Supersymmetry on the Lattice

Alessandra Feo

Universitá di Torino and INFN Torino

Universitá di Pavia
Dipartimento di Fisica Nucleare e Teorica
January 17, 2008

Motivation

Understanding supersymmetric theories is a challenging and fascinating problem

They are expected to be relevant in the next future experiments

Strongly interacting supersymmetric gauge theories are much studied since in many respects they resemble Quantum Chromodynamics

While much is known analytically, the hope is that a discretized formulation of supersymmetric gauge theories would provide information about non-perturbative dynamics and additional information for supersymmetry

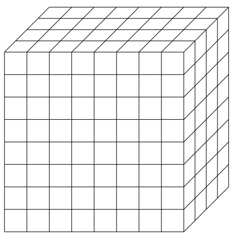
$\Longrightarrow$ lattice formulation

Recent reviews

1. J. Giedt, hep-lat/0701006, (Plenary talk at Lattice 2006), " Advances and applications of lattice supersymmetry".
2. S. Catterall, hep-lat/0509136, (Plenary talk at Lattice 2005), "Dirac-Kahler fermions and exact lattice supersymmetry".
3. A. F., hep-lat/0410012, (Review for MPLA), " Predictions and recent results in SUSY on the lattice'
4. D. B. Kaplan, hep-lat/0309099, (Plenary talk at Lattice 2003), "Recent developments in lattice supersymmetry".
5. A. F., hep-lat/0210015, (Plenary talk at Lattice 2002), "Supersymmetry on the lattice"
6. I. Montvay, hep-lat/0112007, (Review), "Supersymmetric Yang-Mills theory on the lattice"
7. I. Montvay, hep-lat/9709080, (Plenary talk at Lattice 1997), "SUSY on the lattice"

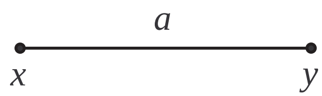
Lattice Gauge theory: General Remarks

Discretization of space-time is achieved introducing an euclidean space-time lattice with spacing a and volume $L^3 \cdot T$. The inverse lattice spacing a^{-1} acts as an UV cutoff.

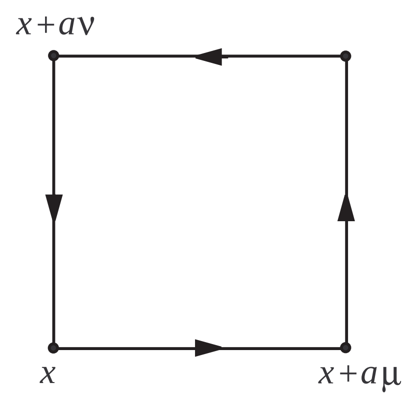


The quark and antiquark fields $\psi(x), \bar{\psi}(x)$ leave in the lattice sites x.

Gauge fields are represented by the link variable $U_\mu(x)$ which are group elements $\in SU(N)$ associated with straight-line path conecting nearest neighbour pairs of lattices sites.



Gauge invariant expressions on the lattice are traces of products of link variables along closed paths. The most elementary one is the *Plaquette variable* 1×1



$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U^\dagger_\mu(x + a\hat{\nu})U^\dagger_\nu(x)$$

that can be used to construct the lattice Yang-Mills action

There is no a unique way to discretize an observable on the lattice and the only request is that have to reduce to the classical value in the continuum limit ($a \to 0$).

Wilson propose the simplest one

$$S_W = \sum_P S_P = \frac{1}{2}\beta \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{2N}\mathrm{Tr}(P_{\mu\nu}(x) + P^\dagger_{\mu\nu}(x))\right)$$

Introducing the gauge field variables by

$$U_\mu(x) \equiv \exp i g_0 a A^b_\mu(x) T^b$$

and using Baker-Campbell-Haussdorf

$$\begin{aligned} P_{\mu\nu} &\simeq e^{i g_o a^2 F_{\mu\nu}(x)} \\ &\simeq 1 + i g_o a F_{\mu\nu}(x) - \frac{g_o^2 a^2}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) \end{aligned}$$

in the limit $a \to 0$

$$S_W = \sum_x \sum_{\mu\nu} \left(a^4 \frac{\beta g_o^2}{2N} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} + O(a^6) \right).$$

So the continuum limit is

$$S_W = \int d^4x \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu},$$

if we define β to be

$$\beta = \frac{2N}{g_0^2}.$$

The expectation value of an observable Ω that depends on the variable U

$$\langle\Omega\rangle = \frac{1}{Z}\int \Pi_{x,\mu}\, dU_\mu(x)\Omega(U)e^{-S_W(U)},$$

and the functional integral Z is determined by requiring $\langle 1\rangle = 1$.

