$

    Subscript 0: Chiral limit

  - Two-point function of two axial-vector currents of the "real world," i.e., for $s=\text{diag}(m_u,m_d,m_s)$, and the "true vacuum" $|0\rangle$,

    $$\langle 0|T[A_\mu^a(x)A_\nu^b(0)]|0\rangle = (-i)^2\frac{\delta^2}{\delta a_a^\mu(x)\delta a_b^\nu(0)}\exp(iZ[v,a,s,p])\Bigg|_{v=a=p=0,s=\text{diag}(m_u,m_d,m_s)}.$$

- Q: But where is QCD?

  A: In $|0\rangle$ and q (solutions to EOM)

  (The actual value of the generating functional for a given configuration of external fields v, a, s, and p reflects the dynamics generated by the QCD Lagrangian.)

- Q: But where is the (infinite) set of all chiral Ward identities?

  A: Ward identities obeyed by the Green functions are equivalent to an invariance of the generating functional under a local transformation of the external fields

- The use of local transformations allows one to also consider divergences of Green functions.

- Q: What do we require of the external fields?

A: We want $\mathcal{L}$ to be a Hermitian Lorentz scalar, to be even under P, C, and T, and to be invariant under local chiral transformations.

What does that imply for the external fields?

- Parity

  Transformation behavior of quark fields

$$q_f(t,\vec{x}) \stackrel{P}{\mapsto} \gamma^0 q_f(t,-\vec{x})$$

  Properties of the Dirac matrices Γ

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

Requirement of parity conservation

$$\mathcal{L}(t,\vec{x}) \stackrel{P}{\mapsto} \mathcal{L}(t,-\vec{x})$$

$\Rightarrow$

$$v^\mu \stackrel{P}{\mapsto} v_\mu, \quad v^\mu_{(s)} \stackrel{P}{\mapsto} v_\mu^{(s)}, \quad a^\mu \stackrel{P}{\mapsto} -a_\mu, \quad s \stackrel{P}{\mapsto} s, \quad p \stackrel{P}{\mapsto} -p.$$

(Change of arguments from $(t,\vec{x})$ to $(t,-\vec{x})$ implied.)

Example:

$$\bar{q}(t,\vec{x})\gamma^\mu v_\mu(t,\vec{x})q(t,\vec{x}) \;\stackrel{P}{\mapsto}\; \bar{q}(t,-\vec{x})\gamma^0\gamma^\mu\tilde{v}_\mu(t,-\vec{x})\gamma^0 q(t,-\vec{x})$$

Tilde denotes the transformed external field.

Make use of table, i.e., $\gamma^0\gamma^\mu\gamma^0 = \gamma_\mu$,

$$\cdots = \bar{q}(t,-\vec{x})\gamma_\mu\tilde{v}_\mu(t,-\vec{x})q(t,-\vec{x}) \stackrel{!}{=} \bar{q}(t,-\vec{x})\gamma_\mu v^\mu(t,-\vec{x})q(t,-\vec{x}).$$

We thus obtain

$$v_\mu(t,\vec{x}) \stackrel{P}{\mapsto} v^\mu(t,-\vec{x}).$$

- **Charge conjugation**

Transformation behavior of the quark fields

$$q_{\alpha,f} \stackrel{C}{\mapsto} C_{\alpha\beta}\bar{q}_{\beta,f}, \quad \bar{q}_{\alpha,f} \stackrel{C}{\mapsto} -q_{\beta,f}C^{-1}_{\beta\alpha},$$

α and β: Dirac spinor indices,

f: flavor index

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -C^{-1} = -C^{\dagger} = -C^T$$

usual charge conjugation matrix

Properties of the Dirac matrices Γ

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$-C\Gamma^T C$	1	$-\gamma^\mu$	$-\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$

Using

$$\begin{aligned}
\bar{q}\Gamma F q &= \bar{q}_{\alpha,f}\Gamma_{\alpha\beta}F_{ff'}q_{\beta,f'} \\
&\overset{C}{\mapsto} -q_{\gamma,f}C^{-1}_{\gamma\alpha}\Gamma_{\alpha\beta}F_{ff'}C_{\beta\delta}\bar{q}_{\delta,f'} \\
&\overset{\text{Fermi statistics}}{=} \bar{q}_{\delta,f'}\underbrace{F_{ff'}}_{F^T_{f'f}}\underbrace{C^{-1}_{\gamma\alpha}\Gamma_{\alpha\beta}C_{\beta\delta}}_{(C^{-1}\Gamma C)^T_{\delta\gamma}}q_{\gamma,f} \\
&= \bar{q}F^T\underbrace{(C^{-1}\Gamma C)^T}_{C^T\Gamma^T C^{-1^T}}q \\
&= -\bar{q}C\Gamma^T C F^T q
\end{aligned}$$

Invariance of $\mathcal{L}_{\rm ext}$ under charge conjugation requires the transformation properties

$$v_\mu \overset{C}{\to} -v_\mu^T, \quad v_\mu^{(s)} \overset{C}{\to} -v_\mu^{(s)T}, \quad a_\mu \overset{C}{\to} a_\mu^T, \quad s,p \overset{C}{\to} s^T, p^T,$$

transposition refers to the flavor space.

- Time reversal: Nothing new

- Local chiral $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R \times \mathrm{U}(1)_V$ transformations

  First step: Rewrite in terms of the left- and right-handed quark fields.

  Exercise

  We first define

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu.$$

1. Make use of the projection operators P_L and P_R and verify

$$\bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu)q = \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_R + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_L.$$

2. Also verify

$$\bar{q}(s - i\gamma_5 p)q = \bar{q}_L(s - ip)q_R + \bar{q}_R(s + ip)q_L.$$

$\Rightarrow$ **QCD Lagrangian with coupling to external fields**

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_L + \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_R \\ &\quad -\bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R. \quad (*)\end{aligned}$$

$(*)$ remains invariant under local transformations

$$\begin{aligned} q_R &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_R(x)q_R, \\ q_L &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_L(x)q_L, \end{aligned}$$

$V_R(x)$ and $V_L(x)$: independent space-time-dependent SU(3) matrices, provided the external fields are subject to the transformations

$$\begin{aligned} r_\mu &\mapsto V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger, \\ l_\mu &\mapsto V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger, \\ v^{(s)}_\mu &\mapsto v^{(s)}_\mu - \partial_\mu\Theta, \\ s+ip &\mapsto V_R(s+ip)V_L^\dagger, \end{aligned}$$

$$s - ip \mapsto V_L(s - ip)V_R^\dagger.$$

(Derivative terms in serve the same purpose as in the construction of gauge theories, i.e., they cancel analogous terms originating from the kinetic part of the quark Lagrangian.)

- Practical implications of the local invariance

  Allows one to also discuss a coupling to external gauge fields in the transition to the EFT.

  1. Coupling of the electromagnetic field to point-like fundamental particles results from gauging a U(1) symmetry. Here, the corresponding U(1) group is to be understood as a subgroup of a local SU(3)$_L \times$ SU(3)$_R$.

  2. Interaction of the light quarks with the charged and neutral gauge bosons of the weak interactions.

Q: What do we have to insert for the external fields to describe the electromagnetic interaction of quarks?

A:

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} : \quad \text{quark charge matrix}$$

Verification

$$\begin{aligned} \mathcal{L}_{\rm ext} &= -e\mathcal{A}_\mu(\bar{q}_L Q\gamma^\mu q_L + \bar{q}_R Q\gamma^\mu q_R) = -e\mathcal{A}_\mu \bar{q}Q\gamma^\mu q \\ &= -e\mathcal{A}_\mu \left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right) \\ &= -e\mathcal{A}_\mu J^\mu. \end{aligned}$$

"SU(2) version" of ChPT:

$$r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, \qquad v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu,$$

because

$$Q = \frac{1}{6}1_{2\times 2} + \frac{\tau_3}{2}.$$

Spontaneous Symmetry Breaking

Example: O(3) sigma model

$$\begin{aligned}\mathcal{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) &= \mathcal{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu \Phi_1, \partial_\mu \Phi_2, \partial_\mu \Phi_3) \\ &= \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i \Phi_i - \frac{\lambda}{4}(\Phi_i \Phi_i)^2\end{aligned}$$

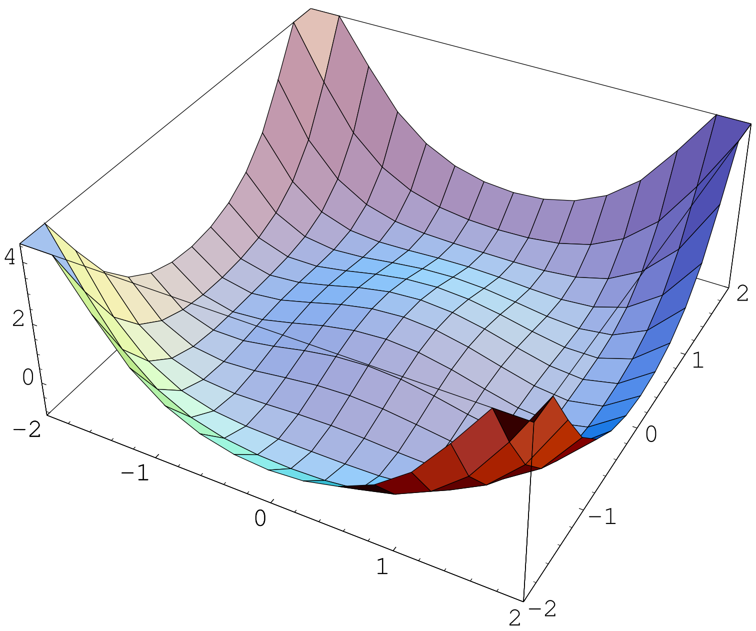
$m^2 < 0$, $\lambda > 0$, Hermitian fields Φ_i

$\mathcal{L}$ invariant under a global "isospin" rotation

$$g \in \textbf{SO(3)}: \; \Phi_i \to \Phi'_i = D_{ij}(g)\Phi_j = (e^{-i\alpha_k T_k})_{ij}\Phi_j$$

$$[T_i, T_j] = i\epsilon_{ijk} T_k$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Two-dimensional rotationally invariant potential: $\mathcal{V}(x,y) = -(x^2+y^2) + \frac{(x^2+y^2)^2}{4}$

Exercise: **Determine the minimum of the potential**

$$\mathcal{V}(\Phi_1,\Phi_2,\Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2$$

We find

$$|\vec{\Phi}_{\rm min}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2+\Phi_2^2+\Phi_3^2}$$

$\vec{\Phi}_{\rm min}$ can point in any direction in isospin space
$\Rightarrow$ non-countably infinite number of degenerate vacua

Spontaneous symmetry breaking (hidden symmetry)

Select a particular direction which, by an appropriate orientation of the internal coordinate frame, we denote as the 3 direction,

$$\vec{\Phi}_{\rm min} = v\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$\vec{\Phi}_{\rm min}$ not invariant under full group $G = \mathbf{SO(3)}$

Rotations about the 1 and 2 axis change $\vec{\Phi}_{\rm min}$

$$T_1\vec{\Phi}_{\rm min} = v\begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}, \quad T_2\vec{\Phi}_{\rm min} = v\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

$\vec{\Phi}_{\rm min}$ invariant under subgroup H of G: rotations about the 3 axis

$$h \in H : \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\rm min} = \vec{\Phi}_{\rm min}, \quad T_3\vec{\Phi}_{\rm min} = 0$$

Exercise: Expand Φ_3 with respect to v: $\Phi_3(x) = v + \eta(x)$

New expression for the potential

$$\tilde{\mathcal{V}} \;=\; \frac{1}{2}(-2m^2)\eta^2 + \lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2 - \frac{\lambda}{4}v^4$$

$$m^2_{\Phi_1} = m^2_{\Phi_2} = 0, \quad m^2_\eta = -2m^2$$

Model-independent feature of the above example:

For each of the two generators T_1 and T_2 which do not annihilate the ground state one obtains a massless Goldstone boson

Number of Goldstone bosons is determined by the structure of the symmetry groups:

- G **symmetry group of the Lagrangian,** n_G **generators**

- H **subgroup with** n_H **generators which leaves the ground state after spontaneous symmetry breaking invariant**

- **# of Goldstone bosons:** $n_G - n_H$

Explicit Symmetry Breaking: A First Look

Modify potential by adding $a\Phi_3$,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3,$$

$m^2 < 0$, $\lambda > 0$, $a > 0$ and real fields Φ_i.

New potential has lower symmetry: O(2) symmetry (rotations about the 3 axis)

Conditions for the new minimum (from $\vec{\nabla}_\Phi \mathcal{V} = 0$) read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0$$

Exercise: Solve using a perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + \mathcal{O}(a^2).$$

Result

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

$\Phi_3^{(0)}$: Result without explicit breaking.

Expand potential with $\Phi_3 = \langle \Phi_3 \rangle + \chi \Rightarrow$

$$m^2_{\Phi_1} = m^2_{\Phi_2} \; = \; a\sqrt{\frac{\lambda}{-m^2}}, \quad \left(m^2_\chi = -2m^2 + 3a\sqrt{\frac{\lambda}{-m^2}} \right).$$

Remarks:

- The Goldstone bosons have acquired a mass.
- Squared masses proportional to a.
- Quantum corrections lead to observables which are nonanalytic in the symmetry breaking parameter a, e.g. $a \ln(a)$ (so-called chiral logarithms).
- Analogue of a in QCD: Quark masses

Spontaneous Symmetry Breaking in QCD

Indications from the Hadron Spectrum

Example: $H_{\rm str}$ is isospin invariant

$$[H_{\rm str}, T_i] = 0, \qquad [T_i, T_j] = i\epsilon_{ijk}T_k$$

Hadrons can be classified as irreducible multiplets of isospin SU(2)

$$\begin{aligned}
T = 0: &\quad d \\
T = \frac{1}{2}: &\quad \begin{pmatrix} p \\ n \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} \\
T = 1: &\quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \\
T = \frac{3}{2}: &\quad \begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}
\end{aligned}$$

- Q: Where does this symmetry come from?

- A: Accidental global symmetry of QCD

Consider linear combinations

$$\begin{aligned} Q_V^a &= Q_R^a + Q_L^a \stackrel{P}{\mapsto} Q_V^a \\ Q_A^a &= Q_R^a - Q_L^a \stackrel{P}{\mapsto} -Q_A^a \end{aligned}$$

Exercise: Commutation relations

$$[Q_V^a, Q_V^b] = i f_{abc} Q_V^c, \quad [Q_V^a, Q_A^b] = i f_{abc} Q_A^c, \quad [Q_A^a, Q_A^b] = i f_{abc} Q_V^c$$

$$[H_{\mathrm{QCD}}^0, Q_V^a] = [H_{\mathrm{QCD}}^0, Q_A^a] = 0$$

Let

$$H^0_{\rm QCD}|\Psi\rangle = E|\Psi\rangle, \quad P|\Psi\rangle = |\Psi\rangle$$

Construct new state $\Phi\rangle = Q_A|\Psi\rangle$ (superscript a suppressed)

$$\begin{aligned} H^0_{\rm QCD}|\Phi\rangle &= H^0_{\rm QCD}Q_A|\Psi\rangle = Q_A \underbrace{H^0_{\rm QCD}|\Psi\rangle}_{E|\Psi\rangle} = E|\Phi\rangle \\ P|\Phi\rangle &= PQ_A|\Psi\rangle = \underbrace{PQ_AP^{-1}}_{-Q_A}\underbrace{P|\Psi\rangle}_{|\Psi\rangle} = -|\Phi\rangle \end{aligned}$$

Not observed in hadronic spectrum

- Q: What's wrong?

- A: We have tacitly assumed that the ground state of QCD is annihilated by Q^a_A.

Solution: spontaneous symmetry breaking

Symmetry of $|0\rangle \neq$ symmetry of $H^0_{\rm QCD}$

- **Coleman theorem:**[4] **The symmetry of the ground state determines the symmetry of the spectrum (reverse argument: infer symmetry of the ground state from the symmetry of the spectrum)**

- **Goldstone theorem:**[5] **To each generator that does not annihilate the ground state exists a massless Goldstone boson**

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[4] S. Coleman, J. Math. Phys. 7, 787 (1966)

[5] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962)

- Here
  - $H^0_{\rm QCD}$ invariant under $G = \mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$
  - $|0\rangle$ invariant under

$$H = \{(V,V)\} \cong \mathrm{SU}(3)_V \quad \text{flavor SU(3)}$$

  - idealized: 8 massless Goldstone bosons π, K, η

Another (sufficient but not necessary) criterion:[6] Nonvanishing scalar quark condensate in the chiral limit.

Analogy with a ferromagnet

$$-\langle\vec{M}\rangle \cdot \vec{H} \leftrightarrow \langle\bar{u}u\rangle_0 m_u + \langle\bar{d}d\rangle_0 m_d + \langle\bar{s}s\rangle_0 m_s$$

[6] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001)

Chiral Perturbation Theory for Mesons

Effective field theory[7]

... if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S−matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. ...

For our purposes:

Most general description of the strong interactions at low energies: $\pi\pi$, πN, NN, etc.

[7] S. Weinberg, Physica A 96, 327 (1979)

Perturbative calculations in effective field theory require **two main ingredients**

(1) Knowledge of the most general effective Lagrangian

(2) Expansion scheme for observables in terms of a consistent power counting

  (a) Tree-level diagrams, loop diagrams, regularization (of infinities)

  (b) Renormalization

  (c) Power counting scheme for renormalized diagrams

Commonly used methods

1. **Expansion in powers of coupling constants (e. g., QED)**

2. **Loop expansion (expansion in $\hbar$)**

3. **Momentum and quark mass expansion, ChPT**

Effective Lagrangian

Starting point of Chiral Perturbation Theory:[8]

Write down the most general Lagrangian in an expansion in (covariant) derivatives ($\rightarrow$ momenta) and quark masses ($\rightarrow$ square of meson masses):

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots$$

- Symmetry group of the Lagrangian as $m_u, m_d, m_s \rightarrow 0$:

$$\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V$$

- Symmetry group of the ground state:

$$H = \text{SU(3)}_V \times \text{U(1)}_V$$

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[8] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158, 142 (1984); Nucl. Phys. B250, 465 (1985)

- 8 pseudoscalar dynamical degrees of freedom which transform as an octet with respect to SU(3)$_V$

- Include explicit chiral symmetry breaking through quark masses as a perturbation

Construction of the lowest–order Lagrangian

Following Gasser and Leutwyler the consequences of the SU(3)$_L\times$SU(3)$_R\times$U(1)$_V$ symmetry of $\mathcal{L}^0_{\mathrm{QCD}}$ are analyzed by

- introducing a coupling to color-neutral, Hermitian external fields:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^0_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ext}} \\ &= \mathcal{L}^0_{\mathrm{QCD}} + \bar{q}\gamma_\mu(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q\end{aligned}$$

where

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v_a^\mu, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a_a^\mu,$$

$$s = \sum_{a=0}^{8} \lambda_a s_a, \quad p = \sum_{a=0}^{8} \lambda_a p_a, \qquad \lambda_0 = \sqrt{2/3}\,\text{diag}(1,1,1)$$

Ordinary three flavor QCD Lagrangian:

$$v^\mu = v^\mu_{(s)} = a^\mu = p = 0, \qquad s = \text{diag}(m_u, m_d, m_s)$$

Introduce

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu$$

Exercise

Using the projection operators P_L and P_R, verify

$$\bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu)q = \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_R + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_L$$

$$\bar{q}(s - i\gamma_5 p)q = \bar{q}_L(s - ip)q_R + \bar{q}_R(s + ip)q_L$$

- and promoting the global symmetry to a local symmetry:

$$L \to V_L(x),\ R \to V_R(x)$$

Building blocks of the effective Lagrangian

Collect Goldstone boson fields in Hermitian, traceless, 3×3 matrix

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

with the Gell-Mann matrices λ_a and $\phi_a(x) = \frac{1}{2}\mathrm{Tr}[\lambda_a \phi(x)]$

Define special unitary matrix

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

Transformation behavior under $G = \mathbf{SU(3)}_L \times \mathbf{SU(3)}_R$, parity P, and charge conjugation C:

$$\begin{aligned} U &\stackrel{G}{\mapsto} V_R U V_L^\dagger \quad * \\ U(\vec{x},t) &\stackrel{P}{\mapsto} U^\dagger(-\vec{x},t) \\ U &\stackrel{C}{\mapsto} U^T \end{aligned}$$

Introduce a covariant derivative and field strength tensors

$$\begin{aligned} D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \stackrel{G}{\mapsto} V_R D_\mu U V_L^\dagger \\ f_{\mu\nu}^R &\equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \stackrel{G}{\mapsto} V_R f_{\mu\nu}^R V_R^\dagger \\ f_{\mu\nu}^L &\equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu] \stackrel{G}{\mapsto} V_L f_{\mu\nu}^L V_L^\dagger \end{aligned}$$

and the linear combination $\chi = 2B_0(s + ip)$

E.g., pure QCD: $\chi = 2B_0 \mathbf{diag}(m_u, m_d, m_s)$

Construct the effective Lagrangian in terms of U, $U^\dagger$, χ, $\chi^\dagger$, $f^R_{\mu\nu}$, $f^L_{\mu\nu}$ and covariant derivatives of these objects.