An efficient way to computing $\langle\Omega\rangle$ would consist in generating a sequence of link variable configurations with a probability distribution given by the Boltzmann factor $e^{-S(C)}$, where $S(C)$ is the action related to this configuration.

Monte Carlo method

$$< \Omega_L > (\beta) \cong \frac{1}{n}\sum_{i=1}^{n} \Omega_L(C_i)$$

with $\{C\}_i, \ (i = 1, \cdots, n)$ denote the link configurations generated.

The Monte Carlo method consists in producing a sequence of configurations $U^{(1)} \to U^{(2)} \to U^{(3)} \to \cdots$ with the appropriate probabilities in a statistical way. This is done *by the computer*.

Fermions on the Lattice

Recently, following the rediscovery of the Ginsparg-Wilson relation (1982), it has emerged that chiral theories can be put on the lattice in a consistent way:

- The overlap (Narayanan-Neuberger 1993,1995,1998)
- Domain wall fermions (Kaplan-Shamir 1992, 1993, 1994)
- Perfect action (Hasenfratz-Niedermayer 1994, 1998).

This was believed to be impossible for a long time [Nielsen-Ninomiya, 1981] the no-go theorem.

A naive formulation of fermions on the lattice fails

$$S_F = \frac{1}{2}\sum_x \sum_\mu \bar{\psi}(x)(\gamma_\mu \Delta_\mu + m)\psi(x) + h.c.$$

and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i\sum_\mu \gamma_\mu \mathrm{sin} k_\mu + m}{\sum_\mu \mathrm{sin}^2 k_\mu + m^2}$$

There is a pole for small k_μ representing the physical particle, but additional poles near $k_\mu = \pm\pi$ appears. S_F describes 16 instead of 1 particle. $\rightarrow$ Doubling problem.

Two popular choices introduced in order to deal with this problem:

- Wilson fermions: Get rids of the doubling species but breaks chiral symmetry explicitily by the Wilson term.

- Staggered fermions (Kogut-Susskind): Reduce from 16 to 4 fermions and for massless fermions a remnant chiral symmetry remains.

In the Wilson formulation the bare mass m is hidden in the hopping parameter by the relation $k = \frac{1}{8r+2m_0}$.

Take Wilson's or staggered fermions for the quarks fields $\psi^f_{\alpha c}(x)$, the complete action is $S = S_W + S_F$. And for an observable we write down

$$\langle \Omega \rangle = \frac{1}{Z} \int \Pi_{x,\mu} dU_\mu(x) \int \Pi_x \bar{d}\psi(x) d\psi(x) \Omega e^{-S_W - S_F},$$

After integrating out the quarks fields the expectation value reads

$$\langle \Omega \rangle = \frac{1}{Z} \int \Pi_{x,\mu} dU_\mu(x) \Pi_f \mathrm{det}(D + m_f) \Omega e^{-S_W},$$

where D is the Dirac operator.

Supersymmetry

Such a symmetry makes:

$$Q|boson\rangle = |fermion\rangle \quad Q|fermion\rangle = |boson\rangle$$

the symmetry generator Q (and its hermitian conjugate $Q^\dagger$) carry spin $\frac{1}{2}$

There is essential one possibility for the SUSY algebra:

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$
$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_\alpha, Q^\dagger_\beta\} = 0$$
$$[P_\mu, Q_\alpha] = [P_\mu, Q^\dagger_\alpha] = 0$$

Also,

- Q, $Q^\dagger$ transform as spinors under the Lorentz group
- Q, $Q^\dagger$ commute with gauge symmetry generators

We may have more than one Q: $Q^i, i = 1, \cdots N$ (extended supersymmetry)

Lattice formulation of Super Yang-Mills theory

- The major obstacle in formulating a supersymmetric theory on the lattice arises from the fact that the supersymmetry algebra, which is actually an extension of the Poincaré algebra, is explicitly broken on the lattice [Dondi and Nicolai 1977] .