$*$ **Remark:**

Consider elements (V_L, V_R) of $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ of the type

- (V, V) : $(V_1, V_1)(V_2, V_2) = (V_1 V_2, V_1 V_2)$
  form subgroup H of G

- $(A^\dagger, A)$: $(A_1^\dagger, A_1)(A_2^\dagger, A_2) \neq ((A_1 A_2)^\dagger, A_1 A_2)$
  no subgroup

If we define the ground state as : $\phi_a(x) = 0$ or $U_0 = 1_{3\times 3}$

- U_0 invariant with respect to (V, V):
  $U_0 \to V U_0 V^\dagger = V V^\dagger U_0 = U_0$

- U_0 not invariant with respect to $(A^\dagger, A)$:
  $U_0 \to A U_0 A = A^2 U_0 \neq U_0$

Construction of invariants

Suppose we have matrices $A, B, C, \ldots$, all of which transform as

$$\begin{aligned} A &\overset{G}{\mapsto} V_R A V_L^\dagger \\ B &\overset{G}{\mapsto} V_R B V_L^\dagger \\ &\ldots \end{aligned}$$

Form invariants by "multiplying" in the following way:

$$
\begin{aligned}
\mathrm{Tr}(AB^\dagger) &\overset{G}{\mapsto} \mathrm{Tr}(V_R A \underbrace{V_L^\dagger V_L}_{1} B^\dagger V_R^\dagger) \\
&= \mathrm{Tr}(V_R^\dagger V_R A B^\dagger) = \mathrm{Tr}(AB^\dagger)
\end{aligned}
$$

- Generalization to more terms is obvious

- Product of invariant traces is invariant

- Assign (chiral) orders:

$$
\begin{aligned}
U &= \mathcal{O}(q^0) \\
D_\mu U &= \mathcal{O}(q) \\
r_\mu, l_\mu &= \mathcal{O}(q) \\
f_{\mu\nu}^{L/R} &= \mathcal{O}(q^2) \\
\chi &= \mathcal{O}(q^2)
\end{aligned}
$$

- List of objects A up to and including order q^2 which transform as $A' = V_R A V_L^\dagger$:

$$U,\ D_\mu U,\ D_\mu D_\nu U,\ \chi,\ U f^L_{\mu\nu},\ f^R_{\mu\nu} U$$

- Construction of chirally invariant expressions (to order q^2):

$$\mathcal{O}(q^0) \ : \ \text{Tr}\left(UU^\dagger\right) = \text{Tr}(1) = const.$$

$$\mathcal{O}(q) \ : \ \text{Tr}\left(D_\mu U U^\dagger\right) = 0$$

important: excludes terms of the type $\text{Tr}[\mathcal{O}(q)] \times \text{Tr}(\ldots)$

$$\mathcal{O}(q^2) \ : \ \text{Tr}\left(D_\mu D_\nu U U^\dagger\right) \ \left(= -\text{Tr}\left[D_\nu U (D_\mu U)^\dagger\right]\right)$$

$$\text{Tr}\left[D_\mu U (D_\nu U)^\dagger\right]$$

$$\text{Tr}\left[U (D_\mu D_\nu U)^\dagger\right] \ \left(= -\text{Tr}\left[D_\mu U (D_\nu U)^\dagger\right]\right)$$

$$\text{Tr}\left(\chi U^\dagger\right)$$

$$\mathrm{Tr}\left(U\chi^\dagger\right)$$
$$\mathrm{Tr}\left[(Uf^L_{\mu\nu})U^\dagger\right] = \mathrm{Tr}\left(f^L_{\mu\nu}\right)$$
$$\mathrm{Tr}\left(f^R_{\mu\nu}\right)$$

- **Lorentz invariance:**

  Indices have to be contracted

$$g^{\mu\nu}f^L_{\mu\nu} = g^{\mu\nu}f^R_{\mu\nu} = 0$$

- **Candidates:**

$$\mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger\right]$$
$$\mathrm{Tr}\left(\chi U^\dagger \pm U\chi^\dagger\right)$$

- **Parity:**

$$\mathcal{L}(\vec{x},t) \overset{P}{\mapsto} \mathcal{L}(-\vec{x},t)$$

Tr$(\chi U^\dagger - U\chi^\dagger)$ has wrong parity

- Charge conjugation [no additional constraint at $\mathcal{O}(q^2)$]

Lowest–order Lagrangian $\mathcal{L}_2$

$$\mathcal{L}_2 = \frac{F_0^2}{4}\text{Tr}\left[D_\mu U (D^\mu U)^\dagger\right] + \frac{F_0^2}{4}\text{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)$$

At $\mathcal{O}(q^2)$ two parameters:

$$F_0 \approx 93\,\text{MeV}, \qquad 3F_0^2 B_0 = -\langle 0|\bar{q}q|0\rangle$$

- $\mathcal{L}_2$ has predictive power!

- However, we first need to discuss the power counting scheme.

Weinberg's power counting for the mesonic sector[9]

Q: How do different diagrams compare?

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q) = \mathcal{O}(q^D)$$

For small enough momenta (and masses) contributions with increasing D become less important

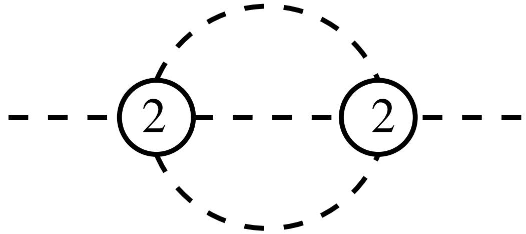
$$\begin{aligned} D &= nN_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k} \\ &= 2 + (n-2)N_L + \sum_{k=1}^{\infty} 2(k-1)N_{2k} \\ &\geq 2 \text{ in 4 dimensions} \end{aligned}$$

- N_L: Number of independent loops

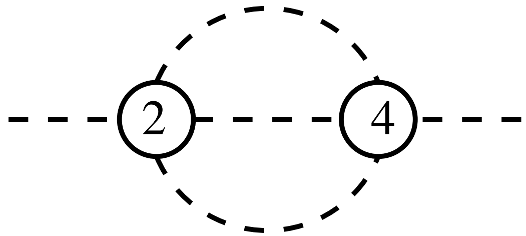
[9] S. Weinberg, Physica A 96, 327 (1979)

- N_I: Number of internal Goldstone boson lines
- N_{2k}: Number of vertices from $\mathcal{L}_{2k}$
- Loops suppressed by $(n-2)N_L$
- Relation between the momentum and loop expansion
- Perturbative scheme in terms of external momenta and quark masses ($\rightarrow$ meson masses^2) which are small compared to some scale [here: $4\pi F_0 = \mathcal{O}(1\text{GeV})$]

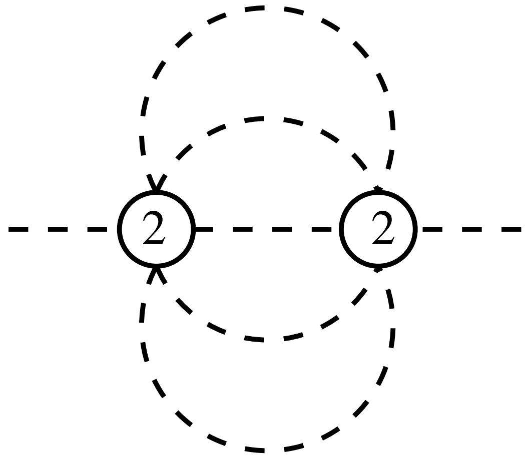
Examples ($n = 4$ dimensions)



$$\begin{aligned} D &= \mathbf{4 \cdot 2 - 2 \cdot 3 + 2 \cdot 2 = 6} \\ &= \mathbf{2 + 2 \cdot 2 + (2 - 2) \cdot 2} \end{aligned}$$



$$D = \mathbf{4 \cdot 2 - 2 \cdot 3 + 1 \cdot 2 + 1 \cdot 4 = 8}$$



$$\boldsymbol{D = 4 \cdot 4 - 2 \cdot 5 + 2 \cdot 2 = 10}$$

Proof:

N_I: # of internal Goldstone boson lines
N_{2k}: # of vertices with $2k$ derivatives or k quark mass terms

- **Internal lines:**

$$\int d^4k \frac{1}{k^2 - M^2 + i\epsilon} \quad \underset{\rightarrow}{M^2 \rightarrow t^2 M^2} \quad \int d^4k \frac{1}{t^2(k^2/t^2 - M^2 + i\epsilon)}$$

$$\underset{=}{k = tl} \quad t^2 \int d^4l \frac{1}{l^2 - M^2 + i\epsilon}$$

- **Vertices with $2k$ derivatives or k quark mass terms:**

$$\delta^4(q) q^{2k} \rightarrow t^{2k-4} \delta^4(q) q^{2k}$$

– since $p \to tp$ if q is an external momentum

– and $k = tl$ if q is an internal momentum (see above)

- These are the rules to calculate $S \sim \delta^4(p)\mathcal{M}$
add 4 to compensate for the overall delta function

- Scaling behavior of the contribution to $\mathcal{M}$ of a given diagram

$$D = 4 + 2N_I + \sum_{k=1}^{\infty} N_{2k}(2k-4)$$

- Relation between # of loops N_L, # of vertices N_V, and # of internal lines:

$$N_I = N_L + N_V - 1 = N_L + \sum_{k=1}^{\infty} N_{2k} - 1$$

- $$D = 2 + \sum_{k=1}^{\infty} (2k-2) N_{2k} + 2N_L \geq 2$$

  In particular, diagrams containing loops are suppressed due to the term $2N_L$

Simple applications at lowest order

Goldstone boson masses due to quark masses

No external sources: $D_\mu U \to \partial_\mu U$, $\chi = 2B_0\mathcal{M}$, $m_u = m_d = \hat{m}$

$$\mathcal{L}_2 \overset{\text{Exercise}}{=} \underbrace{\frac{1}{2}\left(\partial_\mu\pi^0\partial^\mu\pi^0 - M_\pi^2\pi^0\pi^0\right) + \cdots}_{\text{sum of free Lagrangians}} + \underbrace{\mathcal{L}_{\text{int}}}_{O(\phi^4)}$$

Read off

$$\begin{aligned} M_\pi^2 &= 2B_0\hat{m} \\ M_K^2 &= B_0(\hat{m} + m_s) \\ M_\eta^2 &= \frac{2}{3}B_0(\hat{m} + 2m_s) \end{aligned}$$

Remark: Without additional information about B_0 one cannot determine the absolute values of the quark masses.

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{2\hat{m}} \quad \Rightarrow \quad \frac{m_s}{\hat{m}} = 25.9$$

$$\frac{M_\eta^2}{M_\pi^2} = \frac{2m_s + \hat{m}}{3\hat{m}} \quad \Rightarrow \quad \frac{m_s}{\hat{m}} = 24.3$$

Gell-Mann-Okubo formula

$$4M_K^2 = 3M_\eta^2 + M_\pi^2$$

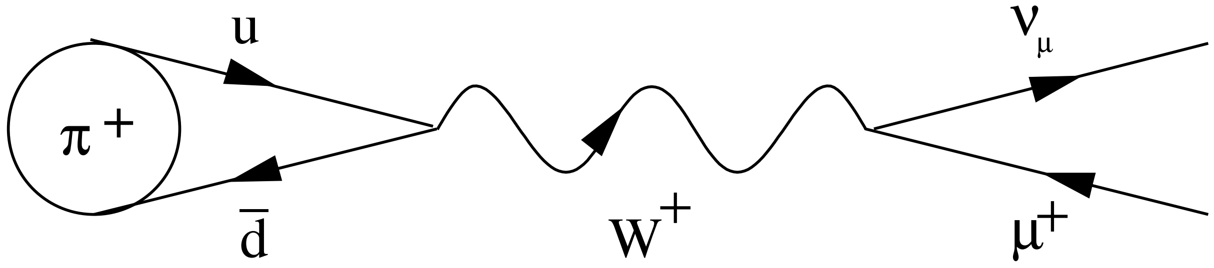
Insert

$$M_K = 496\,\text{MeV}, \qquad M_\pi = 135\,\text{MeV}$$

and "predict"

$M_\eta = 567\,\text{MeV}$, experimental value: 547 MeV

Pion decay $\pi^+ \to \mu^+ \nu_\mu$



- Interaction of quarks with the massive charged weak bosons $\mathcal{W}^\pm_\mu = (\mathcal{W}_{1\mu} \mp i\mathcal{W}_{2\mu})/\sqrt{2}$

$$\mathcal{L}^{(q)}_{\rm CC} = -\frac{g}{2\sqrt{2}}\left\{\mathcal{W}^+_\mu [V_{ud}\bar{u}\gamma^\mu(1-\gamma_5)d + V_{us}\bar{u}\gamma^\mu(1-\gamma_5)s] + h.c.\right\}$$

$$|V_{ud}| = 0.9735 \pm 0.0008, \qquad |V_{us}| = 0.2196 \pm 0.0023$$

Fermi constant is related to the gauge coupling g and the W mass as

$$G_F = \sqrt{2}\frac{g^2}{8M_W^2} = 1.16639(1) \times 10^{-5}\,\text{GeV}^{-2}$$

Set

$$r_\mu = 0, \qquad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}_\mu^+ T_+ + h.c.)$$

in $\mathcal{L}_{\rm ext}$, where

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Coupling of the W bosons to the leptons

$$\mathcal{L}_{\rm CC}^{(l)} = -\frac{g}{2\sqrt{2}}\left[\mathcal{W}_\mu^+ \bar{\nu}_\mu \gamma^\mu (1-\gamma_5)\mu + \mathcal{W}_\mu^- \bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu\right]$$

- Coupling of the W bosons to Goldstone bosons

Insert covariant derivative

$$D_\mu U = \partial_\mu U + iUl_\mu$$

into $\mathcal{L}_2$

$$\begin{aligned}\frac{F_0^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger] &= i\frac{F_0^2}{2}\mathrm{Tr}(l_\mu \partial^\mu U^\dagger U) + \cdots = \frac{F_0}{2}\mathrm{Tr}(l_\mu \partial^\mu \Phi) + \cdots \\ &= -g\frac{F_0}{2}\Big[\mathcal{W}_\mu^+(V_{ud}\partial^\mu \pi^- + V_{us}\partial^\mu K^-) \\ &\quad + \mathcal{W}_\mu^-(V_{ud}\partial^\mu \pi^+ + V_{us}\partial^\mu K^+)\Big]\end{aligned}$$

- Feynman propagator for W bosons

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2} = \frac{g_{\mu\nu}}{M_W^2} + O(\frac{kk}{M_W^4})$$

- Feynman rule for the invariant amplitude for the weak pion decay

$$\mathcal{M} = \underbrace{i\left[-\frac{g}{2\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\rho(1-\gamma_5)v_{\mu^+}\right]}_{\text{leptonic vertex}} \quad \underbrace{\frac{ig_{\rho\sigma}}{M_W^2}}_{W \text{ propagator}} \quad \underbrace{i\left[-g\frac{F_0}{2}V_{ud}(-ip^\sigma)\right]}_{\text{hadronic vertex}}$$

$$= -G_F V_{ud} F_0 \bar{u}_{\nu_\mu} \not{p}(1-\gamma_5) v_{\mu^+}$$

p: four-momentum of the pion

- Decay rate

$$\frac{1}{\tau} = \frac{G_F^2 |V_{ud}|^2}{4\pi} F_0^2 M_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{M_\pi^2}\right)^2$$

- F_0: pion-decay constant in the chiral limit

- Empirical numbers:

$$\begin{aligned} F_\pi &= 92.3\,\text{MeV} \\ F_K &= 113\,\text{MeV} \end{aligned}$$

$\pi\pi$ scattering from $\mathcal{L}_2^{4\phi}$

Consider the Lagrangian

$$\mathcal{L}_2 = \frac{F^2}{4}\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{F^2}{4}\text{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)$$

in the SU(2) sector with

$$\chi = 2B\underbrace{\begin{pmatrix} \hat{m} & 0 \\ 0 & \hat{m} \end{pmatrix}}_{\mathcal{M}}$$

and

$$U = \exp\left(i\frac{\phi}{F}\right), \quad \phi = \sum_{i=1}^{3}\tau_i\phi_i =: \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

Remark on chiral limit:

- In the SU(2) sector it is common to express quantities in the chiral limit without index 0, e. g., F and B. By this one means the SU(2) chiral limit, i. e. $m_u = m_d = 0$ but m_s at its physical value.

- In the SU(3) sector the quantities F_0 and B_0 denote the chiral limit for all three quarks: $m_u = m_d = m_s = 0$.

Substitution $U \leftrightarrow U^\dagger$. $\Rightarrow$ $\mathcal{L}_2$ contains even powers of ϕ only:

$$\mathcal{L}_2 = \mathcal{L}_2^{2\phi} + \mathcal{L}_2^{4\phi} + \cdots .$$

- $\mathcal{L}_2$ does not produce a vertex with 3 Goldstone bosons. $\Rightarrow$ At $D = 2$, no s-, u-, and t-channel pole diagrams.

- At $D = 2$, $\pi\pi$ scattering is generated by a 4 Goldstone boson interaction term.

Expand

$$U = 1 + i\frac{\phi}{F} - \frac{1}{2}\frac{\phi^2}{F^2} - \frac{i}{6}\frac{\phi^3}{F^3} + \frac{1}{24}\frac{\phi^4}{F^4} + \cdots$$

and identify $\mathcal{L}_2^{4\phi}$ as (Exercise)

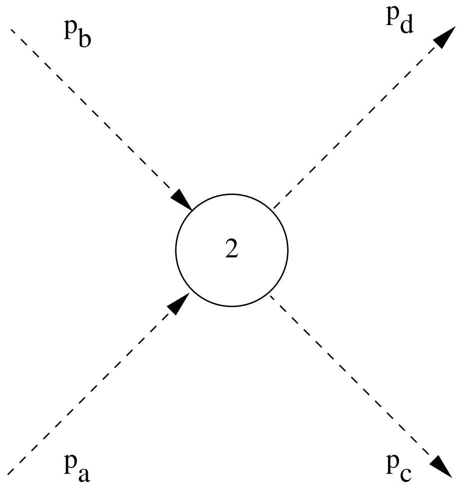
$$\mathcal{L}_2^{4\phi} = \frac{1}{48F^2}\left[\mathrm{Tr}([\phi, \partial_\mu\phi][\phi, \partial^\mu\phi]) + 2B\,\mathrm{Tr}(\mathcal{M}\phi^4)\right].$$

Remark: Substituting $F \to F_0$, $B \to B_0$ and the relevant exprressions for ϕ and the quark mass matrix $\mathcal{M}$ the corresponding formula for SU(3) looks identical.

Insert $\phi = \tau_i\phi_i$. $\Rightarrow$ (Exercise)

$$\begin{aligned}
\mathcal{L}_2^{4\phi} &= -\frac{1}{6F^2}\epsilon_{ijm}\phi_i\partial_\mu\phi_j\epsilon_{klm}\phi_k\partial^\mu\phi_l + \frac{M^2}{24F^2}\phi_i\phi_i\phi_j\phi_j \\
&= \frac{1}{6F^2}(\phi_i\partial^\mu\phi_i\partial_\mu\phi_j\phi_j - \phi_i\phi_i\partial_\mu\phi_j\partial^\mu\phi_j) + \frac{M^2}{24F^2}\phi_i\phi_i\phi_j\phi_j,
\end{aligned}$$

where $M^2 = 2B\hat{m}$.



Feynman rule for Cartesian isospin indices a, b, c, and d:

$$\begin{aligned}\mathcal{M} \;=\; i\Bigg[\frac{1}{6F^2}\Bigg(2\Big[&\delta^{ab}\delta^{cd}(-ip_a - ip_b)\cdot(ip_c + ip_d)\\ &+\delta^{ac}\delta^{bd}(-ip_a + ip_c)\cdot(-ip_b + ip_d)\\ &+\delta^{ad}\delta^{bc}(-ip_a + ip_d)\cdot(-ip_b + ip_c)\Big]\\ -4\Big\{&\delta^{ab}\delta^{cd}\left[(-ip_a)\cdot(-ip_b) + (ip_c)\cdot(ip_d)\right]\\ &+\delta^{ac}\delta^{bd}\left[(-ip_a)\cdot(ip_c) + (-ip_b)\cdot(ip_d)\right]\\ &+\delta^{ad}\delta^{bc}\left[(-ip_a)\cdot(ip_d) + (-ip_b)\cdot(ip_c)\right]\Big\}\Bigg)\end{aligned}$$

$$
\begin{aligned}
& +\frac{M^2}{24F^2}8(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\Big] \\
= \; & \frac{i}{3F^2}\Big\{\delta^{ab}\delta^{cd}[(p_a+p_b)^2+2p_a\cdot p_b+2p_c\cdot p_d+M^2] \\
& +\delta^{ac}\delta^{bd}[(p_a-p_c)^2-2p_a\cdot p_c-2p_b\cdot p_d+M^2] \\
& +\delta^{ad}\delta^{bc}[(p_a-p_d)^2-2p_a\cdot p_d-2p_b\cdot p_c+M^2]\Big\} \\
= \; & \frac{i}{3F^2}\Big[\delta^{ab}\delta^{cd}(3s-p_a^2-p_b^2-p_c^2-p_d^2+M^2) \\
& +\delta^{ac}\delta^{bd}(3t-p_a^2-p_c^2-p_b^2-p_d^2+M^2) \\
& +\delta^{ad}\delta^{bc}(3u-p_a^2-p_d^2-p_b^2-p_c^2+M^2)\Big] \\
= \; & i\left[\delta^{ab}\delta^{cd}\frac{s-M^2}{F^2}+\delta^{ac}\delta^{bd}\frac{t-M^2}{F^2}+\delta^{ad}\delta^{bc}\frac{u-M^2}{F^2}\right] \\
& -\frac{i}{3F^2}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)(\Lambda_a+\Lambda_b+\Lambda_c+\Lambda_d), \quad (1)
\end{aligned}
$$

where $\Lambda_k = p_k^2 - M^2$.

Mandelstam variables

$$s = (p_a + p_b)^2 = (p_c + p_d)^2,$$

$$t = (p_a - p_c)^2 = (p_d - p_b)^2,$$

$$u = (p_a - p_d)^2 = (p_c - p_b)^2$$

and

$$2p_a \cdot p_b = s - p_a^2 - p_b^2, \quad 2p_c \cdot p_d = s - p_c^2 - p_d^2,$$

$$-2p_a \cdot p_c = t - p_a^2 - p_c^2, \quad -2p_b \cdot p_d = t - p_b^2 - p_d^2,$$

$$-2p_a \cdot p_d = u - p_a^2 - p_d^2, \quad -2p_b \cdot p_c = u - p_b^2 - p_c^2.$$

The last line of the Feynman rule disappears, if the external lines satisfy mass shell conditions.

Scattering process $\pi^a(p_a) + \pi^b(p_b) \rightarrow \pi^c(p_c) + \pi^d(p_d)$ at $\mathcal{O}(q^2)$:

$$T = \delta^{ab}\delta^{cd}\frac{s - M_\pi^2}{F_\pi^2} + \delta^{ac}\delta^{bd}\frac{t - M_\pi^2}{F_\pi^2} + \delta^{ad}\delta^{bc}\frac{u - M_\pi^2}{F_\pi^2}$$

We replaced

$$F \longrightarrow F_\pi, \qquad F_\pi = F(1 + \mathcal{O}(q^2)),$$
$$M^2 \longrightarrow M_\pi^2, \qquad M_\pi^2 = M^2(1 + \mathcal{O}(q^2)),$$

because the difference is of $\mathcal{O}(q^4)$ in T.