  In particular the Super Poincaré algebra is given by the anti-commutator of a supercharge Q_α and its conjugate Q_β yields the generator of infinitesimal translations P_μ. Schematically,

  $$\left\{ Q_\alpha, Q^\dagger_\beta \right\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

  On the lattice there are no infinitesimal translations and therefore the supersymmetry algebra must be broken.

Ordinary Poincaré algebra is also broken by the lattice but the hyper-cubic crystal symmetry forbids relevant operators which could spoil the Poincaré symmetry in the continuum limit $\rightarrow$

The Poincaré invariance is achieved automatically in the continuum limit without fine tuning since operators that violate Poincaré invariance are all irrelevant.

However, in the case of the super Poincaré algebra, the lattice crystal group is not enough to guarantee the absence of supersymmetry violating operators.

Failure of the Leibniz rule

On the lattice the Leibniz rule does not hold anymore. [Fujikawa, hep-th/0205095]

$$
\begin{aligned}
&\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x)) = \\
&= \frac{1}{a}(f(x+a) - f(x))g(x) + \frac{1}{a}f(x)(g(x+a) - g(x)) \\
&+ a\frac{1}{a}(f(x+a) - f(x))\frac{1}{a}(g(x+a) - g(x)) \\
&= (\nabla f(x))g(x) + f(x)(\nabla g(x)) + a(\nabla f(x))(\nabla g(x))
\end{aligned}
$$

the breaking of supersymmetry is of order $O(a)$.

- If the supersymmetric theory contains scalar mass terms they break supersymmetry. Since these operators are relevant fine tuning is needed in order to cancel their contributions.

- A naive regularization of fermions results in the doubling problem [Nielsen and Ninomiya, 1981] $\longrightarrow$ wrong number of fermions and violation of the balance between bosons and fermions

Without exact lattice supersymmetry one might hope to construct non-supersymmetric lattice theories with a supersymmetric continuum limit.

This is the case of the Wilson fermion approach for the $4d$ $N = 1$ supersymmetric Yang-Mills theory where the only operator that violates supersymmetry is a fermion mass term.

By tunning the fermion mass to the supersymmetric limit one recovers supersymmetry in the continuum limit [Curci and Veneziano, 1987; I. Montvay, hep-lat/0112007, hep-lat/9510042; Feo, hep-lat/0305020]

Alternatively, using domain wall fermions [Kaplan and Schmaltz hep-lat/0002030] or overlap fermions [Huet, Narayanan, Neuberger, hep-th/9602176] this fine tunning is not required.

In the case of theories with extended supersymmetries the fine tuning of coupling constant is neither feasible nor theoretically practical.

Due to difficulties in realizing exact supersymmetry on the lattice, all that remain us it to realize part of the supercharges as an exact symmetry on the lattice:

This exact lattice supersymmetry is expected to play a key role to restore continuum supersymmetry without (or with less) fine tuning of the parameters of the action.

Two ways to study SUSY on the lattice

- Construct non-SUSY lattice theories with a SUSY continuum limit.
  - N=1 Super-Yang Mills
    (with Wilson fermions – Curci and Veneziano formulation)

- Keep some exact algebra of SUSY on the lattice in order to recover the continuum limit with no (or less) fine tuning of parameters of the action.
  - N=1 Super-Yang Mills
    (with Domain Wall-fermions – Kaplan formulation) → the zero gluino mass term is achieved without find tuning

Outline

- Lattice formulation of $N = 1$ SYM theory
  - Curci-Veneziano: Wilson fermions
  - Kaplan-Schmaltz: Domain wall fermions
- Numerical simulations
  - Chiral symmetry breaking
  - Low-lying mass spectrum
  - SUSY WTi $\rightarrow$ Renormalization constants for the supercurrent.
- Exact supersymmetry on the lattice?
  - $N = 1$ Wess-Zumino
  - $N = 2$ SYM theory in $d = 2$
- Conclusions and perspectives