Consider (theoretical) limit $M_\pi^2, s, t, u \longrightarrow 0$:

$$T \longrightarrow 0$$

- Goldstone bosons interact "weakly" at low energies.

Isospin symmetry. $\Rightarrow$ Most general parametrization

$$T = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{bc}A(u,t,s)$$

with $A(s,t,u) = A(s,u,t)$.

Isospin channels:

$$\begin{aligned} T^{I=0} &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ T^{I=1} &= A(t,u,s) - A(u,s,t) \\ T^{I=2} &= A(t,u,s) + A(u,s,t) \end{aligned}$$

s-wave scattering lengths

$$\begin{aligned} T^{I=0}|_{\rm thr} &= 32\pi a_0^0 \\ T^{I=2}|_{\rm thr} &= 32\pi a_0^2 \end{aligned}$$

- $\pi^+\pi^+$ **scattering described by** $T^{I=2}$.

- **Other physical reactions may be determined using the appropriate Clebsch-Gordan coefficients.**

Evaluate T matrices at threshold. $\Rightarrow$ s-wave $\pi\pi$ scattering lenghts[10]

$$T^{I=0}|_{\rm thr} = 32\pi a_0^0, \quad T^{I=2}|_{\rm thr} = 32\pi a_0^2.$$

Lower index 0: s wave; upper index: Isospin.

($T^{I=1}|_{\rm thr}$ vanishes because of Bose symmetry.)

Prediction at $\mathcal{O}(q^2)$:

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}.$$

At threshold

$$s_{\rm thr} = (2M_\pi)^2$$

and thus

$$A(s_{\rm thr}, t_{\rm thr}, u_{\rm thr}) = \frac{3M_\pi^2}{F_\pi^2}.$$

---

[10] The convention in ChPT differs by a factor $(-M_\pi)$ from the usual definition of a scattering length in the effective range expandion.

- $I = 0$: Consider linear combination

$$
\begin{aligned}
&[3A(s,t,u) + A(t,u,s) + A(u,s,t)]_{\text{thr}} \\
&\quad = [2A(s,t,u) + A(s,t,u) + A(t,u,s) + A(u,s,t)]_{\text{thr}} \\
&\quad = \frac{6M_\pi^2}{F_\pi^2} + \frac{[s+t+u-3M_\pi^2]_{\text{thr}}}{F_\pi^2} \\
&\quad = \frac{7M_\pi^2}{F_\pi^2}
\end{aligned}
$$

- $I = 2$: Consider linear combination

$$
\begin{aligned}
&[A(t,u,s) + A(u,s,t)]_{\text{thr}} \\
&\quad = [A(t,u,s) + A(u,s,t) + A(s,t,u) - A(s,t,u)]_{\text{thr}} \\
&\quad = \frac{M_\pi^2}{F_\pi^2} - \frac{3M_\pi^2}{F_\pi^2} \\
&\quad = -\frac{2M_\pi^2}{F_\pi^2}.
\end{aligned}
$$

- $\Rightarrow$ Famous results of current algebra for the scattering lengths:[11]

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.156, \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045.$$

($F_\pi = 93.2$ MeV and $M_\pi = 139.57$ MeV)

- Absolute prediction of chiral symmetry! Once we know F_π (from pion decay) we can predict the scattering lengths.

- Different from Wigner-Eckart theorem which predicts relations among processes of the same type.

---

[11] S. Weinberg, Phys. Rev. Lett. 17, 616 (1966)

Experimental data

$\pi^{\pm}p \to \pi^{\pm}\pi^{+}n$:[12] $a_0^0 = 0.204 \pm 0.014\,(\text{stat}) \pm 0.008\,(\text{syst})$,

$K^{+} \to \pi^{+}\pi^{-}e^{+}\nu_e$:[13] $a_0^0 = 0.216 \pm 0.013\,(\text{stat}) \pm 0.002\,(\text{syst}) \pm 0.002\,(\text{theor})$,

$\pi^{+}\pi^{-}$ atom lifetime:[14] $|a_0^0 - a_0^2| = 0.264^{+0.033}_{-0.020}$,

$K^{\pm} \to \pi^{\pm}\pi^{0}\pi^{0}$:[15] $a_0^0 - a_0^2 = 0.268 \pm 0.010\,(\text{stat}) \pm 0.004\,(\text{syst}) \pm 0.013\,(\text{ext})$,

$$a_0^2 = -0.041 \pm 0.022\,(\text{stat}) \pm 0.014\,(\text{syst}).$$

---

[12] M. Kermani et al. [CHAOS Collaboration], Phys. Rev. C 58, 3431 (1998)

[13] S. Pislak et al., Phys. Rev. D 67, 072004 (2003)

[14] B. Adeva et al. [DIRAC Collaboration], Phys. Lett. B 619, 50 (2005)

[15] J. R. Batley et al. [NA48/2 Collaboration], Phys. Lett. B 633, 173 (2006)

Predictions for the s-wave scattering lengths at $\mathcal{O}(q^6)$[16]

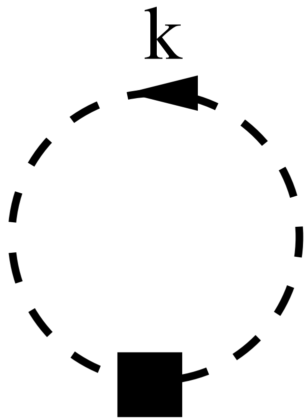
$$a_0^0 = \overbrace{0.156}^{\mathcal{O}(q^2)} + \overbrace{\underbrace{0.039}_{\text{L}} + \underbrace{0.005}_{\text{anal.}}}^{\mathcal{O}(q^4)\text{: }+28\%} + \overbrace{\underbrace{0.013}_{k_i} + \underbrace{0.003}_{\text{L}} + \underbrace{0.001}_{\text{anal.}}}^{\mathcal{O}(q^6)\text{: }+8.5\%} = \overbrace{0.217}^{\text{total}},$$

$$a_0^0 - a_0^2 = \overbrace{0.201}^{\mathcal{O}(q^2)} + \overbrace{\underbrace{0.036}_{\text{L}} + \underbrace{0.006}_{\text{anal.}}}^{\mathcal{O}(q^4)\text{: }+21\%} + \overbrace{\underbrace{0.012}_{k_i} + \underbrace{0.003}_{\text{L}} + \underbrace{0.001}_{\text{anal.}}}^{\mathcal{O}(q^6)\text{: }+6.6\%} = \overbrace{0.258}^{\text{total}}.$$

[16] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B 374, 210 (1996)

Dimensional regularization: Basics

Simple example



$$I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+}, \qquad k^2 = k_0^2 - \vec{k}^{\,2}$$

Introduce

$$a \equiv \sqrt{\vec{k}^{\,2} + M^2} > 0$$

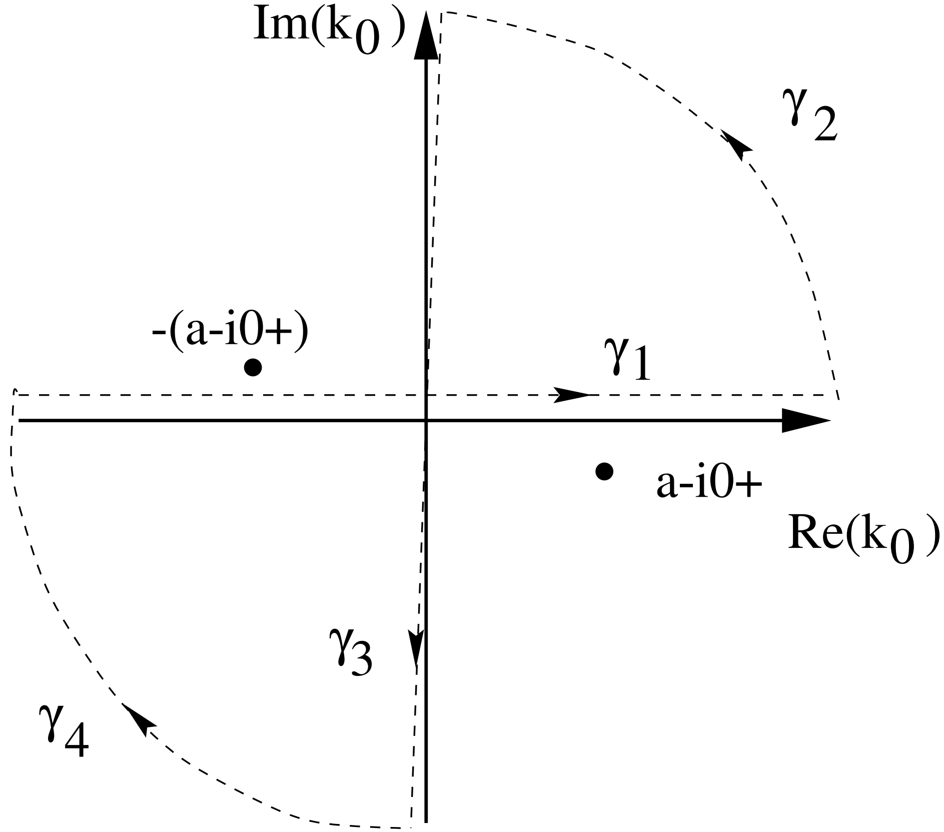
and define

$$f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^+)]}$$

Consider f in the complex k_0 plane and make use of Cauchy's theorem

$$\oint_C dz f(z) = 0$$

for functions which are differentiable in every point inside the closed contour C



$$0 = \sum_{i=1}^{4} \int_{\gamma_i} dz f(z)$$

$$\begin{aligned}
\int_{\gamma_1} f(z)dz &= \int_{-\infty}^{\infty} f(t)dt \\
\int_{\gamma_2} f(z)dz &= 0 \\
\int_{\gamma_3} f(z)dz &= \int_{\infty}^{-\infty} f(it)idt \\
\int_{\gamma_4} f(z)dz &= 0
\end{aligned}$$

$\Rightarrow$ the so-called Wick rotation

$$\int_{-\infty}^{\infty} f(t)dt = -i \int_{\infty}^{-\infty} dt f(it) = i \int_{-\infty}^{\infty} dt f(it)$$

Intermediate result

$$
\begin{aligned}
I &= \frac{1}{(2\pi)^4} i \int_{-\infty}^{\infty} dk_0 \int d^3k \frac{i}{(ik_0)^2 - \vec{k}^{\,2} - M^2 + i0^+} \\
&= \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + M^2 - i0^+}
\end{aligned}
$$

$l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$ denotes a Euclidian scalar product

- I diverges for large values of l (ultraviolet divergence)

- $M^2 \to 0$: I diverges for small values of l (infrared divergence)

The degree of divergence can be estimated by simply counting the powers of momenta.

If the integral behaves asymptotically as

$$\int d^4l/l^2 : \quad \text{diverges quadratically}$$

$$\int d^4l/l^3 : \quad \text{diverges linearly}$$

$$\int d^4l/l^4 : \quad \text{diverges logarithmically}$$

I diverges quadratically

Dimensional regularization: Generalize from 4 to n dimensions and introduce polar coordinates

$$\begin{aligned} l_1 &= l\cos(\theta_1) \\ l_2 &= l\sin(\theta_1)\cos(\theta_2) \\ l_3 &= l\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ &\vdots \\ l_{n-1} &= l\sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{n-1}) \\ l_n &= l\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}) \end{aligned}$$

$$0 \le l, \qquad \theta_i \in [0,\pi], i = 1,\cdots,n-2, \qquad \theta_{n-1} \in [0,2\pi]$$

A general integral is then symbolically of the form

$$\int d^n l \cdots = \int_0^\infty l^{n-1} dl$$
$$\times \int_0^{2\pi} d\theta_{n-1} \int_0^\pi d\theta_{n-2} \sin(\theta_{n-2}) \cdots \int_0^\pi d\theta_1 \sin^{n-2}(\theta_1) \cdots$$

If the integrand does not depend on the angles, the angular integration can explicitly be carried out:

$$\int d\Omega_n = 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

Example

$$n = 3: \qquad 4\pi = 2\frac{\pi}{1/2} = 2\frac{\pi^{3/2}}{\sqrt{\pi}/2} = 2\frac{\pi^{3/2}}{\Gamma(3/2)}$$

We define the integral for n dimensions (n integer) as

$$I_n(M^2, \mu^2) = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+}$$

Scale μ: Unit of mass, 't Hooft parameter, renormalization scale (integral has the same dimension for arbitrary n)

Integral formally reads

$$I_n(M^2,\mu^2) = \mu^{4-n} \underbrace{2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}}_{\text{angular integration}} \frac{1}{(2\pi)^n} \underbrace{\int_0^\infty dl \frac{l^{n-1}}{l^2+M^2}}_{\text{elementary}}$$

$$= \mu^{4-n}\, 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{(2\pi)^n} \frac{1}{2}(M^2)^{\frac{n}{2}-1} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right)}{\underbrace{\Gamma(1)}_{1}}$$

$$= \frac{\mu^{4-n}}{(4\pi)^{\frac{n}{2}}}(M^2)^{\frac{n}{2}-1}\Gamma\left(1-\frac{n}{2}\right)$$

$\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z=-n$, $n=0,1,2,\cdots$, where it possesses simple poles with residue $(-1)^n/n!$

$a^z = \exp[\ln(a)z]$, $a \in R^+$ is an analytic function in C

Define (as a function of a complex variable n)

$$I(M^2,\mu^2,n) = \frac{M^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{n}{2}}\Gamma\left(1-\frac{n}{2}\right)$$

As $n \to 4$ Gamma function has a pole $\Rightarrow$ $I(M^2,\mu^2,n)$ has a pole

How is this pole is approached?

Important property: $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma\left(1-\frac{n}{2}\right) = \frac{\Gamma\left(1-\frac{n}{2}+1\right)}{1-\frac{n}{2}} = \frac{\Gamma\left(2-\frac{n}{2}+1\right)}{\left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)} = \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)}{(-1)\left(1-\frac{\epsilon}{2}\right)\frac{\epsilon}{2}}$$

where $\epsilon \equiv 4-n$.

$$a^x = \exp[\ln(a)x] = 1+\ln(a)x+O(x^2)$$

$$I(M^2,\mu^2,n) = \frac{M^2}{16\pi^2}\left[-\frac{2}{\epsilon}\underbrace{-\Gamma'(1)}_{\gamma_E}-1-\ln(4\pi)+\ln\left(\frac{M^2}{\mu^2}\right)+O(\epsilon)\right]$$

Summary

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n-4)$$

where

$$\underbrace{R}_{\widetilde{\text{MS}}} = \underbrace{\underbrace{\frac{2}{n-4}}_{\textbf{MS}} - [\ln(4\pi) + \Gamma'(1)]}_{\overline{\text{MS}}} - 1$$

The Chiral Lagrangian at $\mathcal{O}(q^4)$

S. Weinberg, The Quantum Theory of Fields, Vol. I, Chap. 12

... the cancellation of ultraviolet divergences does not really depend on renormalizability; as long as we include every one of the infinite number of interactions allowed by symmetries, the so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.

Conclusion: Adjust (renormalize) parameters of $\mathcal{L}_4$ to cancel one-loop infinities!

$$
\begin{aligned}
L_i &= L_i^r + \frac{\Gamma_i}{32\pi^2} R, \qquad i = 1, \cdots, 10 \\
H_i &= H_i^r + \frac{\Delta_i}{32\pi^2} R, \qquad i = 1, 2
\end{aligned}
$$

$\mathcal{L}_4$ of Gasser and Leutwyler:[17]

$$
\begin{aligned}
\mathcal{L}_4 \;=\;& L_1 \left\{ \mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] \right\}^2 + L_2 \mathrm{Tr}\left[D_\mu U (D_\nu U)^\dagger \right] \mathrm{Tr}\left[D^\mu U (D^\nu U)^\dagger \right] \\
& + L_3 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger \right] \\
& + L_4 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger \right] \mathrm{Tr}\left(\chi U^\dagger + U \chi^\dagger \right) \\
& + L_5 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger) \right] \\
& + L_6 \left[\mathrm{Tr}\left(\chi U^\dagger + U \chi^\dagger \right) \right]^2 + L_7 \left[\mathrm{Tr}\left(\chi U^\dagger - U \chi^\dagger \right) \right]^2 \\
& + L_8 \mathrm{Tr}\left(U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger \right) \\
& - i L_9 \mathrm{Tr}\left[f^R_{\mu\nu} D^\mu U (D^\nu U)^\dagger + f^L_{\mu\nu} (D^\mu U)^\dagger D^\nu U \right] \\
& + L_{10} \mathrm{Tr}\left(U f^L_{\mu\nu} U^\dagger f^{\mu\nu}_R \right) \\
& + H_1 \mathrm{Tr}\left(f^R_{\mu\nu} f^{\mu\nu}_R + f^L_{\mu\nu} f^{\mu\nu}_L \right) \\
& + H_2 \mathrm{Tr}\left(\chi \chi^\dagger \right)
\end{aligned}
$$

---

[17] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985)

Coefficient	Empirical Value	Γ_i
L_1^r	0.4 ± 0.3	$\frac{3}{32}$
L_2^r	1.35 ± 0.3	$\frac{3}{16}$
L_3^r	-3.5 ± 1.1	0
L_4^r	-0.3 ± 0.5	$\frac{1}{8}$
L_5^r	1.4 ± 0.5	$\frac{3}{8}$
L_6^r	-0.2 ± 0.3	$\frac{11}{144}$
L_7^r	-0.4 ± 0.2	0
L_8^r	0.9 ± 0.3	$\frac{5}{48}$
L_9^r	6.9 ± 0.7	$\frac{1}{4}$
L_{10}^r	-5.5 ± 0.7	$-\frac{1}{4}$

The renormalized coefficients L_i^r depend on the scale μ introduced by dimensional regularization and their values at two different scales μ_1 and μ_2 are related by

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln\left(\frac{\mu_1}{\mu_2}\right)$$

Present status of the mesonic Lagrangian [SU(3)×SU(3)][18] (π, K, η)

$$\underbrace{2}_{\mathcal{O}(q^2)} \quad + \quad \underbrace{10+2}_{\mathcal{O}(q^4)} \quad + \quad \underbrace{90+4+23}_{\mathcal{O}(q^6)} \quad + \quad \cdots$$

- q: Small quantity such as a pion mass
- Even powers
- Two-loop level

---

[18] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985);
H. W. Fearing and S. Scherer, Phys. Rev. D 53, 315 (1996);
J. Bijnens, G. Colangelo, G. Ecker, JHEP 02, 020 (1999);
T. Ebertshäuser, H. W. Fearing, S. Scherer, Phys. Rev. D 65, 054033 (2002);
J. Bijnens, L. Girlanda, P. Talavera, Eur. Phys. J. C 23, 539 (2002)

Masses at $\mathcal{O}(q^4)$

Definition of the propagator of a (pseudo-) scalar field:

$$i\Delta(p) = \int d^4x e^{-ip\cdot x} \langle 0|T\left[\Phi_0(x)\Phi_0(0)\right]|0\rangle$$

index 0: bare unrenormalized field

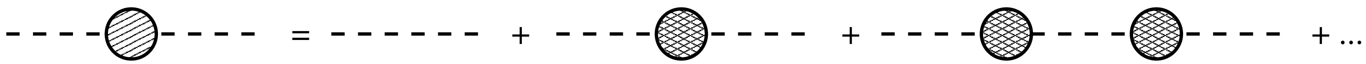
At lowest order ($D = 2$) the propagator simply reads

$$i\Delta(p) = \frac{i}{p^2 - M_0^2 + i0^+}$$

with lowest-order masses M_0

$$\begin{aligned} M_{\pi,2}^2 &= 2B_0\hat{m} \\ M_{K,2}^2 &= B_0(\hat{m} + m_s) \\ M_{\eta,2}^2 &= \frac{2}{3}B_0\left(\hat{m} + 2m_s\right) \end{aligned}$$

Full propagator in terms of the so-called proper self-energy insertions $-i\Sigma(p^2)$



Summation via a geometric series

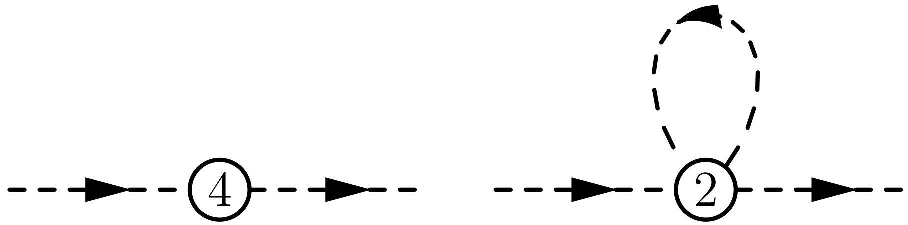
$$
\begin{aligned}
i\Delta(p) &= \frac{i}{p^2 - M_0^2 + i0^+} \\
&\quad + \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[-i\Sigma(p^2)] \frac{i}{p^2 - M_0^2 + i0^+}}_{x} \\
&\quad + \cdots \\
&= \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[1 + x + x^2 + \cdots]}_{1/(1-x)} \\
&= \frac{i}{p^2 - M_0^2 - \Sigma(p^2) + i0^+}
\end{aligned}
$$

$-i\Sigma(p^2)$: one-particle-irreducible diagrams

Definition of the physical mass

$$
M^2 - M_0^2 - \Sigma(M^2) \stackrel{!}{=} 0
$$

Self-energy contributions at $D = 4$:



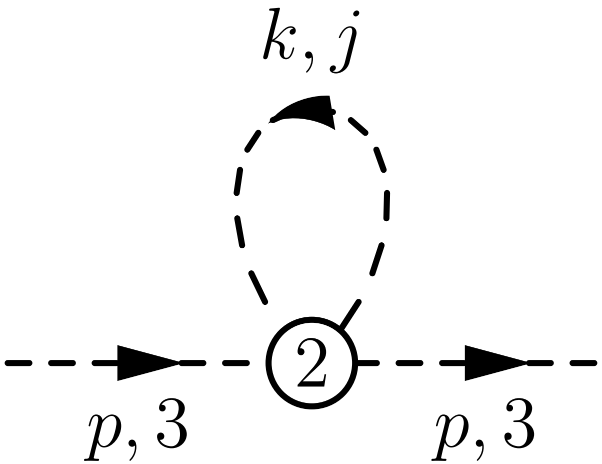
$$\mathcal{L}_{\text{int}} = \mathcal{L}_4^{2\phi} + \mathcal{L}_2^{4\phi}$$

At $\mathcal{O}(q^4)$ the self-energies are of the form

$$\Sigma_\phi(p^2) = A_\phi + B_\phi p^2$$

A_ϕ and B_ϕ receive a tree-level contribution from $\mathcal{L}_4$ and a one-loop contribution with a vertex from $\mathcal{L}_2$

Example: pion-loop contribution to the π^0 self-energy



$$\frac{i}{6F_0^2}(-4p^2 + M_{\pi,2}^2) I(M_{\pi,2}^2, \mu^2, n)$$

diverges as $n \to 4$

Example:

$$
\begin{aligned}
A_\pi &= \frac{M^2_{\pi,2}}{F_0^2}\Big\{ \underbrace{-\frac{1}{6}I(M^2_{\pi,2}) - \frac{1}{6}I(M^2_{\eta,2}) - \frac{1}{3}I(M^2_{K,2})}_{\text{one-loop contribution}} \\
&\quad \underbrace{+32[(2m+m_s)B_0L_6 + mB_0L_8]}_{\text{contact contribution}} \Big\}
\end{aligned}
$$

$$
B_\pi = \frac{2}{3}\frac{I(M^2_{\pi,2})}{F_0^2} + \frac{1}{3}\frac{I(M^2_{K,2})}{F_0^2} - \frac{16B_0}{F_0^2}\left[(2m+m_s)L_4 + mL_5\right]
$$

Masses at $\mathcal{O}(q^4)$

$$
\begin{aligned}
M^2 &= M_0^2 + \Sigma(M^2) = M_0^2 + A + BM^2 \\
&= \frac{M_0^2 + A}{1-B} = M_0^2(1+B) + A + \mathcal{O}(q^6)
\end{aligned}
$$

because $A = \mathcal{O}(q^4)$ and $\{B, M_0^2\} = \mathcal{O}(q^2)$

Final result

$$M_{\pi,4}^2 = M_{\pi,2}^2\{1+\frac{M_{\pi,2}^2}{32\pi^2F_0^2}\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)-\frac{M_{\eta,2}^2}{96\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right)$$

$$+\frac{16}{F_0^2}[(2m+m_s)B_0(2L_6^r-L_4^r)+mB_0(2L_8^r-L_5^r)]\}$$

Remarks:

1. Expressions for the masses are finite. The bare coefficients L_i of the Lagrangian of Gasser and Leutwyler must be infinite in order to cancel the infinities resulting from the divergent loop integrals.

2. At any order $\mathcal{O}(q^{2n})$ the masses of the Goldstone bosons vanish, if the quark masses are sent to zero.

3. The quark masses appear in combination with B_0. No absolute statements about quark masses possible without knowledge about B_0.

4. Analytic terms $\sim m_q^2$ multiplied by the renormalized low-energy coupling constants L_i^r.

5. Non-analytic terms $\sim m_q^2 \ln(m_q)$ (so-called chiral logarithms) contain no new constants.

6. Physical observables do not depend on the scale μ.

Wess-Zumino-Witten effective action and "anomalous" processes

- Q: Is a symmetry of classical physics necessarily a symmetry of quantum physics?

- A: No! Quantum fluctuations can break classical symmetries. $\Rightarrow$ misleading name "anomaly"

Problem: Lagrangians $\mathcal{L}_2$ and $\mathcal{L}_4$ have a larger symmetry than the "real world"[19]

Consider

$$\phi(x) \mapsto -\phi(x) \Leftrightarrow U = \exp(i\phi/F_0) \leftrightarrow \exp(-i\phi/F_0) = U^\dagger$$

[19] E. Witten, Nucl. Phys. B223, 422 (1983)

E.g., "pure QCD:"

$$\frac{F_0^2}{4}\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \;\mapsto\; \frac{F_0^2}{4}\text{Tr}(\partial_\mu U^\dagger \partial^\mu U) = \frac{F_0^2}{4}\text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

Analogy

$$f(x) = f(-x)$$

$\Rightarrow \mathcal{L}_2$ ("pure QCD") contains interaction terms with an even number of Goldstone bosons only (even intrinsic parity):

Cannot describe, e.g, $K^+K^- \to \pi^+\pi^-\pi^0$

Analogously: $\mathcal{L}_2$ and $\mathcal{L}_4$ cannot describe $\pi^0 \to \gamma\gamma$

- What's wrong?

Witten: Add the simplest term possible which breaks the symmetry of having only an even number of Goldstone bosons at the Lagrangian level

Equation of motion

$$\partial_\mu \left(\frac{F_0^2}{2} U \partial^\mu U^\dagger \right) + \lambda \epsilon^{\mu\nu\rho\sigma} U \partial_\mu U^\dagger U \partial_\nu U^\dagger U \partial_\rho U^\dagger U \partial_\sigma U^\dagger = 0$$

λ is a (purely imaginary) constant

Surprise: Action functional corresponding to the new term cannot be written as the four-dimensional integral of a Lagrangian expressed in terms of U and its derivatives

Mathematical trick: Extend the range of definition of the fields to a hypothetical fifth dimension,

$$
\begin{aligned}
U(y) &= \exp\left(i\alpha\frac{\phi(x)}{F_0}\right) \\
y^i &= (x^\mu, \alpha),\ i = 0, \cdots, 4,\ 0 \le \alpha \le 1
\end{aligned}
$$

Minkowski space is defined as the surface of the five-dimensional space for $\alpha = 1$

Wess-Zumino-Witten action[20] in the absence of external fields (denoted by a superscript 0):

$$
\begin{aligned}
S^0_{\text{ano}} &= n S^0_{\text{WZW}} \\
S^0_{\text{WZW}} &= -\frac{i}{240\pi^2}\int_0^1 d\alpha \int d^4x \epsilon^{ijklm} \text{Tr}\left(\mathcal{U}^L_i \cdots \mathcal{U}^L_m\right)
\end{aligned}
$$

where

$$
\epsilon_{01234} = -\epsilon^{01234} = 1, \quad \mathcal{U}^L_i = U^\dagger \partial U / \partial y^i, \quad \lambda = in/(48\pi^2)
$$

[20] J. Wess and B. Zumino, Phys. Lett. B 37, 95 (1971), E. Witten, Nucl. Phys. B223, 422 (1983)

Witten uses topological arguments to show that n must be an integer

O. Bär and U.-J. Wiese, Nucl. Phys. B609, 225 (2001):

"Traditional" argument relating n with number of colors N_c is wrong!

Consequences of S^0_{WZW}:

$$U(y) = 1 + i\alpha\phi(x)/F_0 + O(\phi^2)$$

$$\begin{aligned} S^{5\phi}_{\mathrm{WZW}} &= \frac{1}{240\pi^2 F_0^5}\int_0^1 d\alpha \int d^4x \epsilon^{ijklm} \\ &\quad \times \mathrm{Tr}[\partial_i(\alpha\phi)\partial_j(\alpha\phi)\partial_k(\alpha\phi)\partial_l(\alpha\phi)\partial_m(\alpha\phi)] \\ &= \cdots \\ &= \frac{1}{240\pi^2 F_0^5}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\phi\partial_\mu\phi\partial_\nu\phi\partial_\rho\phi\partial_\sigma\phi) \end{aligned}$$

- “Ordinary action” in four space-time dimensions.

- Constructed by Wess and Zumino order by order.

- Describes interactions of an odd number of Goldstone bosons.

- WZW action takes care of the chiral anomaly in QCD.

- Q: How can one identify n?

- A: Introduce a coupling to electromagnetism.

In the presence of external fields there will be an additional term in the anomalous action,

$$S_{\text{ano}} = S^{0}_{\text{ano}} + S^{\text{ext}}_{\text{ano}} = n(S^{0}_{\text{WZW}} + S^{\text{ext}}_{\text{WZW}})$$

$$S_{\rm WZW}^{\rm ext} = -\frac{i}{48\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(Z_{\mu\nu\rho\sigma})$$

with

$$\begin{aligned}
Z_{\mu\nu\rho\sigma} \;=\; & \frac{1}{2}\,Ul_\mu U^\dagger r_\nu U l_\rho U^\dagger r_\sigma \\
& + U l_\mu l_\nu l_\rho U^\dagger r_\sigma - U^\dagger r_\mu r_\nu r_\rho U l_\sigma \\
& + iU\partial_\mu l_\nu l_\rho U^\dagger r_\sigma - iU^\dagger \partial_\mu r_\nu r_\rho U l_\sigma \\
& + i\partial_\mu r_\nu U l_\rho U^\dagger r_\sigma - i\partial_\mu l_\nu U^\dagger r_\rho U l_\sigma \\
& - i\mathcal{U}_\mu^L l_\nu U^\dagger r_\rho U l_\sigma + i\mathcal{U}_\mu^R r_\nu U l_\rho U^\dagger r_\sigma \\
& - i\mathcal{U}_\mu^L l_\nu l_\rho l_\sigma + i\mathcal{U}_\mu^R r_\nu r_\rho r_\sigma \\
& + \frac{1}{2}\mathcal{U}_\mu^L U^\dagger \partial_\nu r_\rho U l_\sigma - \frac{1}{2}\mathcal{U}_\mu^R U\partial_\nu l_\rho U^\dagger r_\sigma \\
& + \frac{1}{2}\mathcal{U}_\mu^L U^\dagger r_\nu U\partial_\rho l_\sigma - \frac{1}{2}\mathcal{U}_\mu^R U l_\nu U^\dagger \partial_\rho r_\sigma \\
& - \mathcal{U}_\mu^L \mathcal{U}_\nu^L U^\dagger r_\rho U l_\sigma + \mathcal{U}_\mu^R \mathcal{U}_\nu^R U l_\rho U^\dagger r_\sigma \\
& + \mathcal{U}_\mu^L l_\nu \partial_\rho l_\sigma - \mathcal{U}_\mu^R r_\nu \partial_\rho r_\sigma \\
& + \mathcal{U}_\mu^L \partial_\nu l_\rho l_\sigma - \mathcal{U}_\mu^R \partial_\nu r_\rho r_\sigma
\end{aligned}$$

$$+\frac{1}{2}\,\mathcal{U}^L_\mu l_\nu \mathcal{U}^L_\rho l_\sigma - \frac{1}{2}\,\mathcal{U}^R_\mu r_\nu \mathcal{U}^R_\rho r_\sigma$$
$$-i\mathcal{U}^L_\mu \mathcal{U}^L_\nu \mathcal{U}^L_\rho l_\sigma + i\mathcal{U}^R_\mu \mathcal{U}^R_\nu \mathcal{U}^R_\rho r_\sigma$$

Abbreviations $\mathcal{U}^L_\mu = U^\dagger \partial_\mu U$ and $\mathcal{U}^R_\mu = U \partial_\mu U^\dagger$

Special case:

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu$$

corresponds to

$$\mathcal{L}_{\gamma qq} = -e\mathcal{A}_\mu \left[\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right]$$

if

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$
$$\overset{\text{B\&W}}{\longmapsto} \begin{pmatrix} \frac{1}{2N_c} + \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2N_c} - \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2N_c} - \frac{1}{2} \end{pmatrix}$$

WZW in the presence of electromagnetism:

$$n\mathcal{L}_{\mathrm{WZW}}^{\mathrm{ext}} = -en\mathcal{A}_\mu J^\mu + i\frac{ne^2}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\mathcal{A}_\rho\mathcal{A}_\sigma \mathrm{Tr}[2Q^2(U\partial_\mu U^\dagger - U^\dagger\partial_\mu U) - QU^\dagger Q\partial_\mu U + QUQ\partial_\mu U^\dagger]$$

The current

$$J^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{48\pi^2}\mathrm{Tr}(Q\partial_\nu UU^\dagger\partial_\rho UU^\dagger\partial_\sigma UU^\dagger + QU^\dagger\partial_\nu UU^\dagger\partial_\rho UU^\dagger\partial_\sigma U), \quad \epsilon_{0123} = 1,$$

by itself is not gauge invariant

The additional terms containing two $\mathcal{A}$'s are required to obtain a gauge-invariant action

How can one identify n?

Find the interaction Lagrangian which is relevant to the decay $\pi^0 \to \gamma\gamma$:

$$U = 1 + i\text{diag}(\pi^0, -\pi^0, 0)/F_0 + \cdots,$$

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\frac{n}{N_c}\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\frac{\pi^0}{F_0}$$

Invariant amplitude

$$\mathcal{M} = i\frac{n}{N_c}\frac{e^2}{4\pi^2 F_0}\epsilon^{\mu\nu\rho\sigma}q_{1\mu}\epsilon^*_{1\nu}q_{2\rho}\epsilon^*_{2\sigma}$$

Decay rate

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{\alpha^2 M^3_{\pi^0}}{64\pi^3 F_0^2}\frac{n^2}{N_c^2} = 7.6\,\text{eV} \times \left(\frac{n}{N_c}\right)^2$$

in good agreement with the experimental value (7.7 ± 0.6) eV for $n = N_c$

- **Bär und Wiese: No indication for $N_c = 3$!**

- **One should rather look at $\eta \to \pi^+\pi^-\gamma$**

- **But: Important sub-leading terms which are needed to account for the experimental decay widths and decay spectra**[21]

[21] B. Borasoy and E. Lipartia, Phys. Rev. D 71, 014027 (2005)

Chiral Perturbation Theory for Baryons

- Interaction of pions and nucleons[22]

- Most general Lagrangian

$$\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \cdots$$

Transformation properties of the fields

Nucleon field

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

Transformation behavior under isospin SU(2)

$$\Psi \mapsto V\Psi$$

[22] J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988); A. Krause, Helv. Phys. Acta 63, 3 (1990)

Introduce

$$u^2(x) = U(x)$$

Define $K(V_L, V_R, U)$ in terms of

$$u(x) \mapsto u'(x) = \sqrt{V_R U V_L^\dagger} \equiv V_R u K^{-1}(V_L, V_R, U)$$

i.e.

$$K(V_L, V_R, U) = u'^{-1} V_R u = \sqrt{V_R U V_L^\dagger}^{-1} V_R \sqrt{U}$$

Transformation properties under local SU$(2)_L \times$ SU(2)$_R \times$ U(1)$_V$

$$\begin{pmatrix} U \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} V_R U V_L^\dagger \\ \exp[-i\Theta] K(V_L, V_R, U)\Psi \end{pmatrix}$$

Exercise: Verify

$$K(V_L', V_R', V_R U V_L^\dagger) K(V_L, V_R, U) = K((V_L' V_L), (V_R' V_R), U)$$

We need a covariant derivative of the nucleon field with

$$D'_\mu \Psi' = \exp(-i\Theta) K(V_L, V_R, U) D_\mu \Psi \quad (*)$$

Introduce

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right]$$

and define

$$D_\mu \Psi = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})\Psi$$

Exercise: Verify $(*)$

Define

$$u_\mu \equiv i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]$$

Behavior under parity

$$u_\mu \overset{P}{\mapsto} i\left[u(\partial^\mu - il^\mu)u^\dagger - u^\dagger(\partial^\mu - ir^\mu)u\right] = -u^\mu$$

Exercise: Using

$$u' = V_R u K^\dagger = K u V_L^\dagger$$

show that, under SU(2)$_L \times$ SU(2)$_R \times$ U$(1)_V$, u_μ transforms as

$$u_\mu \mapsto K u_\mu K^\dagger$$

Counting scheme for the (new) elements of baryon chiral perturbation theory[23]

$$\begin{aligned}
&\Psi, \bar{\Psi} = \mathcal{O}(q^0) \\
&D_\mu \Psi = \mathcal{O}(q^0) \\
&(i\not{D} - m)\Psi = \mathcal{O}(q) \\
&1, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(q^0) \\
&\gamma_5 = \mathcal{O}(q)
\end{aligned}$$

order given is the minimal one

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[23] A. Krause, Helv. Phys. Acta 63, 3 (1990)

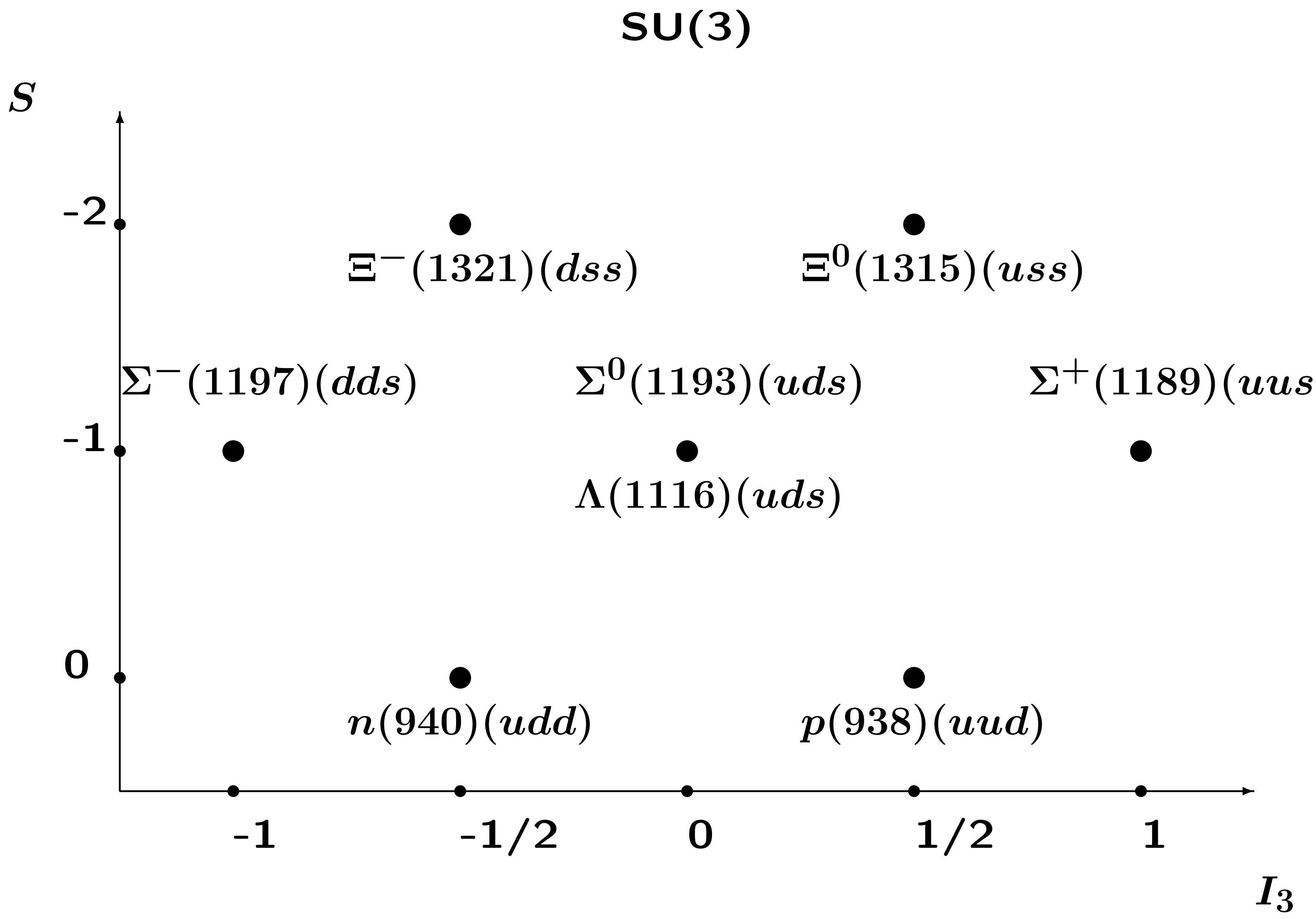
$$\mathcal{L}^{(1)}_{\pi N} = \bar{\Psi}\left(i\not{D} - m + \frac{\overset{\circ}{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi$$

$$= \bar{\Psi}\left(i\gamma_\mu\partial^\mu - \boxed{m} - \frac{1}{2}\frac{\boxed{\overset{\circ}{g}_A}}{F}\gamma_\mu\gamma_5\tau^a\partial^\mu\pi^a\right)\Psi + \cdots$$

Two parameters not determined by chiral symmetry:

- nucleon mass m in the chiral limit
- axial-vector coupling constant $\overset{\circ}{g}_A$ in the chiral limit

[Physical nucleon mass: $m_N = 939$ MeV. Theoretical analysis: $m \approx 883$ MeV (at fixed $m_s \neq 0$). Physical axial-vector coupling constant from neutron beta decay: $g_A = 1.267$.]