$N = 1$ SYM dynamics: open questions

The basic feature of $N = 1$ SYM dynamics (similar to QCD):

- Confinement
- Spontaneous chiral symmetry breaking

  $U(1)_\lambda \stackrel{anomaly}{\longrightarrow} Z_{2N_c} \stackrel{spontaneous}{\longrightarrow} Z_2$

  Gluino condensation: $\langle \lambda\lambda \rangle \neq 0$

- Low-lying mass spectrum
- SUSY Ward-Takahashi identity (WTi) (anomaly?)

This are **non-perturbative effects**
$\Longrightarrow$ Lattice formulation

The Model

The continuum action of $N = 1$ SYM and gauge group $SU(N_c)$ reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \bar{\lambda}^a \gamma_\mu (\mathcal{D}_\mu \lambda)^a + m_{\tilde{g}} \bar{\lambda}^a \lambda^a \, ,$$

$\lambda = \lambda^a T^a$ is a Majorana spinor in the adjoint representation of the gauge group that satisfies the Majorana condition

$$\bar{\lambda} = \lambda^T C \, , \qquad \lambda = C \bar{\lambda}^T$$

The gluon fields are represented by

$$\begin{aligned} A_\mu &= -ig A^a_\mu T^a \\ F_{\mu\nu} &= -ig F^a_{\mu\nu} T^a \\ \mathcal{D}_\mu \lambda^a &= \partial_\mu \lambda^a + g f_{abc} A^b_\mu \lambda^c \, . \end{aligned}$$

The action has for $m_{\tilde{g}} = 0$ a supersymmetry respect to the SUSY transformations.

The continuum SUSY transformations read

$$\begin{aligned}\delta A_\mu(x) &= -2g\bar{\lambda}(x)\gamma_\mu\varepsilon \\ \delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}F_{\rho\tau}(x)\varepsilon \\ \delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}F_{\rho\tau}(x)\end{aligned}$$

where $\sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau]$ and ε is a global Grassmann parameter with Majorana properties.

These transformations relate fermions and bosons.

They leave the action invariant and commute with the gauge transformations so that the resulting Noether current $S_\mu(x)$ is gauge invariant.

For $N = 1$ SYM theory the supercurrent is

$$S_\mu = -F^a_{\rho\tau}\sigma_{\rho\tau}\gamma_\mu\lambda^a\,.$$

Classically the Noether theorem is conserved

$$\partial_\mu S_\mu = 0\,,$$

(if the fields satisfy the eq. of motion). Furthermore, it fulfills a spin 3/2 constraint

$$\gamma_\mu S_\mu = 0\,.$$

SUSY WTi

The existence of the renormalized supercurrent S_μ^R is assumed

$$\partial_\mu S_\mu^R = 2m_R\chi_R$$

where

$$\chi_R = Z_\chi \chi \,, \qquad \chi \equiv \frac{1}{2} F_{\mu\nu}^a \sigma_{\mu\nu} \lambda^a \,.$$

m_R is the renormalized gluino mass.

- SUSY occurs for $m_R = 0$.
- The non-vanishing of m_R describes a **soft breaking** of SUSY.

Chiral symmetry breaking

Introducing a non zero gluino mass term

$$\mathcal{L}_{mass} = m_{\tilde{g}} \bar{\lambda}^a \lambda^a$$

breaks SUSY softly $\rightarrow$ Non-renormalization theorem and cancellation of divergencies are preserved

Girardello & Grisaru '82

In the massless case, the global chiral symmetry is $U(1)_\lambda$

$$\lambda \rightarrow e^{-i\varphi\gamma_5}\lambda\,, \qquad \bar{\lambda} \rightarrow \bar{\lambda}e^{-i\varphi\gamma_5}\,.$$

It is anomalous ($J^5_\mu = \bar{\lambda}\gamma_\mu\gamma_5\lambda$)

$$\partial_\mu J^5_\mu = \frac{N_c g^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\,.$$