SU(3)
S
-2
-1
0
Ξ⁻(1321)(dss)
Ξ⁰(1315)(uss)
Σ⁻(1197)(dds)
Σ⁰(1193)(uds)
Σ⁺(1189)(uus)
Λ(1116)(uds)
n(940)(udd)
p(938)(uud)
-1
-1/2
0
1/2
1
I3

$$B = \sum_{a=1}^{8} \frac{\lambda_a B_a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}$$

Transformation behavior under flavor SU(3)$_V$

$$B \mapsto VBV^\dagger$$

Transformation behavior under SU(3)$_L \times$ SU(3)$_R$

$$\begin{pmatrix} U \\ B \end{pmatrix} \mapsto \begin{pmatrix} U' \\ B' \end{pmatrix} = \begin{pmatrix} V_R U V_L^\dagger \\ K(V_L, V_R, U) B K^\dagger(V_L, V_R, U) \end{pmatrix}$$

Covariant derivative

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B]$$

Most general Lagrangian at $\mathcal{O}(q)$

$$\mathcal{L}_{MB}^{(1)} = \text{Tr}\left[\bar{B}\left(i\not{D} - M_0\right)B\right] - \frac{D}{2}\text{Tr}\left(\bar{B}\gamma^\mu\gamma_5\{u_\mu, B\}\right) - \frac{F}{2}\text{Tr}\left(\bar{B}\gamma^\mu\gamma_5[u_\mu, B]\right)$$

Three parameters not determined by chiral symmetry:

- Mass of the baryon octet in the chiral limit M_0

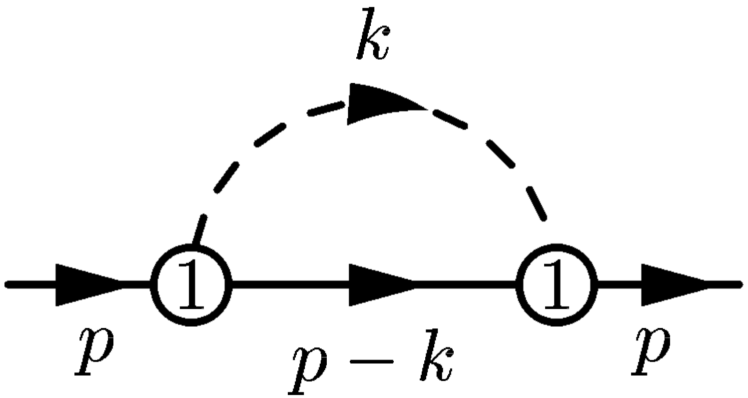
- D and F may be determined by fitting the semi-leptonic decays $B \to B' + e^- + \bar{\nu}_e$ at tree level:

$$D = 0.80, \quad F = 0.50$$

Power counting and renormalization: Outline of the problem

- **Power counting:** Associate chiral order D with a diagram
  - Loop integration in n dimensions $\sim \mathcal{O}(q^n)$
  - Vertex from $\mathcal{L}_{2k} \sim \mathcal{O}(q^{2k})$
  - Vertex from $\mathcal{L}_{\pi N}^{(k)} \sim \mathcal{O}(q^k)$
  - Nucleon propagator $\sim \mathcal{O}(q^{-1})$
  - Pion propagator $\sim \mathcal{O}(q^{-2})$

- **Example: Contribution to nucleon mass**



Goal: $D = n \cdot 1 - 2 \cdot 1 - 1 \cdot 1 + 2 \cdot 1 = n - 1$

Result

$$\Sigma = -\frac{3g_{A0}^2}{4F_0^2}[(\not p + m)I_N + M^2(\not p + m)I_{N\pi}(-p, 0) + \cdots]$$

Apply $\widetilde{\mathrm{MS}}$ renormalization scheme

$$\Sigma_r = -\frac{3g_{Ar}^2}{4F_r^2}[M^2(\not p + m) \underbrace{I_{N\pi}^r(-p,0)}_{-\frac{1}{16\pi^2}+\cdots} + \cdots] = \mathcal{O}(q^2)$$

GSS:[24] It turns out that loops have a much more complicated low-energy structure if baryons are included. Because the nucleon mass m_N does not vanish in the chiral limit, the mass scale m (nucleon mass in the chiral limit) occurs in the effective Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ ... . This complicates life a lot.

BKKM:[25] Stated differently, the consistent power counting scheme present in the mesonic sector is altered when baryons are included and one no longer has a one-to-one mapping between the loop and small-momentum expansion.

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[24] J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988)

[25] V. Bernard, N. Kaiser, J, Kambor, U.-G. Meißner, Nucl. Phys. B388, 315 (1992)

Solutions

Solution 1: Heavy-Baryon Approach [26]

- **Trick: Let $p = mv + k_p$, $v^2 = 1$, $v^0 \geq 1$ [Often $v^\mu = (1,0,0,0)$]**

$$\Psi(x) = e^{-imv\cdot x}(\mathcal{N}_v + \mathcal{H}_v)$$

with

$$\begin{aligned} \mathcal{N}_v &= e^{+imv\cdot x}\frac{1}{2}(1+\not{v})\Psi \\ \mathcal{H}_v &= e^{+imv\cdot x}\frac{1}{2}(1-\not{v})\Psi \end{aligned}$$

- **Using the equation of motion for $\mathcal{H}_v$, one can eliminate $\mathcal{H}_v$ and obtain a Lagrangian for $\mathcal{N}_v$**

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[26] E. Jenkins and A. V. Manohar, Phys. Lett. B 255, 558 (1991)

- To lowest order

$$\widehat{\mathcal{L}}^{(1)}_{\pi N} = \bar{\mathcal{N}}_v (iv \cdot D + g_A S_v \cdot u) \mathcal{N}_v + \mathcal{O}(1/m)$$

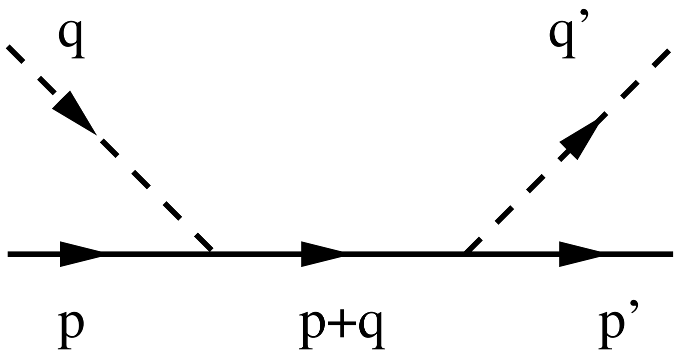
$1/m$ expansion of the Lagrangian similar to a Foldy-Wouthuysen expansion

- Power counting works as in the mesonic sector ($\widetilde{\text{MS}}$ scheme)

- But ...

- **... problems with analyticity**

Simple example (T. Becher, Chiral Dynamics 2000)



Singularity due to the nucleon pole in the s channel is understood in terms of the relativistic propagator

$$\frac{1}{(p+q)^2 - m_N^2} = \frac{1}{2p\cdot q + M_\pi^2}$$

pole at $\quad 2p\cdot q = -M_\pi^2$

Heavy-baryon type of expansion (with $p^\mu = m_N v^\mu$)

$$\frac{1}{2p\cdot q + M_\pi^2} = \frac{1}{2m_N}\,\frac{1}{v\cdot q + \frac{M_\pi^2}{2m_N}}$$
$$= \frac{1}{2m_N}\,\frac{1}{v\cdot q}\,\left(1 - \frac{M_\pi^2}{2m_N v\cdot q} + \cdots\right)$$

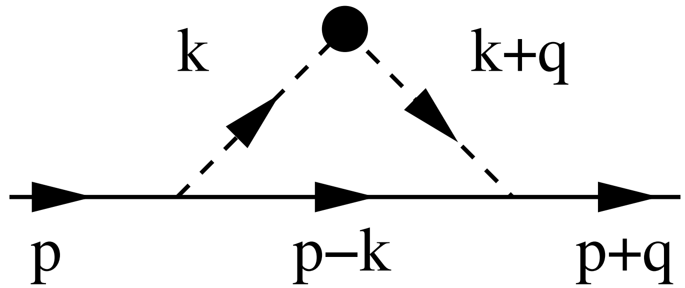
To any finite order the heavy-baryon expansion produces poles at

$$v\cdot q = 0$$

instead of a simple pole at

$$v\cdot q = -M_\pi^2/(2m_N)$$

Second example: Triangle diagram



- **Problems in the analytic behavior of form factors in the time-like region**

- **Example: Scalar form factor**

Solution 2: Infrared regularization[27]

Special treatment of one-loop integrals based on the Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}$$

$$\begin{aligned} a &= (p-k)^2 - m^2 + i0^+ \\ b &= k^2 - M^2 + i0^+ \end{aligned}$$

$$H = \int_0^1 dz \cdots \; = \; \int_0^\infty dz \cdots - \int_1^\infty dz \cdots \equiv I + R$$

- I: power counting o.k.

- R: violates power counting; regular, i.e., can be absorbed in counterterms

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[27] T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999)

Infrared regularization in more detail.

Consider dimensionally regularized one-loop integral[28]

$$
\begin{aligned}
&H(p^2, m^2, M^2; n)\\
&\equiv -i\int \frac{d^nk}{(2\pi)^n}\frac{1}{[(p-k)^2-m^2+i0^+][k^2-M^2+i0^+]}\\
&= -i\int \frac{d^nk}{(2\pi)^n}\frac{1}{[k^2-2p\cdot k+\underbrace{(p^2-m^2)}_{\mathcal{O}(q)}+i0^+][k^2-\underbrace{M^2}_{\mathcal{O}(q^2)}+i0^+]}.
\end{aligned}
$$

Qualitative discussion:

- Ultraviolet behavior:

  Estimate of degree of divergence: For large values of k integrand behaves as k^{n-1}/k^4. $\Rightarrow$

[28]Note the minus sign. Factor μ^{4-n} omitted.

- $n = 4$: Logarithmic divergence (dim. reg.: $1/(n-4)$).
- $n < 4$: Integral converges.

- Infrared behavior: Consider limit $M^2 \to 0$.
  - $n = 4$: Integral is infrared regular for both $p^2 = m^2$ and $p^2 \neq m^2$, because, for small momenta, the integrand behaves as k^3/k^3 and k^3/k^2, respectively.
  - For $n = 3$ the integral is infrared regular for $p^2 \neq m^2$ but singular for $p^2 = m^2$.
  - For any smaller value of n it is infrared singular for arbitrary p^2.
  - Infrared singularity as $M^2 \to 0$ originates in the region, where the integration variable k is small, i.e.,

of the order $\mathcal{O}(q)$. Counting powers of momenta, we (naively) expect this part to be of order $\mathcal{O}(q^{n-3})$.

- Intermediate region:

  On the other hand, for loop momenta of the order of and larger than the nucleon mass we expect power counting to fail, because the momentum of the nucleon propagating in loop integral is not constrained to be small.

Explicit evaluation of integral $H(p^2, m^2, M^2; n)$:

Feynman parameterization:

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}$$

with $a = (p-k)^2 - m^2 + i0^+$ and $b = k^2 - M^2 + i0^+$.

Interchange the order of integrations:

$$H = -i\int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - 2k\cdot pz + (p^2 - m^2)z + zM^2 + i0^+]^2}.$$

Perform the shift $k \to k + zp$:

$$H(p^2, m^2, M^2; n) = -i\int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - A(z) + i0^+]^2},$$

where

$$A(z) = z^2 p^2 - z(p^2 - m^2 + M^2) + M^2.$$

Make use of (Exercise)

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^p}{(k^2 - A)^q} = \frac{i(-)^{p-q}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma\left(p + \frac{n}{2}\right)\Gamma\left(q - p - \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(q)} A^{p+\frac{n}{2}-q}$$

in dim. reg.

Apply to H with $p = 0$ and $q = 2$. $\Rightarrow$

Intermediate result:

$$H(p^2, m^2, M^2; n) = \frac{1}{(4\pi)^{\frac{n}{2}}}\Gamma\left(2 - \frac{n}{2}\right)\int_0^1 dz[A(z) - i0^+]^{\frac{n}{2}-2}.$$

Discussion of relevant properties at the threshold:

$$\begin{aligned} p^2_{\rm thr} &= (m+M)^2, \\ A_{\rm thr}(z) &= z^2(m+M)^2 - z[(m+M)^2 - m^2 + M^2] + M^2 \\ &= [z(m+M) - M]^2 \geq 0, \\ z_0 &= M/(m+M), \qquad A_{\rm thr}(z_0) = 0. \end{aligned}$$

Splitting integration interval into $[0, z_0]$ and $[z_0, 1]$, we have, for $n > 3$,

$$\begin{aligned} \int_0^1 dz[A_{\rm thr}(z)]^{\frac{n}{2}-2} &= \int_0^{z_0} dz[M - z(m+M)]^{n-4} \\ &\quad + \int_{z_0}^1 dz[z(m+M) - M]^{n-4} \\ &= \frac{1}{(n-3)(m+M)}(M^{n-3} + m^{n-3}). \end{aligned}$$

Analytic continuation for arbitrary n:

$$H((m+M)^2, m^2, M^2; n) = \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \left(\frac{M^{n-3}}{m+M} + \frac{m^{n-3}}{m+M}\right).$$

Discussion

- The first term, proportional to M^{n-3}, is defined as the so-called infrared singular part I.

- As $M \to 0$, I behaves as in the qualitative discussion above.

- $M \to 0$ implies $p_{\mathrm{thr}}^2 \to m^2$. I is singular for $n \leq 3$.

- The second term, proportional to m^{n-3}, is defined as the so-called infrared regular part R.

- Can be thought of as originating from an integration region where k is of order m.

- For non-integer n the infrared singular part contains non-integer powers of M.

- Expansion of the regular part always contains non-negative integer powers of M only.

Formal definition of the infrared singular and regular parts (for arbitrary p^2).

Introduce the dimensionless variables

$$\begin{aligned} \alpha &= \frac{M}{m} = \mathcal{O}(q), \\ \Omega &= \frac{p^2 - m^2 - M^2}{2mM} = \mathcal{O}(q^0). \end{aligned}$$

Rewrite $A(z)$ as

$$A(z) = m^2[z^2 - 2\alpha\Omega z(1-z) + \alpha^2(1-z)^2] \equiv m^2 C(z).$$

$\Rightarrow$ H is now given by

$$H(p^2, m^2, M^2; n) = \kappa(m; n) \int_0^1 dz [C(z) - i0^+]^{\frac{n}{2}-2},$$

where

$$\kappa(m; n) = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}} m^{n-4}.$$

- Infrared singularity originates from small values of z, where $C(z)$ goes to zero as $M \to 0$.

- Isolate divergent part by scaling integration variable $z \equiv \alpha x$. Upper limit $z = 1$ in Feynman parameterization corresponds to $x = 1/\alpha \to \infty$ as $M \to 0$.

- Define integral I having the same infrared singularity as H. To that end replace upper limit by ∞:

$$\begin{aligned} I &\equiv \kappa(m;n)\int_0^\infty dz[C(z)-i0^+]^{\frac{n}{2}-2} \\ &= \kappa(m;n)\alpha^{n-3}\int_0^\infty dx[D(x)-i0^+]^{\frac{n}{2}-2}, \end{aligned}$$

where

$$D(x)=1-2\Omega x+x^2+2\alpha x(\Omega x-1)+\alpha^2x^2.$$

(The pion mass M is not sent to zero.)

- Define regular part of H as

$$R\equiv -\kappa(m;n)\int_1^\infty dz[C(z)-i0^+]^{\frac{n}{2}-2},$$

so that

$$H=I+R.$$

- Q: Do these definitions indeed reproduce the behavior for $p^2_{\rm thr}$?

  A: Yes!

  Verification: $\Omega_{\rm thr} = 1$.

- Threshold value of the infrared singular part:

$$I_{\rm thr} = \kappa(m;n)\alpha^{n-3}\int_0^\infty dx\left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2},$$

  which converges for $n < 3$.

  In order to continue the integral to $n > 3$, we write

$$\left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2} =$$
$$= \frac{(1+\alpha)x-1}{(1+\alpha)(n-4)}\frac{d}{dx}\left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2},$$

and make use of a partial integration

$$\int_0^\infty dx \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2} =$$

$$\left[\frac{(1+\alpha)x-1}{(1+\alpha)(n-4)}\left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2}\right]_0^\infty$$

$$-\frac{1}{n-4}\int_0^\infty dx \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2}.$$

For $n < 3$, the first expression vanishes at the upper limit and, at the lower limit, yields $1/[(1+\alpha)(n-4)]$.

Bringing the second expression to the left-hand side, we may then continue the integral analytically as

$$\int_0^\infty dx \left\{[(1+\alpha)x-1]^2 - i0^+\right\}^{\frac{n}{2}-2} = \frac{1}{(n-3)(1+\alpha)},$$

so that we obtain for I_{thr}

$$I_{\text{thr}} = \kappa(m;n)\alpha^{n-3}\frac{1}{(n-3)(1+\alpha)} = \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{M^{n-3}}{m+M}.$$

Agrees with the infrared singular part discussed above.

- Threshold value of the regular part obtained by analytic continuation from $n < 3$ to $n > 3$:

$$
\begin{aligned}
R_{\text{thr}} &= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\int_1^\infty dz[z(m+M)-M_\pi]^{n-4} \\
&= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\frac{1}{(n-3)(m+M)}(\infty^{n-3}-m^{n-3}) \\
&\overset{n\leq 3}{=} \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{m^{n-3}}{m+M}.
\end{aligned}
$$

Again, agrees with the regular part discussed above.

- Distinction between I and R:

  For non-integer values of n, the chiral expansion of I gives rise to non-integer powers of small quantities.

  Regular part R may be expanded in an ordinary Taylor series.

  I satisfies power counting; R does not.

  Basic idea of the infrared regularization: Replace general integral H by its infrared singular part I, and drop the regular part R.

  In the low-energy region H and I have the same analytic properties.

  Contribution of R, which is of the type of an infinite series in the momenta, can be included by adjusting the coefficients of the most general effective Lagrangian.

- Generalization to arbitrary one-loop graph.
  - Reduce tensor integrals involving an expression of the type $k^{\mu_1}\cdots k^{\mu_2}$ in the numerator to scalar loop integrals of the form

$$-i\int\frac{d^nk}{(2\pi)^n}\frac{1}{a_1\cdots a_m}\frac{1}{b_1\cdots b_n},$$

    $a_i=(q_i+k)^2-M^2+i0^+$: Inverse meson propagators;

    $b_i=(p_i-k)^2-m^2+i0^+$ Inverse nucleon propagators;

    q_i: four-momenta of $\mathcal{O}(q)$;

    p_i: four-momenta which are not far off the nucleon mass shell, i.e., $p_i^2=m^2+\mathcal{O}(q)$.

  - Using the Feynman parameterization, combine all nucleon propagators separately and all pion propagators separately.

– Write the result such that it is obtained by applying $(m-1)$ and $(n-1)$ partial derivatives with respect to M^2 and m^2, respectively, to a master formula.

Simple illustration:

$$\frac{1}{a_1 a_2} = \int_0^1 dz \frac{1}{[a_1 z + a_2(1-z)]^2} = \frac{\partial}{\partial M^2} \int_0^1 dz \frac{1}{a_1 z + a_2(1-z)},$$

where $a_i = (q_i + k)^2 - M^2 + i0^+$.

Expressions become more complicated for larger numbers of propagators!

Relevant property of the above procedure:

Result of combining the meson propagators is of the type $1/A$ with $A = (k+q)^2 - M^2 + i0^+$, where q is a linear combination of the m momenta q_i, with an analogous expression $1/B$ for the nucleon propagators.

– Finally, treat expression

$$-i \int \frac{d^n k}{(2\pi)^n} \frac{1}{AB}$$

in complete analogy to H: Combine denominators and identify infrared singular and regular pieces are identified by writing

$$\int_0^1 dz \cdots = \int_0^\infty dz \cdots - \int_1^\infty dz \cdots .$$

- Q: Does the infrared regularization respect the constraints of chiral symmetry as expressed through the chiral Ward identities?

  A: Yes

  The argument is as follows.

  * Total nucleon-to-nucleon transition amplitude is chirally symmetric.

    (Invariant under a local transformation of the external fields.)

  * Calculation within EFT:

    Contribution from all the tree-level diagrams is chirally symmetric so that the loop contribution must also be chirally symmetric.

  * Dim. reg.: Statement holds for an arbitrary n.

Now: Separation into infrared singular and regular parts amounts to distinguishing between contributions of non-integer and non-negative integer powers in the momentum expansion.

These powers do not mix for arbitrary n. $\Rightarrow$ Infrared singular and regular parts must be separately chirally symmetric.

Finally, regular part can be expanded in powers of either momenta or quark masses, and thus may as well be absorbed in the (modified) tree-level contribution.

Solution 3: Extended on-mass-shell (EOMS) scheme[29]

Main idea: Perform additional subtractions such that renormalized diagrams satisfy the power counting

Motivation for this approach[30]

Terms violating the power counting are analytic in small quantities (and can thus be absorbed in a renormalization of counterterms)

- Example (chiral limit)

$$H(p^2, m^2; n) = -\int \frac{d^n k}{(2\pi)^n} \frac{i}{[(k-p)^2 - m^2 + i0^+][k^2 + i0^+]}$$

[29] T. Fuchs, J. Gegelia, G. Japaridze, S. S., Phys. Rev. D 68, 056005 (2003)

[30] J. Gegelia and G. Japaridze, Phys. Rev. D 60, 114038 (1999)

Small quantity

$$\Delta = \frac{p^2 - m^2}{m^2} = \mathcal{O}(q)$$

We want the renormalized integral to be of order

$$D = n - 1 - 2 = n - 3$$

Result of integration

$$H \sim F(n, \Delta) + \Delta^{n-3} G(n, \Delta)$$

F and G are hypergeometric functions; analytic in Δ for arbitrary n

Observation[31]

F corresponds to first expanding the integrand in small quantities and then performing the integration

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[31] J. Gegelia, G. Japaridze, K. S. Turashvili, Theor. Math. Phys. 101, 1313 (1994)

⇒ **Algorithm**: Expand integrand in small quantities and subtract those (integrated) terms whose order is smaller than suggested by the power counting

Here:

$$
\begin{aligned}
H^{\rm subtr} &= -\int \frac{d^n k}{(2\pi)^n} \frac{i}{(k^2 - 2k\cdot p + i0^+)(k^2 + i0^+)}\Bigg|_{p^2=m^2} \\
&= -2\bar{\lambda} + \frac{1}{16\pi^2} + O(n-4)
\end{aligned}
$$

where

$$
\bar{\lambda} = \frac{m^{n-4}}{(4\pi)^2}\left\{\frac{1}{n-4} - \frac{1}{2}\left[\ln(4\pi) + \Gamma'(1) + 1\right]\right\}
$$

$$
H^R = H - H^{\rm subtr} = \mathcal{O}(q^{n-3})
$$

General case including pion mass

$$\left.\frac{1}{\left(k^2 - 2k\cdot p + i0^+\right)\left(k^2 + i0^+\right)}\right|_{p^2=m^2}$$

$$+(p^2 - m^2)\left[\frac{1}{2m^2}\frac{1}{\left(k^2 - 2k\cdot p + i0^+\right)^2} + \cdots\right]_{p^2=m^2}$$

$$+M^2\left.\frac{1}{\left(k^2 - 2k\cdot p + i0^+\right)\left(k^2 + i0^+\right)^2}\right|_{p^2=m^2}$$

$$+\cdots$$

Remarks:

- (Axial) Vector mesons can be consistently included[32]
  - Improved phenomenology[33]
  - Larger radius of convergence
- IR renormalization can be reformulated in terms of EOMS[34]
  - Formal equivalence shown at one-loop level