The anomaly leaves a Z_{2N_c} subgroup of $U(1)_\lambda$ unbroken

These transformations are equivalent to

$$m_{\tilde{g}} \to m_{\tilde{g}} e^{-2i\varphi\gamma_5}, \quad \Theta_{SYM} \to \Theta_{SYM} - 2N_c\varphi$$

In the SUSY case $m_{\tilde{g}} = 0$, $U(1)_\lambda$ symmetry is unbroken if

$$\varphi = \varphi_k \equiv \frac{k\pi}{N_c}, \qquad (k = 0, 1, \cdots, 2N_c - 1)$$

Z_{2N_c} is expected to be spontaneously broken to Z_2 by $\langle\lambda\lambda\rangle \neq 0$

Witten '82

Consequence of this spontaneous chiral symmetry breaking is

$\Longrightarrow$ First order phase transition at $m_{\tilde{g}} = 0$

$\Rightarrow$ **Existence of N_c degenerate ground states with different orientations of the gluino condensate**

$$\langle\lambda\lambda\rangle = c\Lambda^3 e^{\frac{2\pi i k}{N_c}} \qquad (k = 0, \cdots, N_c - 1)$$

Dependence of the gauge group

- **case $SU(2)$:** Two degenerate ground states with opposite signs of the gluino condensate, $\langle\lambda\lambda\rangle < 0\,, \quad \langle\lambda\lambda\rangle > 0$

- **case $SU(3)$:** There are three degenerate vacua at $k = k_c$

Magnitude of the gluino condensate

The value of the gluino condensate in $N = 1$ SYM theory has been calculated using two different methods (which gives differents results) the $\frac{4}{5}$ puzzle !

- Based on weak-coupling instanton (WCI) calculations

$$\left\langle \lambda\lambda \right\rangle = c\Lambda^3$$

$$\Lambda = M_{PV} \left(\frac{16\pi^2}{3N_c g^2} \right)^{1/3} \exp\left(-\frac{8\pi^2}{3N_c g^2} \right)$$

with $c = 6$.

Affleck, Dine & Seiberg '84
Novikov, Shifman, Vainshtein & Zakharov '85
Shifman & Vainshtein '88

- Based on strong-coupling instanton (SCI) calculations with $c = \frac{4}{5}$

Novikov, Shifman, Vainshtein & Zakharov '83
Rossi & Veneziano '84
Amati, Rossi & Veneziano '84

Discussions about the two methods

Amati, Konishi, Meurice, Rossi & Veneziano '88
Kovner & Shifman '97
Hollowood, Khoze, Lee & Mattis '00
Ritz & Vainshtein '00

More recently, a third elegant method has been used where the gluino condensate is directly calculated in the semiclassical approx.

Davies, Hollowood, Khoze & Mattis '99
Davies, Hollowood & Khoze '00

This method gives results in agreement with the weak-coupling calculations (Affleck et al.) and

Finnell & Pouliot '95

Still another method makes use of exact solution solutions of $N = 2$ SYM theories

Seiberg & Witten '94
Argyres & Faraggi '95

One first compute the gluino condensate in the $N = 2$ SYM perturbed by the adjoint mass term and then decouples the adjoint matter fields, going back to pure $N = 1$ SYM. Results are equivalent to the weak-coupling instanton calculation.

Also has been applied to softly $N = 2$ SYM

Konishi & Ricco '03

confirm the correctness of the weak-coupling instanton calculation.

Light hadron spectrum

Veneziano and Yankielowicz '82 proposed an effective action to study the low energy behavior of the SYM theory.

To construct the action they identify all degrees of freedom they expect to govern the low energy dynamics. These are gauge invariant and colorless composite fields

$$F^a_{\mu\nu}F^a_{\mu\nu}$$
$$F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$$
$$\bar{\lambda}^a\lambda^a$$
$$\sigma_{\mu\nu}F^a_{\mu\nu}\lambda^a$$

The first three operators are known in QCD while $\chi = \sigma_{\mu\nu}F^a_{\mu\nu}\lambda^a$ is a new type of composite operator formed by the gluino λ and the gauge field F.