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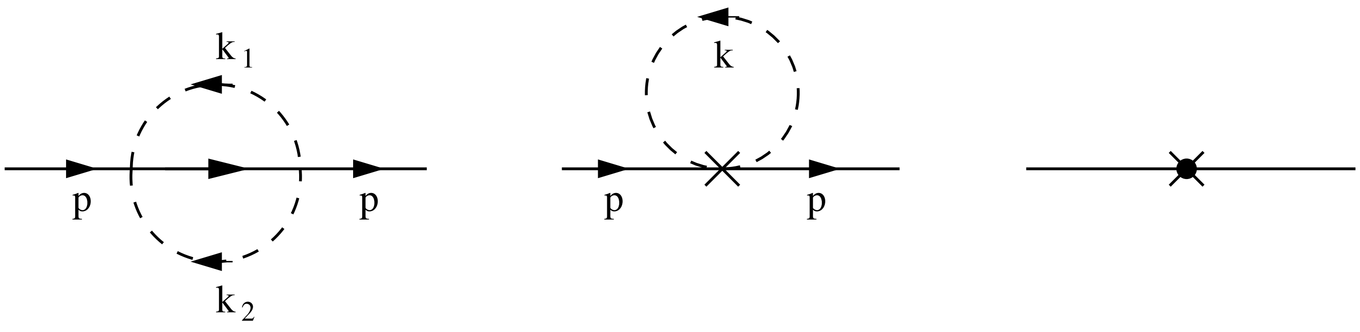
[32] T. Fuchs, M. R. Schindler, J. Gegelia, and S. Scherer, Phys. Lett. B 575, 11 (2003)

[33] M. R. Schindler, J. Gegelia, and S. Scherer, Eur. Phys. J. A 26, 1 (2005)

[34] M. R. Schindler, J. Gegelia, and S. Scherer, Phys. Lett. B 586, 258 (2004)

– **Known integrals tested**

- **One two-loop example has been explicitly tested**[35]



(a) (b) (c)

[35] M. R. Schindler, J. Gegelia, and S. Scherer, Nucl. Phys. B682, 367 (2004)

The EOMS approach in more detail

Calculation of the nucleon mass up to and including order $\mathcal{O}(q^3)$

Full propagator

$$S_0(p) = \frac{1}{\not{p} - m_0 - \Sigma_0(\not{p})} \equiv \frac{1}{\not{p} - m - \Sigma(\not{p})}$$

- m_0 bare mass
- m nucleon mass in the chiral limit
- $\Sigma_0(\not{p})$ self energy

Definition of the nucleon mass

$$m_N - m_0 - \Sigma_0(m_N) = m_N - m - \Sigma(m_N) = 0$$

- Tree-level contribution

Recall πN Lagrangian at order $\mathcal{O}(q^2)$

$$\begin{aligned}
\mathcal{L}_{\pi N}^{(2)} &= c_1 \,\text{Tr}(\chi_+)\bar{\Psi}\Psi - \frac{c_2}{4m^2}\left[\bar{\Psi}\,\text{Tr}(u_\mu u_\nu) D^\mu D^\nu \Psi + \text{H.c.}\right] \\
&+ \bar{\Psi}\left[\frac{c_3}{2}\text{Tr}(u_\mu u^\mu) + i\frac{c_4}{4}[u_\mu, u_\nu] + c_5\left[\chi_+ - \frac{1}{2}\text{Tr}(\chi_+)\right]\right. \\
&\qquad \left. + \frac{c_6}{2} f_{\mu\nu}^+ + \frac{c_7}{2} v_{\mu\nu}^{(s)}\right]\sigma^{\mu\nu}\Psi
\end{aligned}$$

Only c_1 term contributes to the self energy

$$-4c_1 M^2$$

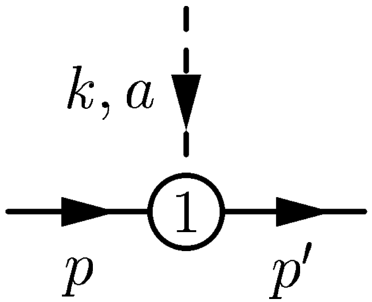
No contact contributions from the Lagrangian $\mathcal{L}_{\pi N}^{(3)}$

- **Loop contributions**

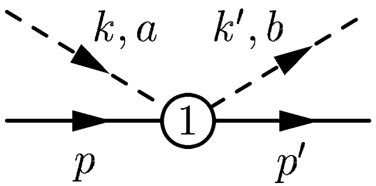
Expand $\mathcal{L}_{\pi N}^{(1)}$ up to and including two pion fields:

$$\mathcal{L}_{\rm int}^{(1)} = -\frac{1}{2}\frac{\overset{\circ}{g}_{A0}}{F_0}\bar{\Psi}\gamma^\mu\gamma_5\tau^b\partial_\mu\phi^b\Psi - \frac{1}{4F_0^2}\bar{\Psi}\gamma^\mu\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}\Psi$$

Feynman rules

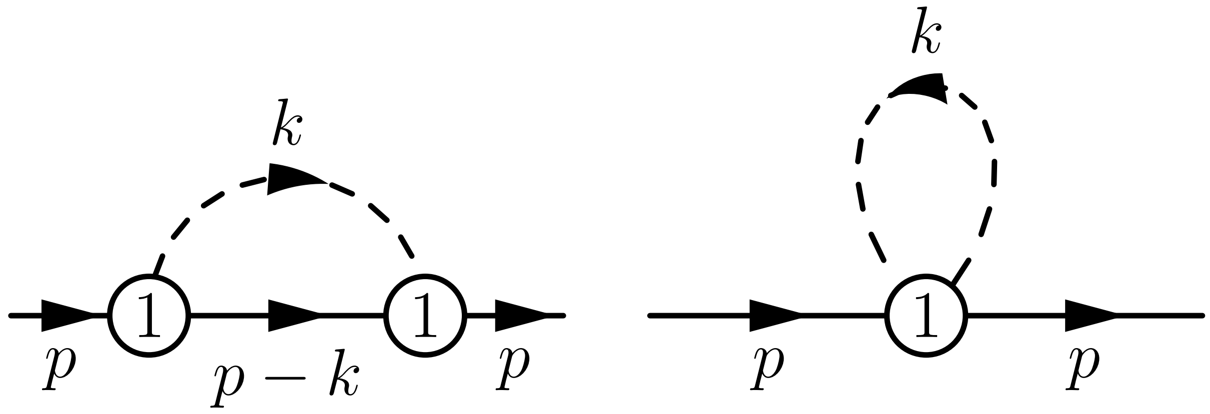


$$-\frac{\overset{\circ}{g}_{A0}}{2F_0}\,\not{k}\gamma_5\tau_a$$



$$\frac{1}{4F_0^2}\,(\not{k}+\not{k}')\epsilon_{abc}\tau_c$$

Two types of loop contributions at order $\mathcal{O}(q^3)$



Second diagram does not contribute: $\epsilon_{aac} = 0$

Feynman rules + propagators + $\tau_a \tau_a = 3$

$$
\begin{aligned}
i\Delta_\pi(p) &= \frac{i}{p^2 - M^2 + i0^+} \\
iS_N(p) &= i\,\frac{\not{p} + m - i0^+}{p^2 - m^2 + i0^+}
\end{aligned}
$$

$\Rightarrow$ **contribution of the first diagram in dim. reg.**

$$-i\Sigma^{\text{loop}}(\not p) =$$

$$-i\frac{3\,\overset{\circ}{g}{}^2_{A0}}{4F_0^2}\, i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{\not k(\not p - m - \not k)\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

Simplify numerator using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

$$-(\not p + m)\underbrace{k^2}_{k^2 - M^2 + M^2} + (p^2 - m^2)\not k - \left[(p-k)^2 - m^2\right]\not k$$

Intermediate result

$$\Sigma^{\text{loop}}(\not p) = \frac{3\,\overset{\circ}{g}{}^2_{A0}}{4F_0^2}\left\{-(\not p + m)\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p-k)^2 - m^2 + i0^+]}\right.$$

$$-(\not p + m)M^2\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

$$+(p^2 - m^2)\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

$$\left.-\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{\not k}{[k^2 - M^2 + i0^+]}\right\}$$

Last term vanishes (integrand odd)

Convention

$$I_{N\cdots\pi\cdots}(p_1, \cdots, q_1, \cdots)$$

$$= \mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k+p_1)^2 - m^2 + i0^+]\cdots[(k+q_1)^2 - M^2 + i0^+]\cdots}$$

To determine the vector integral use the ansatz

$$\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} = p_\mu C$$

Multiply by $p^\mu \Rightarrow$

$$C = \frac{1}{2p^2} \left[I_N - I_\pi + (p^2 - m^2 + M^2) I_{N\pi}(-p, 0) \right]$$

Using the above convention the loop contribution to the nucleon self energy reads

$$\begin{aligned}\Sigma^{\text{loop}}(\not{p}) \;=\; & -\frac{3 \overset{\circ}{g}{}_{A0}^{2}}{4F_0^2} \{ (\not{p} + m) I_N + (\not{p} + m) M^2 I_{N\pi}(-p, 0) \\ & -(p^2 - m^2) \frac{\not{p}}{2p^2} \left[I_N - I_\pi + (p^2 - m^2 + M^2) I_{N\pi}(-p, 0) \right] \}\end{aligned}$$

The explicit expressions for the integrals are given by

$$
\begin{aligned}
I_\pi &= \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] \\
I_N &= \frac{m^2}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right)\right] \\
I_{N\pi}(p,0) &= \frac{1}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right) - 1\right. \\
&\quad \left. + \frac{p^2 - m^2 - M^2}{p^2}\ln\left(\frac{M}{m}\right) + \frac{2mM}{p^2}F(\Omega)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
R &= \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1] \\
\Omega &= \frac{p^2 - m^2 - M^2}{2mM}
\end{aligned}
$$

and

$$
F(\Omega) = \begin{cases} \sqrt{\Omega^2 - 1}\ln\left(-\Omega - \sqrt{\Omega^2 - 1}\right), & \Omega \leq -1, \\ \sqrt{1 - \Omega^2}\ \arccos(-\Omega), & -1 \leq \Omega \leq 1, \\ \sqrt{\Omega^2 - 1}\ln\left(\Omega + \sqrt{\Omega^2 - 1}\right) - i\pi\sqrt{\Omega^2 - 1}, & 1 \leq \Omega\,. \end{cases}
$$

- Σ^{loop} contains divergences as $n \to 4$ (the terms proportional to R) $\Rightarrow$ needs to be renormalized

- For convenience: $\mu = m$

- $\widetilde{\text{MS}}$ renormalization:
  - drop terms proportional to R
  - replace all bare coupling constants ($c_1, \overset{\circ}{g}_{A0}, F_0$) with the renormalized ones, now indicated by a subscript r

⇒ $\widetilde{\mathrm{MS}}$ renormalized self energy contribution

$$\Sigma_r^{\mathrm{loop}}(\not p) = -\frac{3\,\overset{\circ}{g}{}_{Ar}^2}{4F_r^2}\Big\{(\not p + m)M^2 I_{N\pi}^r(-p,0) \\ -(p^2-m^2)\frac{\not p}{2p^2}\Big[(p^2-m^2+M^2)I_{N\pi}^r(-p) - I_\pi^r\Big]\Big\}$$

Using

$$I_{N\pi}^r(-p,0) = -\frac{1}{16\pi^2} + \cdots$$

⇒ contribution of $\mathcal{O}(q^2)$

Solve for the nucleon mass

$$\begin{aligned} m_N &= m + \Sigma_r^{\mathrm{contact}}(m_N) + \Sigma_r^{\mathrm{loop}}(m_N) \\ &= m - 4c_{1r}M^2 + \Sigma_r^{\mathrm{loop}}(m_N) \end{aligned}$$

- $m_N - m = \mathcal{O}(q^2)$

- We need $\Sigma_r^{\text{loop}}(m_N)$ to $\mathcal{O}(q^3)$

- Expansion of $I^r_{N\pi}$

$$\arccos(-\Omega) = \frac{\pi}{2} + \cdots$$

$$I^r_{N\pi} = \frac{1}{16\pi^2}\left(-1 + \frac{\pi M}{m} + \cdots\right)$$

- This yields

$$m_N = m - 4c_{1r}M^2 + \boxed{\frac{3\overset{\circ}{g}{}^2_{Ar} M^2}{32\pi^2 F_r^2} m} - \frac{3\,\overset{\circ}{g}{}^2_{Ar}\, M^3}{32\pi^2 F_r^2}$$

- Power counting problem

Solution

Term violating the power counting is analytic in small quantities and can thus be absorbed in counter terms

Rewrite

$$c_{1r} = c_1 + \delta c_1, \quad \delta c_1 = \frac{3mg_A^2}{128\pi^2 F^2} + \cdots$$

Final result for the nucleon mass at order $\mathcal{O}(q^3)$

$$m_N = m - 4c_1 M^2 - \frac{3 \overset{\circ}{g}{}_A^2 \, M^3}{32\pi^2 F^2}$$

Infrared regularization reformulated[36]

Basic idea

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}$$

$$\begin{aligned} a &= (k-p)^2 - m^2 + i0^+ \\ b &= k^2 - M^2 + i0^+ \end{aligned}$$

$$H = \int_0^1 dx \cdots = \int_0^\infty dx \cdots - \int_1^\infty dx \cdots \equiv I + R$$

In R expand the integrand in small momenta and masses and interchange summation and integration[37]

$\Rightarrow$ integrals over x of the type

$$I_i = -\int_1^\infty dx x^{n+i}, \quad i \text{ integer number}$$

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[36] M. R. Schindler, J. Gegelia, and S. Scherer, Phys. Lett. B 586, 258 (2004)

[37] T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999)

I_i are calculated by analytic continuation from the domain of n in which they converge, i.e.

$$I_i = - \left. \frac{x^{n+i+1}}{n+i+1} \right|_1^\infty = \frac{1}{n+i+1}$$

EOMS:

- Expand integrand in small momenta and masses
- Interchange summation and integration

$\Rightarrow$ exactly the same expansion as for the IR regular part of the IR regularization with the only difference that instead of the integrals I_i we now have

$$J_i = \int_0^1 dx x^{n+i}$$

Calculating these integrals by analytical continuation from the domain of n in which they converge, we obtain:

$$J_i = \left.\frac{x^{n+i+1}}{n+i+1}\right|_0^1 = \frac{1}{n+i+1}$$

IR and EOMS renormalization of two-loop diagrams[38]

For simplicity: toy model Lagrangian (no spin or chiral structure):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\pi\partial^\mu\pi - M^2\pi^2) + \frac{1}{2}(\partial_\mu\Psi\partial^\mu\Psi - m^2\Psi^2) - \frac{g}{4}\pi^2\Psi^2 + \mathcal{L}_1$$

Power counting:

- Loop integration in n dimensions $\sim \mathcal{O}(q^n)$
- $\pi\Psi$ vertex $\sim \mathcal{O}(q^0)$

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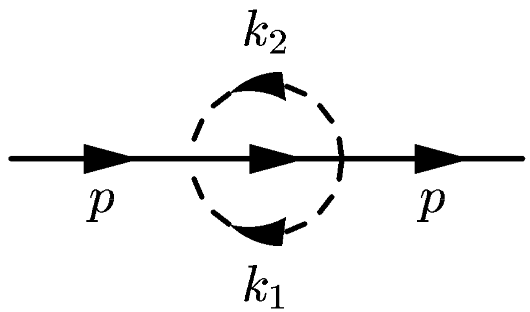
[38] M. R. Schindler, J. Gegelia, and S. Scherer, Nucl. Phys. B682, 367 (2004)

- π **propagator** $\sim \mathcal{O}(q^{-2})$

- Ψ **propagator** $\sim \mathcal{O}(q^{-1})$

Example:



(1)

$$D = 2 \cdot n - 2 \cdot 2 - 1 = 2n - 5$$

Dimensional counting analysis:[39]

$$\Sigma_\Psi = F(p^2, m^2, M^2, n) + M^{n-2} G(p^2, m^2, M^2, n) + M^{2n-4} H(p^2, m^2, M^2, n)$$

F, G, H are analytic in M^2

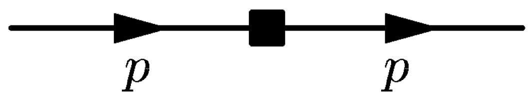
$$F(p^2, m^2, M^2, n) = -\frac{g^2}{2(2\pi)^{2n}} \left[f^{(1)}(p^2, m^2, M^2, n) + f^{(2)}(p^2, m^2, M^2, n) \right]$$

[39] J. Gegelia, G. S. Japaridze, K. S. Turashvili, Theor. Math. Phys. 101, 1313 (1994)

$$
\begin{aligned}
f^{(1)} &= \sum_{i,j=0}^{\infty} (M^2)^{i+j} \sum_{l=0}^{\infty} (p^2 - m^2)^l f^{(1)}_{ij,l}(m^2, n) \\
f^{(2)} &= \sum_{i,j=0}^{\infty} (M^2)^{i+j} (p^2 - m^2)^{2n-5-2i-2j} \sum_{l=0}^{\infty} (p^2 - m^2)^l f^{(2)}_{ij,l}(m^2, n)
\end{aligned}
$$

- **Nonanalytic part in $p^2 - m^2$ satisfies power counting**

- **Analytic part violates power counting, but can be absorbed in counterterms $\Rightarrow$ diagram (2)**



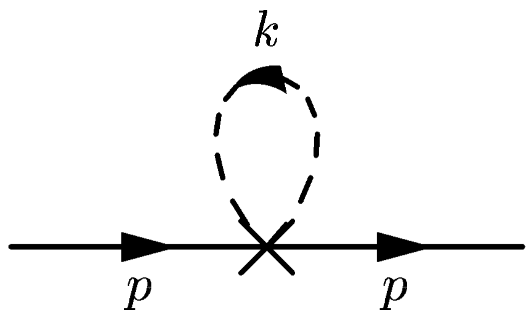
(2)

$$M^{n-2}G(p^2, m^2, M^2, n) =$$

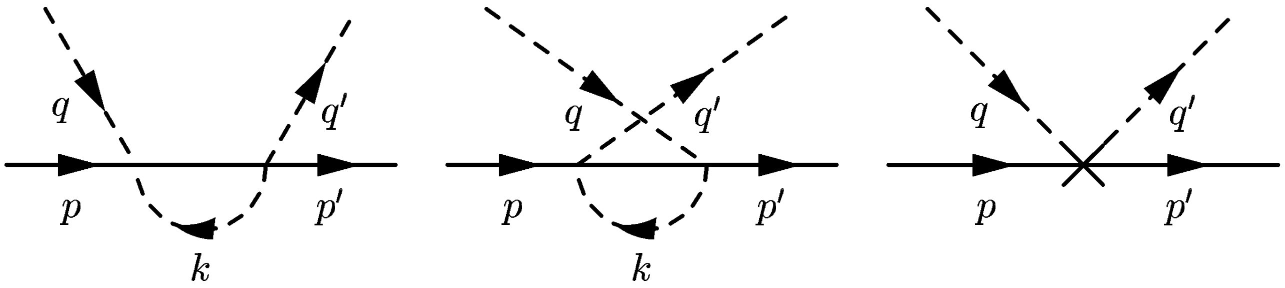
$$-g^2 \boxed{M^{n-2}} \frac{m^{n-4}\Gamma(2-n/2)}{(4\pi)^{n/2}(n-3)} \left[1 - \frac{1}{2m^2}(p^2 - m^2)\right] \frac{\Gamma(1-n/2)}{(4\pi)^{n/2}} + \cdots$$

Problem (at first sight): Part nonanalytic in M^2 that violates power counting

But: Counterterms from renormalization of one-loop subdiagrams generate contributions which exactly cancel the part in $M^{n-2}G(p^2, m^2, M^2, n)$ that violates power counting $\Rightarrow$ diagram (3)



(3)



(a) (b) (c)

$$M^{2n-4}H(p^2, m^2, M^2, n)$$

Only terms that satisfy power counting

Overall: All terms violating the power counting can be absorbed in counterterms or are canceled by contributions stemming from renormalization of one-loop subdiagrams

The road is open for consistent two-loop calculations

Applications

Mass of the nucleon at $\mathcal{O}(q^3)$

- GSS ($\widetilde{\mathrm{MS}}$) [40]

$$m_N = m - 4c_1^r M^2 + \frac{3g_{Ar}^2 M^2}{32\pi^2 F_r^2} m - \frac{3g_{Ar}^2 M^3}{32\pi F_r^2}$$

$$c_1^0 = c_1^r - \frac{3\overset{\circ}{g}{}_A^2}{128\pi^2} R$$

- EOMS [41]

$$c_1^r = c_1 + \frac{3mg_A^2}{128\pi^2 F^2} + \cdots$$

$$m_N = m - 4c_1 M^2 - \frac{3g_A^2 M^3}{32\pi F^2} + \mathcal{O}(M^4)$$

[40] J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988)

[41] T. Fuchs, J. Gegelia, G. Japaridze, S. Scherer, Phys. Rev. D 68, 056005 (2003)

Mass of the nucleon at $\mathcal{O}(q^4)$[42]

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\left(\frac{M}{m}\right) + k_4 M^4 + O(M^5)$$

$$k_1 = -4c_1, \quad k_2 = -\frac{3\overset{\circ}{g}_A^{\,2}}{32\pi F^2},$$

$$k_3 = \frac{3}{32\pi^2 F^2}\left(8c_1 - c_2 - 4c_3 - \frac{\overset{\circ}{g}_A^{\,2}}{m}\right),$$

$$k_4 = \frac{3\overset{\circ}{g}_A^{\,2}}{32\pi^2 F^2 m}(1 + 4c_1 m) + \frac{3}{128\pi^2 F^2}c_2 - 16e_{38} - 2e_{115} - 2e_{116}.$$

$$m = [938.3 - 74.8 + 15.3 + 4.7 + 1.6 - 2.3]\,\text{MeV} = 882.8\,\text{MeV}$$

$$\boxed{\Delta m = 55.5\,\text{MeV}}$$

Remark: $m = m_N(m_u = 0, m_d = 0, m_s)$

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[42] T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)

σ term[43]