The fields can be combined to form the chiral superfield

$$S(x,\theta) = \phi(x) + \sqrt{2}\theta\chi(x) + \theta\theta F(x)$$

$$\phi = \frac{\beta(g)}{2g}(\psi_w)^\alpha(\psi_w)_\alpha\,,\;\sqrt{2}\chi_\alpha = -\frac{\beta(g)}{2g}(-i(\psi_w)_\alpha D + (\sigma^{\mu\nu}\psi_w)_\alpha F_{\mu\nu})\,,$$

$$F = -\frac{\beta(g)}{g}\left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\psi_w\sigma^\mu\partial_\mu\bar{\psi}_w - \frac{i}{8}F_{\mu\nu}\varepsilon_{\mu\nu\rho\tau}F_{\rho\tau} + \frac{i}{2}\partial_\mu J^5_\mu + \frac{1}{2}D^2\right\}.$$

The effective VY action is

$$\mathcal{L}_{eff} = \frac{1}{\alpha}(S^\dagger S)^{1/3}|_D + \gamma[(S\log\frac{S}{\mu^3} - S)|_F + h.c.]\,.$$

Expanding the effective action around its minimum, it is found the low-lying spectrum forming a supermultiplet of the Wess-Zumino type consist of

- A scalar meson $\phi = \bar{\lambda}^a \lambda^a$. In analogy with QCD. (The gluino is in the adjoint representation). We call this particle $a - f_0$.

- A massive Majorana fermion $\chi = \sigma_{\mu\nu} F^a_{\mu\nu} \lambda^a$ called gluino-glue.

- A pseudoscalar meson $\phi_p = \bar{\lambda}^a \gamma_5 \lambda^a$, called the $a - \eta\prime$.

It is not clear why glueballs should be absent in the low-lying spectrum.

The introduction of a non-zero gluino mass breaks susy softly and leads to a splitting of the multiplet.

How the spectrum of glueballs, gluinoballs and gluino-glueballs are influenced by the soft SUSY breaking due to a non-zero gluino mass $m_{\tilde{g}} \neq 0$

$$
\begin{aligned}
M_{a-\eta'} &= N_c \alpha \Lambda + \frac{40\pi^2 |m_{\tilde{g}}|}{3N_c} + \cdots \\
M_{a-\chi} &= N_c \alpha \Lambda + \frac{48\pi^2 |m_{\tilde{g}}|}{3N_c} + \cdots \\
M_{a-f_0} &= N_c \alpha \Lambda + \frac{56\pi^2 |m_{\tilde{g}}|}{3N_c} + \cdots
\end{aligned}
$$

Evans, Hsu & Schwetz '97

The range of applicability of the linear mass formulae is not known because the unknown magnitude of the constants and of the higher order terms.

The question how to include glueballs in the low energy scenario has been addressed by

Farrar, Gabadadze & Schwetz '98

They introduce an extra term in the effective action that gives the dynamics for the glueballs.

For unbroken supersymmetry the masses of these two supermultiplets are not identical. The heavier supermultiplet corresponds to the VY multiplet. The lighter one contains

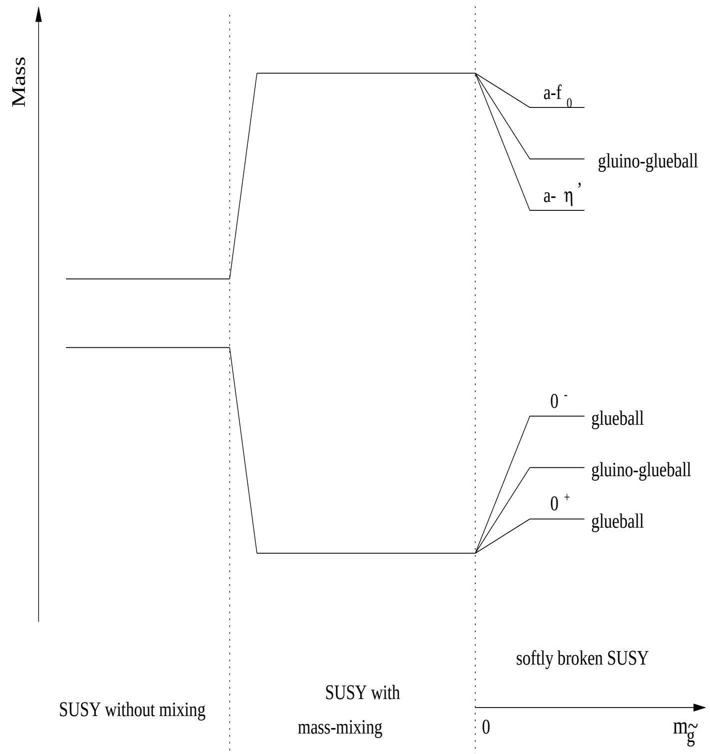
- A 0^-glueball $\approx F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