Definition of the so-called sigma commutator

$$\sigma^{ab}(x) \equiv [Q_A^a(x_0), [Q_A^b(x_0), \mathcal{H}_{\mathrm{sb}}(x)]], \qquad a, b = 1, 2, 3$$

where

$$\mathcal{H}_{\mathrm{sb}} = \bar{q} M q = m_q(\bar{u}u + \bar{d}d)$$

Measure of explicit symmetry breaking

$$\sigma \equiv \frac{1}{2m_N} \langle p | \sigma^{11}(0) | p \rangle$$

[43] T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)

$$\sigma = \sigma_1 M^2 + \sigma_2 M^3 + \sigma_3 M^4 \ln\left(\frac{M}{m}\right) + \sigma_4 M^4 + O(M^5)$$

$$\sigma_1 = -4c_1$$

$$\sigma_2 = -\frac{9\overset{\circ}{g}_A{}^2}{64\pi F^2}$$

$$\sigma_3 = \frac{3}{16\pi^2 F^2}\left(8c_1 - c_2 - 4c_3 - \frac{\overset{\circ}{g}_A{}^2}{m}\right)$$

$$\sigma_4 = \frac{3}{8\pi^2 F^2}\left[\frac{3\overset{\circ}{g}_A{}^2}{8m} + c_1(1 + 2\overset{\circ}{g}_A{}^2) - \frac{c_3}{2}\right] + \alpha$$

$$\sigma = 45\,\text{MeV} = (74.8 - 22.9 - 9.4 - 2.0 + 4.5)\,\text{MeV}$$

Hellmann-Feynman theorem

Consider a Hermitian operator $H(\lambda)$ with

$$
\begin{aligned}
H(\lambda)|\alpha(\lambda)\rangle &= E(\lambda)|\alpha(\lambda)\rangle, \quad (1)\\
\langle\alpha(\lambda)|\alpha(\lambda)\rangle &= 1. \quad (2)
\end{aligned}
$$

Then

$$
\frac{\partial E(\lambda)}{\partial\lambda} = \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle \quad (3)
$$

Because

$$
\begin{aligned}
\frac{\partial E(\lambda)}{\partial\lambda} &\overset{(1),(2)}{=} \frac{\partial}{\partial\lambda}\langle\alpha(\lambda)|H(\lambda)|\alpha(\lambda)\rangle\\
&= \frac{\partial\langle\alpha(\lambda)|}{\partial\lambda}H(\lambda)|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|H(\lambda)\frac{\partial|\alpha(\lambda)\rangle}{\partial\lambda}\\
&\overset{(1)}{=} E(\lambda)\frac{\partial}{\partial\lambda}\langle\alpha(\lambda)|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle\\
&\overset{(2)}{=} \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle
\end{aligned}
$$

Here: Multiply (3) with λ and identify

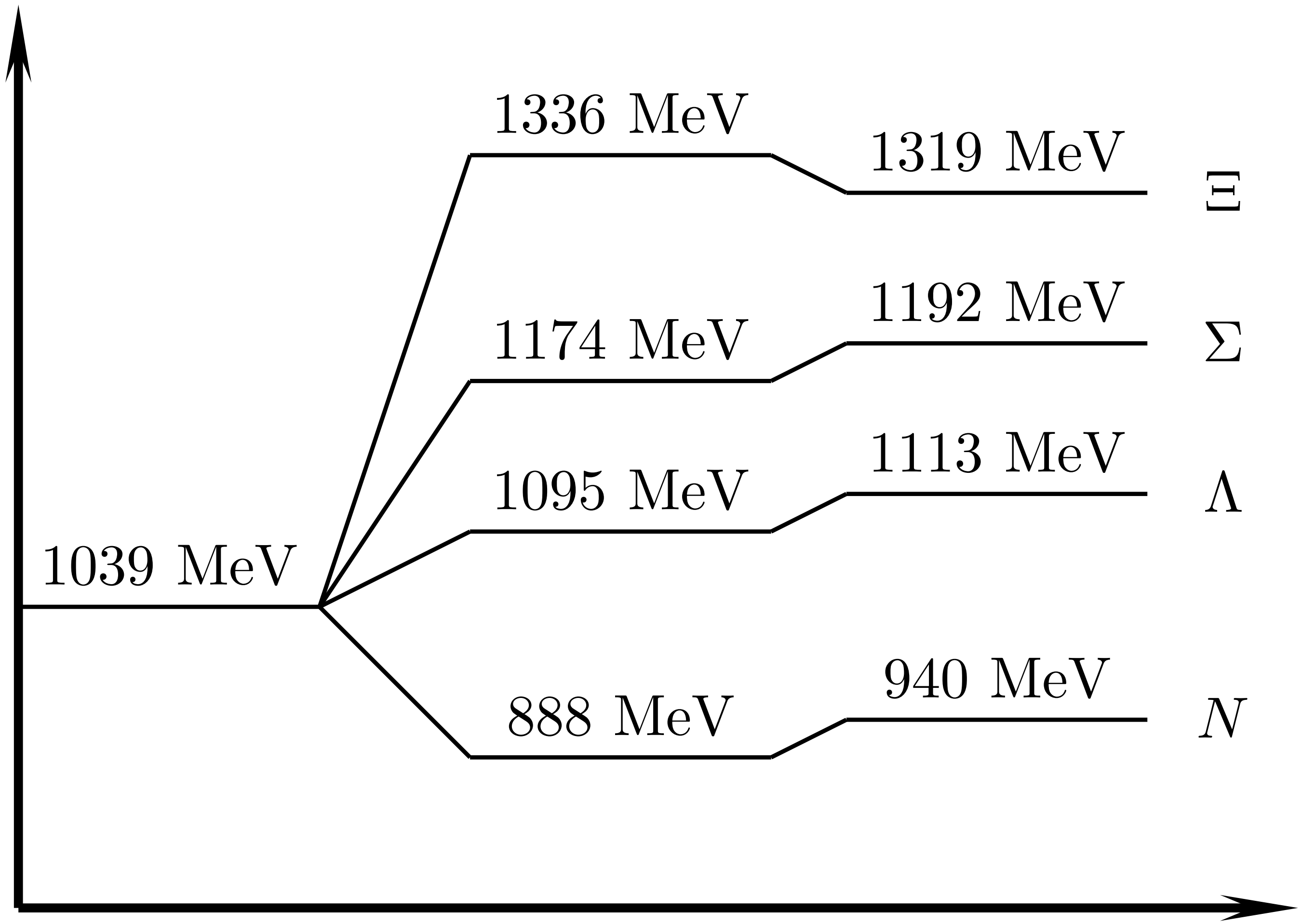
$$\begin{aligned} \lambda &\rightarrow \hat{m}, \\ |\alpha(\lambda)\rangle &\rightarrow |N(\hat{m})\rangle, \\ E(\lambda) &\rightarrow m_N(\hat{m}), \\ \frac{\partial H}{\partial \lambda} &\rightarrow \frac{\partial \mathcal{H}_{\rm QCD}}{\partial \hat{m}} = \bar{u}u + \bar{d}d. \end{aligned}$$

Note that $M^2 = 2B\hat{m}$ and thus

$$\sigma = M^2 \frac{\partial m_N}{\partial M^2}.$$

Exercise: Using the expressions for k_i and σ_i verify the Hellmann-Feynman theorem applied to the sigma term and the nucleon mass.

Masses of the baryon octet at $\mathcal{O}(q^3)$[44]



$SU(3)_L \times SU(3)_R$ $SU(2)_L \times SU(2)_R$ $SU(2)_V$

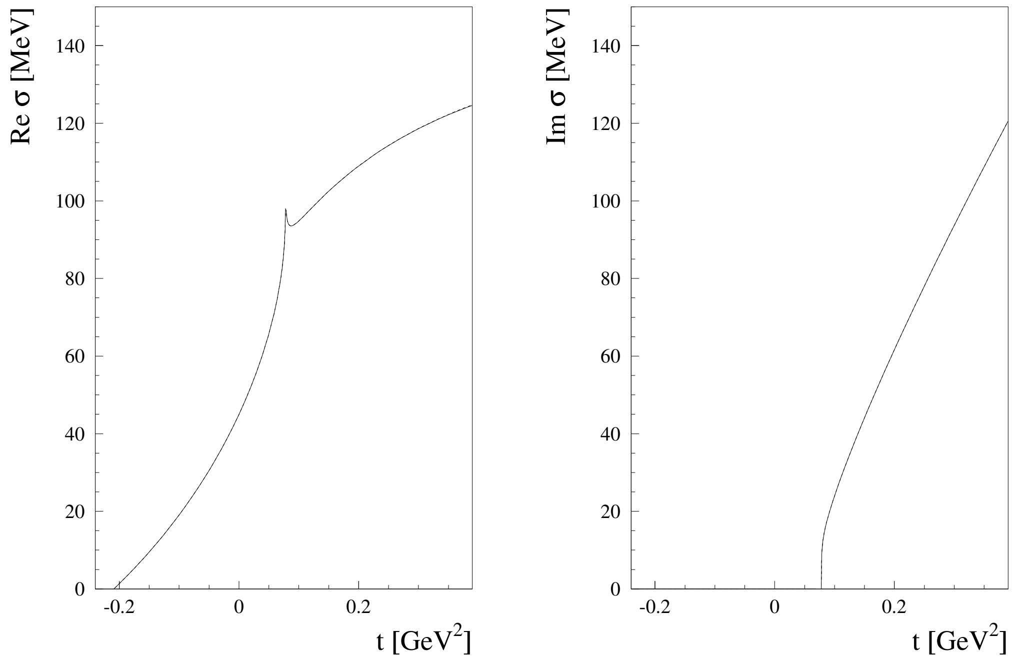
[44] B. C. Lehnhart, J. Gegelia, S. Scherer, J. Phys. G 31, 1 (2005)

Scalar form factor[45]

Definition of the scalar form factor

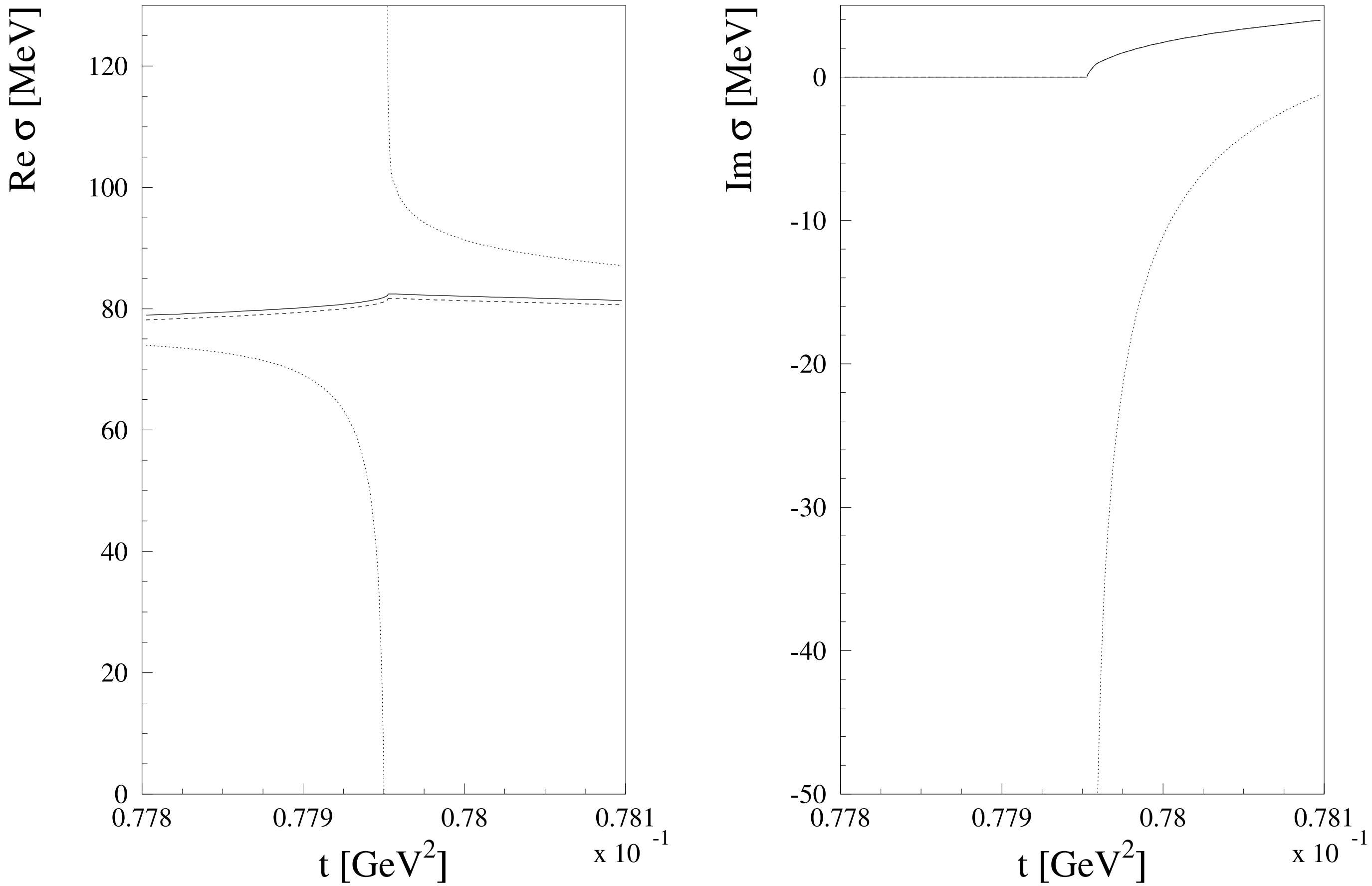
$$\langle N(p')|\hat{m}[\bar{u}(0)u(0)+\bar{d}(0)d(0)]|N(p)\rangle = \bar{u}(p')u(p)\sigma(t)$$

Form factor $\sigma(t)$ at $\mathcal{O}(q^4)$



[45] T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)

Form factor $\sigma(t)$ at $\mathcal{O}(q^3)$



Electromagnetic form factors

Electromagnetic current operator

$$J^\mu(x) = \frac{2}{3}\,\bar{u}(x)\gamma^\mu u(x) - \frac{1}{3}\,\bar{d}(x)\gamma^\mu d(x) + \cdots = \bar{q}\,Q\,q + \cdots$$

Definition of Dirac and Pauli form factors

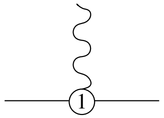
$$\langle N(p')|J^\mu(0)|N(p)\rangle = \bar{u}(p')\left[\gamma^\mu F_1(Q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m_N}\,F_2(Q^2)\right]u(p)$$

$$q^\mu = p'^\mu - p^\mu, \qquad Q^2 = -q^2$$

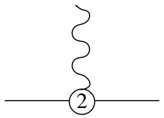
Sachs form factors

$$G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4m_N^2}F_2^N(Q^2)$$

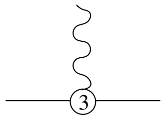
$$G_M^N(Q^2) = F_1^N(Q^2) + F_2^N(Q^2)$$



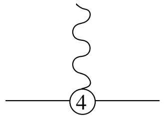
(1)



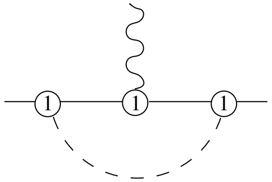
(2)



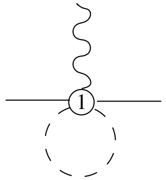
(3)



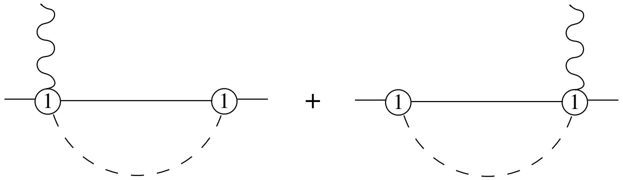
(4)



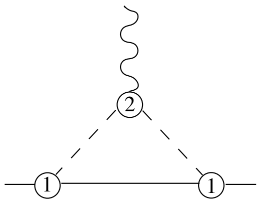
(5)



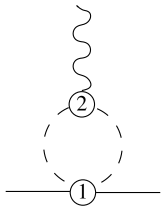
(6)



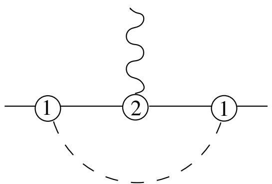
(7)



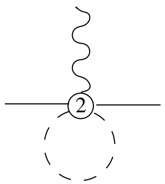
(8)



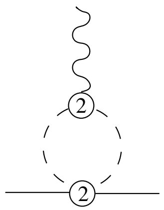
(9)



(10)

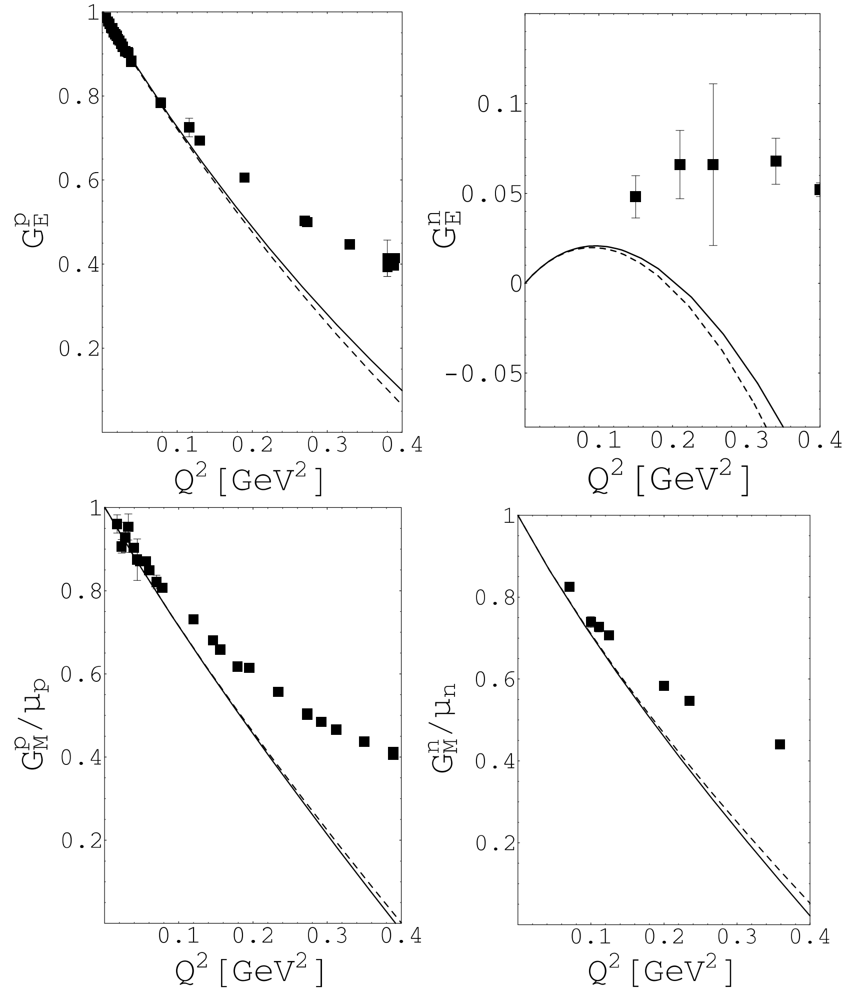


(11)



(12)

Electromagnetic form factors [46]



[46] B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001); T. Fuchs, J. Gegelia, S. S., J. Phys. G 30, 1407 (2004); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)

Vector meson dominance model $\longrightarrow$ Important contributions to the electromagnetic form factors [47]

In standard ChPT: Vector meson contributions in low-energy constants

$$\frac{1}{q^2 - M_V^2} = -\frac{1}{M_V^2}\left[1 + \frac{q^2}{M_V^2} + \left(\frac{q^2}{M_V^2}\right)^2 + \mathcal{O}(q^6)\right]$$

Inclusion of vector mesons $\Rightarrow$ re-summation of higher-order contributions

---

[47] B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001)

Additional Lagrangians[48]

$$\mathcal{L}_{\pi V}^{(3)} = -f_\rho \mathrm{Tr}(\rho^{\mu\nu} f_{\mu\nu}^{+}) - f_\omega \omega^{\mu\nu} f_{\mu\nu}^{(s)} - f_\phi \phi^{\mu\nu} f_{\mu\nu}^{(s)} + \cdots$$

$$V_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu, \quad V = \rho, \omega, \phi, \quad \nabla_\mu V_\nu = \partial_\mu V_\nu + [\Gamma_\mu, V_\nu]$$

$$\mathcal{L}_{NV}^{(0)} = \frac{1}{2} \sum_{V=\rho,\omega,\phi} g_V \bar{\Psi} \gamma^\mu V_\mu \Psi$$

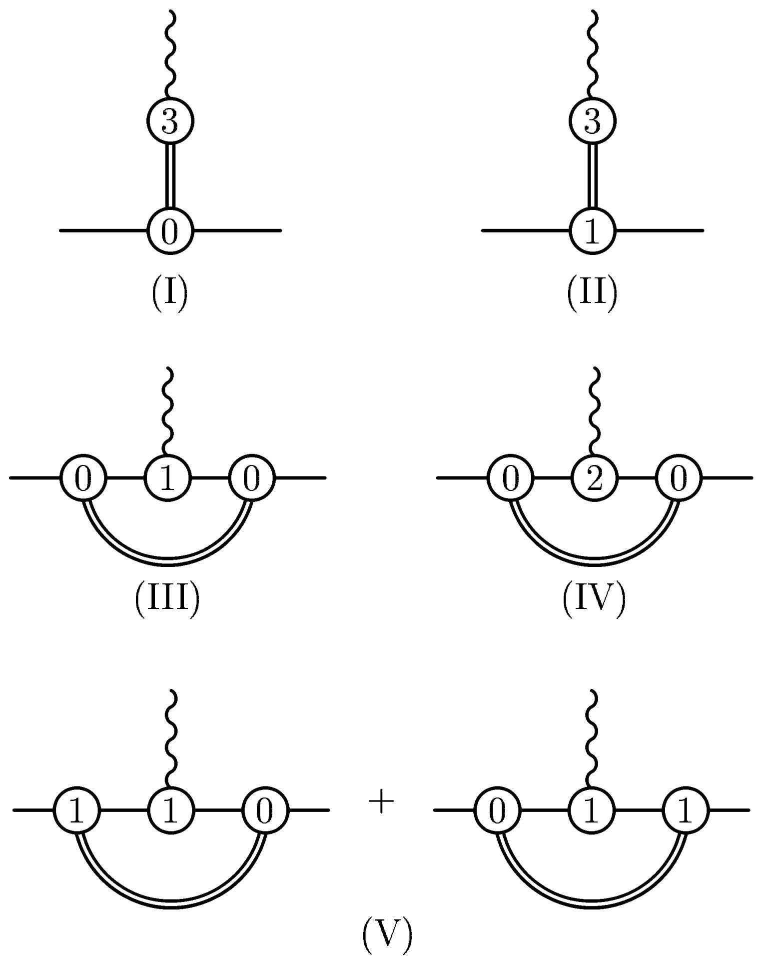
$$\mathcal{L}_{NV}^{(1)} = \frac{1}{4} \sum_{V=\rho,\omega,\phi} G_V \bar{\Psi} \sigma^{\mu\nu} V_{\mu\nu} \Psi$$

Additional rules:

- Vector meson propagator $\sim \mathcal{O}(q^0)$
- Vertex from $\mathcal{L}_V^{(i)} \sim \mathcal{O}(q^i)$

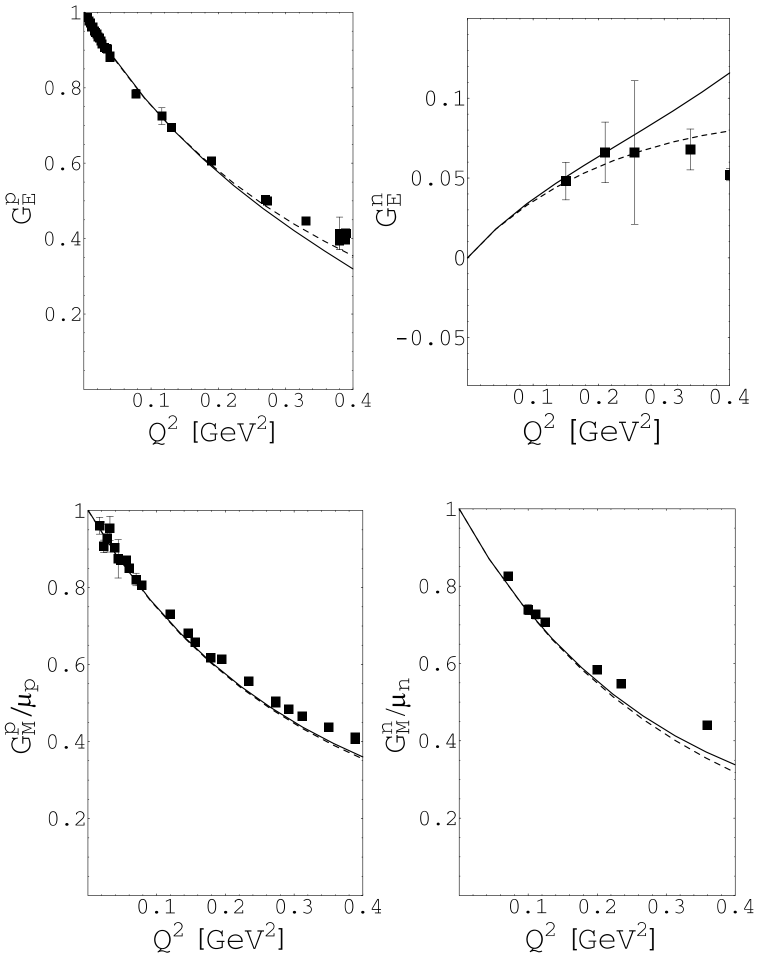
---

[48] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, and E. de Rafael, Phys. Lett. B 223, 425 (1989)



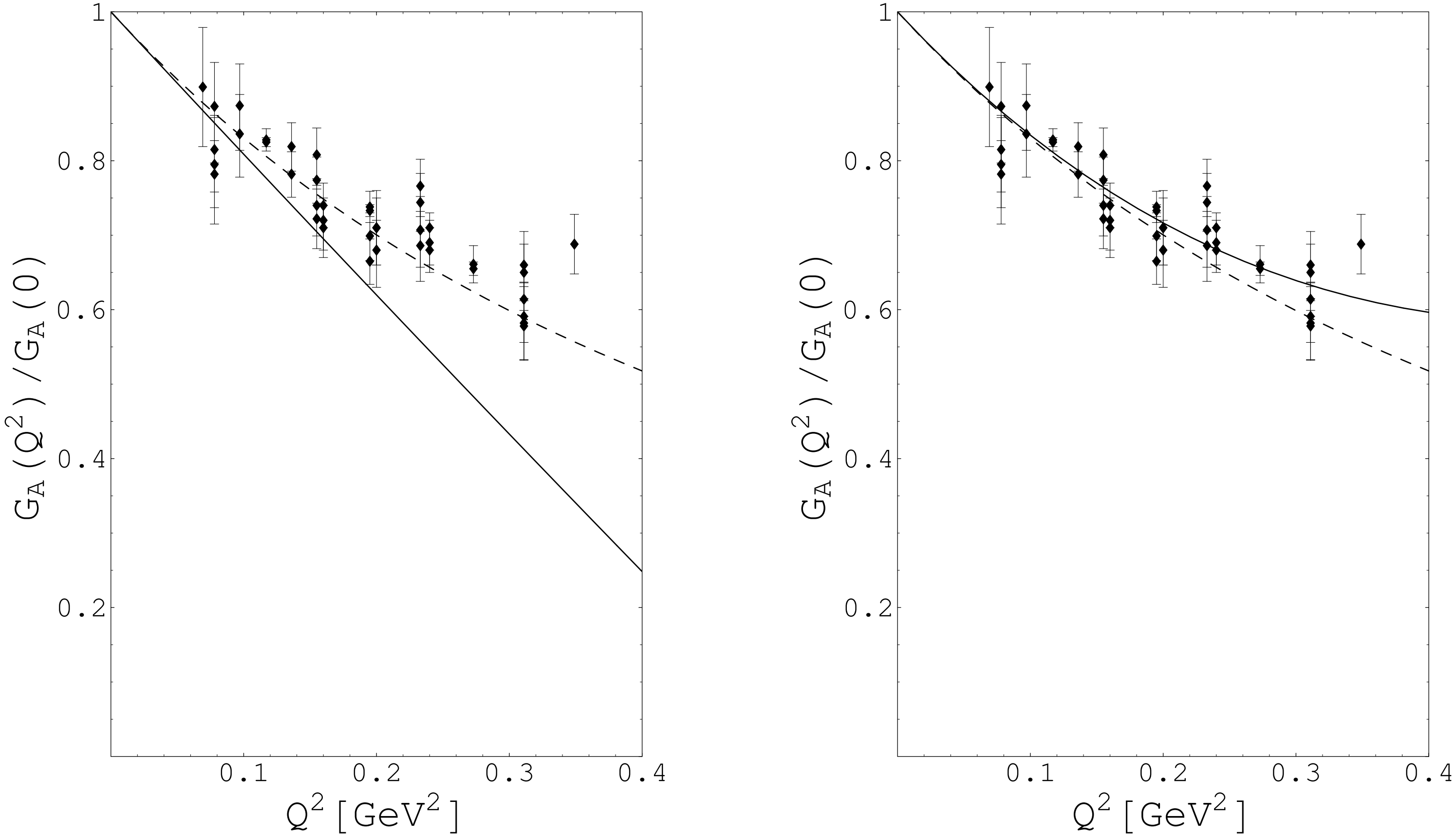
Feynman diagrams involving vector mesons contributing to the electromagnetic form factors up to and including $\mathcal{O}(q^4)$

E.m. form factors including vector mesons at $\mathcal{O}(q^4)$ [49]



[49] M. R. Schindler, J. Gegelia, and S. S., Eur. Phys. J. A 26, 1 (2005); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)

Axial form factor G_A including the a_1 meson at $\mathcal{O}(q^4)$ [50]



[50] M. R. Schindler and S. S., arXiv:hep-ph/0608325, M. R. Schindler, T. Fuchs, and S. S., in preparation

Universality of the ρ meson coupling in EFT [51]

Chirally invariant effective Lagrangian including vector mesons

$$\mathcal{L} = \mathcal{L}_{\text{basic}} + \mathcal{L}_{\text{ct}} + \tilde{\mathcal{L}}_1$$

$$\begin{aligned}
\mathcal{L}_{\text{basic}} &= \text{free Lagrangians} \\
&+ \boxed{g_{\rho\pi\pi}}\ \varepsilon^{abc}\pi^a\partial_\mu\pi^b\rho^{c\mu} \\
&- \boxed{g}\ \varepsilon^{abc}\partial_\mu\rho^a_\nu\rho^{b\mu}\rho^{c\nu} \\
&- \frac{1}{4}\boxed{g^2}\ \varepsilon^{abc}\varepsilon^{ade}\rho^b_\mu\rho^c_\nu\rho^{d\mu}\rho^{e\nu} \\
&+ \boxed{g_{\rho NN}}\ \bar{\Psi}\gamma^\mu\frac{\tau^a}{2}\Psi\rho^a_\mu
\end{aligned}$$

---

[51] D. Djukanovic, M. R. Schindler, J. Gegelia, G. Japaridze, S. S., Phys. Rev. Lett. 93, 122002 (2004)

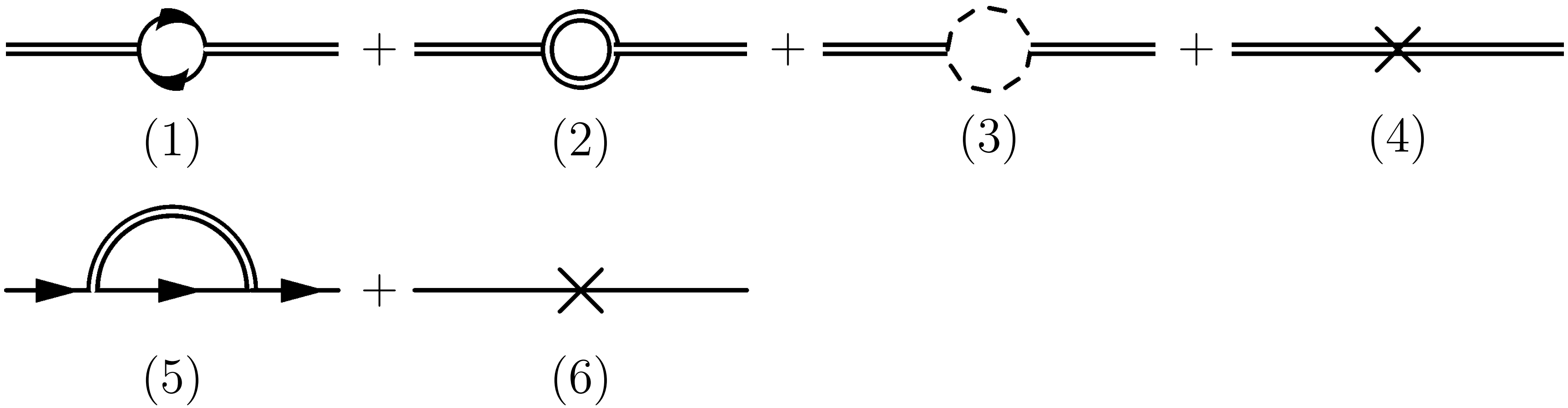
Chiral symmetry $\Rightarrow$ $\boxed{g_{\rho NN} = g}$ [52]

Counterterm Lagrangian

$$\begin{aligned}
\mathcal{L}_{\rm ct} &= -\frac{\delta Z_\rho}{4} A^a_{\mu\nu} A^{a\mu\nu} \\
&+ \left[\delta g + g\left(\frac{\delta Z_\rho}{2} + \delta Z_\Psi\right)\right] \bar{\Psi}\gamma^\mu \frac{\tau^a}{2}\Psi \rho^a_\mu \\
&+ \left[\delta g_{\rho\pi\pi} + g_{\rho\pi\pi}\left(\frac{\delta Z_\rho}{2} + \delta Z_\pi\right)\right] \varepsilon^{abc}\pi^a \partial_\mu \pi^b \rho^{c\mu} \\
&- \left(\delta g + \frac{3}{2} g \delta Z_\rho\right) \varepsilon^{abc} \partial_\mu \rho^a_\nu \rho^{b\mu} \rho^{c\nu}
\end{aligned}$$

[52] S. Weinberg, Phys. Rev. 166, 1568 (1968)

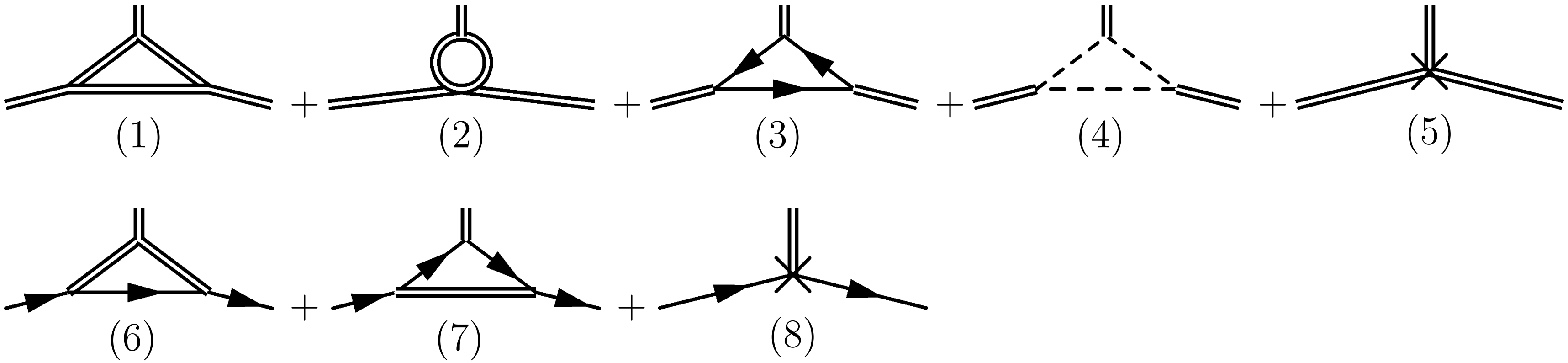
Evaluate self energy diagrams



$$
\begin{aligned}
\frac{1}{2}\delta Z_\rho &= z_1\, g^2 + z_2\, g^2_{\rho\pi\pi} \qquad (*) \\
\frac{1}{2}\delta Z_\Psi &= w_1\, g^2 \qquad (**)
\end{aligned}
$$

EFT approach generates consistency conditions among the renormalized parameters of the Lagrangian

Evaluate renormalized vertex diagrams



Consider $\rho\rho\rho$ vertex function $\Rightarrow$

Demand: Divergent loop contribution and counterterm contribution cancel

$$\Gamma_1\, g^3 + \Gamma_2\, g^3_{\rho\pi\pi} - \delta g - \frac{3}{2} g \delta Z_\rho = 0$$

Insert $(*)$ $\Rightarrow$

$$\boxed{\delta g} = \Gamma_1\, g^3 + \Gamma_2\, g^3_{\rho\pi\pi} - 3 z_1\, g^3 - 3 z_2\, g g^2_{\rho\pi\pi} \equiv F(g, g_{\rho\pi\pi})$$

Consider $\rho\bar{\Psi}\Psi$ vertex function $\Rightarrow$

Demand: Divergent loop contribution and counterterm contribution cancel

$$D_1\, g^3 + \delta g + g\frac{1}{2}\delta Z_\rho + g\delta Z_\Psi = 0$$

Insert $(**)$ $\Rightarrow$

$$\boxed{\delta g} = -D_1\, g^3 - (z_1 + 2w_1)g^3 - z_2\, g g^2_{\rho\pi\pi} \equiv G(g, g_{\rho\pi\pi})$$

$\Rightarrow$ consistency condition: $F = G$

Nontrivial solution

$$\boxed{g_{\rho\pi\pi} = g} \qquad \text{universality}$$

Next step: Chiral symmetry [53]

$$g_{\rho\pi\pi} = \frac{M_\rho^2}{2gF^2}$$

Combine with universality $\Rightarrow$

$$g^2 = \frac{M_\rho^2}{2F^2}$$

KSRF relation [54]

---

[53] S. Weinberg, Phys. Rev. 166, 1568 (1968);
G. Ecker et al., Phys. Lett. B 223, 425 (1989)

[54] K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966);
Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966)

- **Kawarabayashi & Suzuki**

PCAC + current algebra $\Rightarrow$

$$g g_{\rho\pi\pi} = \frac{M_\rho^2}{2F^2}$$

NB: Chiral Ward identity

$$\begin{aligned} T[\partial^x_\mu A^\mu_i(x)\partial^y_\nu A^\nu_j(y)] &= \partial^x_\mu \partial^y_\nu T[A^\mu_i(x)A^\nu_j(y)] \\ &+\partial^x_\mu \delta^4(y-x) i\varepsilon_{ijk} V^\mu_k(y) \\ &+\delta^4(x-y) i\hat{m}\delta_{ij}\bar{q}(x)q(x) \end{aligned}$$

universality as an extra assumption $\Rightarrow$ KSRF relation

- Riazuddin & Fayyazuddin

PCAC + current algebra (but a different Ward identity similar to the Adler-Gilman relation of pion electroproduction, makes use of the analogue of the Goldberger-Treiman relation) $\Rightarrow$

$$g g_{\rho\pi\pi} = \frac{M_\rho^2}{2F^2}$$

Dynamical assumption: $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$ dominated by the ρ meson pole + CVC hypothesis $\Rightarrow$ universality $\Rightarrow$ KSRF relation

Quantum electrodynamics for vector mesons [55]

Inclusion of the electromagnetic interaction

$$\begin{aligned}\mathcal{L}_{\rm basic} &= \ldots - i\,\boxed{e}\,A_\mu(\rho^{-\mu\nu}\rho^{+}_{\nu} - \rho^{+\mu\nu}\rho^{-\nu}) \\ &+ \frac{1}{2}\boxed{c}\,F_{\mu\nu}\rho^{0\mu\nu} \\ &- i\,\boxed{\kappa}\,F_{\mu\nu}\rho^{+\mu}\rho^{-\nu} + \ldots\end{aligned}$$

consistency condition $\Rightarrow$ $\boxed{\kappa = e}$, $\boxed{c = e/g}$

- Gyromagnetic ratio of the ρ^+: $g = 2$
- $M_{\rho^0} - M_{\rho^\pm} \sim 1\ \mathrm{MeV}$ using KSRF

---

[55] D. Djukanovic, M. R. Schindler, J. Gegelia, S. Scherer, Phys. Rev. Lett. 95, 012001 (2005)

Inclusion of the $\Delta(1232)$ into ChPT [56]

$$\Delta(1232): \quad I(J^P) = \frac{3}{2}(\frac{3}{2}^+)$$

Description in terms of a vector-spinor isovector-isospinor

$$\Psi_{\mu,\alpha;i,m}$$

Too many components $\Rightarrow$ Constraints

Dirac's analysis using the Hamiltonian method:

$$L(q,\dot{q}) \quad \rightarrow \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \rightarrow \quad H(q,p) = p_i \dot{q}_i - L$$

But

$$\Phi_m(q,p) = 0 \quad \text{primary constraints}$$

Introduce constraints in terms of Lagrange multipliers into Hamiltonian

$$H_T = H + u_m \Phi_m$$

---

[56] C. Hacker, N. Wies, J. Gegelia, S. S., Phys. Rev. C 72, 055203 (2005); N. Wies, J. Gegelia, S. S., Phys. Rev. D 73, 094012 (2006)

Consider time evolution (in terms of Poisson brackets)

$\{H_T, \Phi_m\} = 0 \quad \Rightarrow$ new (secondary) constraints

Iterate until all Lagrange multipliers have been solved.

In a consistent theory

initial # of d.o.f $-$ # of constraints $=$ correct # of d.o.f.

$\Rightarrow$ Restrictions on the possible interaction terms

Example [57]

$$
\begin{aligned}
\mathcal{L}_{\pi\Delta} &= -\bar{\Psi}^{\mu}[\frac{g_1}{2} g_{\mu\nu}\gamma^{\alpha}\gamma_5\partial_{\alpha}\phi \\
&\quad + \frac{g_2}{2}(\gamma_{\mu}\partial_{\nu}\phi + \partial_{\mu}\phi\gamma_{\nu})\gamma_5 \\
&\quad + \frac{g_3}{2}\gamma_{\mu}\gamma^{\alpha}\gamma_5\gamma_{\nu}\partial_{\alpha}\phi]\Psi^{\nu}
\end{aligned}
$$

[57] T. R. Hemmert, B. R. Holstein, J. Kambor, J. Phys. G 24, 1831 (1998)

Analysis of constraints [58] [59]

$$\begin{aligned} g_2 &= A g_1\,, \\ g_3 &= -\frac{1}{2}(1+2A+3A^2)g_1 \end{aligned}$$

Applications so far

- **Mass of the nucleon**
- **Pole of the Δ**
- **πN scattering** [60]

---

[58](A parameter of the lowest-order Lagrangian)

[59]N. Wies, J. Gegelia, S. S., Phys. Rev. D 73, 094012 (2006)

[60]N. Wies, thesis, Mainz, 2005

- Magnetic moment of the Δ resonance [61]

$$\mu = \frac{1}{2}\,\mu_\Delta^{(s)} + T_3\,\mu_\Delta^{(v)} = \left[\frac{1}{2}\left(1+\kappa_\Delta^{(s)}\right) + T_3\left(1+\kappa_\Delta^{(v)}\right)\right]\frac{e}{2m_\Delta}$$

Numerical results

$$\begin{aligned}\kappa_\Delta^{(s)} &= d_1 + 0.23 + \mathcal{O}(q^4),\\ \kappa_\Delta^{(v)} &= d_2 - 0.22 + i\,0.37 + \mathcal{O}(q^4)\end{aligned}$$

Compare with nucleon

$$\begin{aligned}\kappa_N^{(s)} &= \bar{c}_7 + \mathcal{O}(q^4),\\ \kappa_N^{(v)} &= \bar{c}_6 - 0.62 + \mathcal{O}(q^4)\end{aligned}$$

---

[61] C. Hacker, N. Wies, J. Gegelia, S. S., Eur. Phys. J. A 28, 5 (2006)

Experiment [62]

$$\mu_{\Delta^{++}} = (3.7 - 7.5)\mu_N,$$
$$\mu_{\Delta^{+}} = (2.7^{+1.0}_{-1.3}(\text{stat.}) \pm 1.5(\text{syst.}) \pm 3(\text{theor.}))\,\mu_{\rm N}$$

SU(6) symmetry:

$$\mu_{\Delta^{++}} = 6\mu_N,$$
$$\mu_{\Delta^{+}} = 3\mu_N$$

[62] PDG, $\pi^+ p \to \pi^+ p\gamma$; M. Kotulla et al., Phys. Rev. Lett. 89, 272001 (2002), $\gamma p \to p\pi^0\gamma'$

Thanks to my collaborators

- Dalibor Djukanovic
- Prof. Dr. Harold W. Fearing (TRIUMF)
- Dr. Thomas Fuchs
- Dr. Jambul Gegelia
- Christian Hacker
- Dr. George Japaridze (Clark Atlanta University)
- Björn C. Lehnhart
- Matthias R. Schindler
- Natalia Wies