- A gluino-glue ground state.

- A 0^+glueball $\approx F_{\mu\nu}F_{\mu\nu}$.

In the low effective action of FGS there is a possible non-zero mixing between the states in the two light supermultiplets. In particular there can be a mixing of the $a - f_0$ gluinoball and 0^+ glueball.

Masses



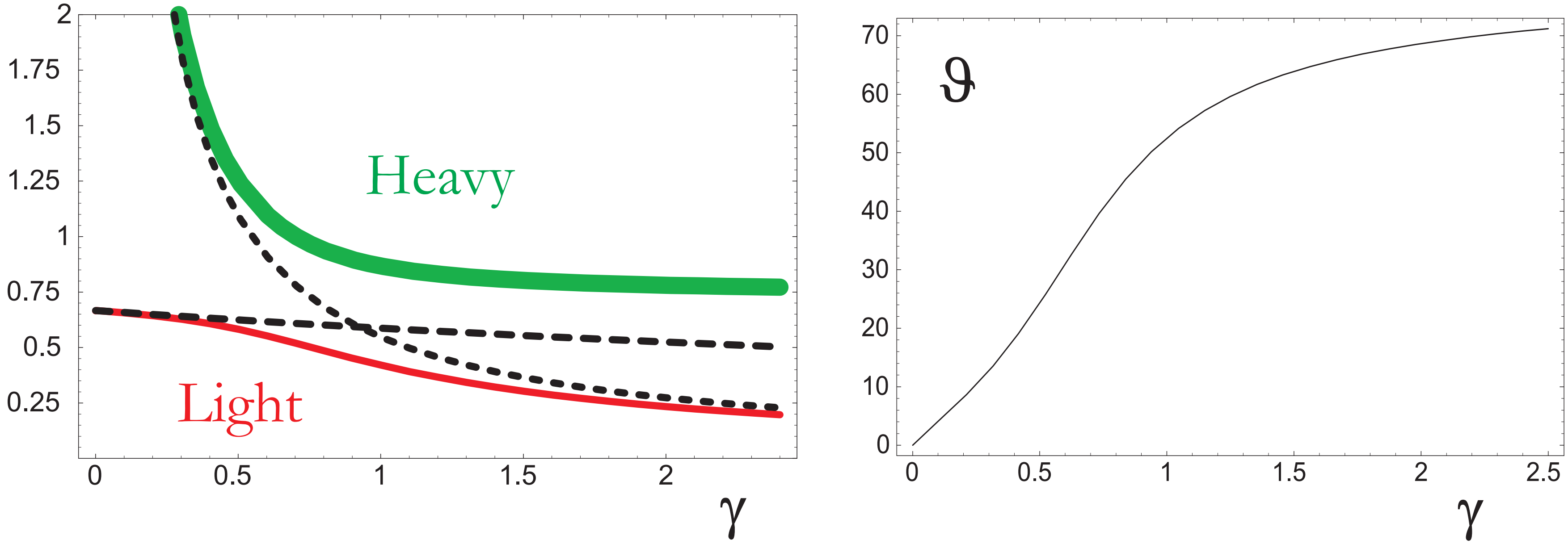
Mass
a-f 0
gluino-glueball
a- η '
0 -
glueball
gluino-glueball
0 +
glueball
softly broken SUSY
SUSY without mixing
SUSY with
mass-mixing
0
m g̃

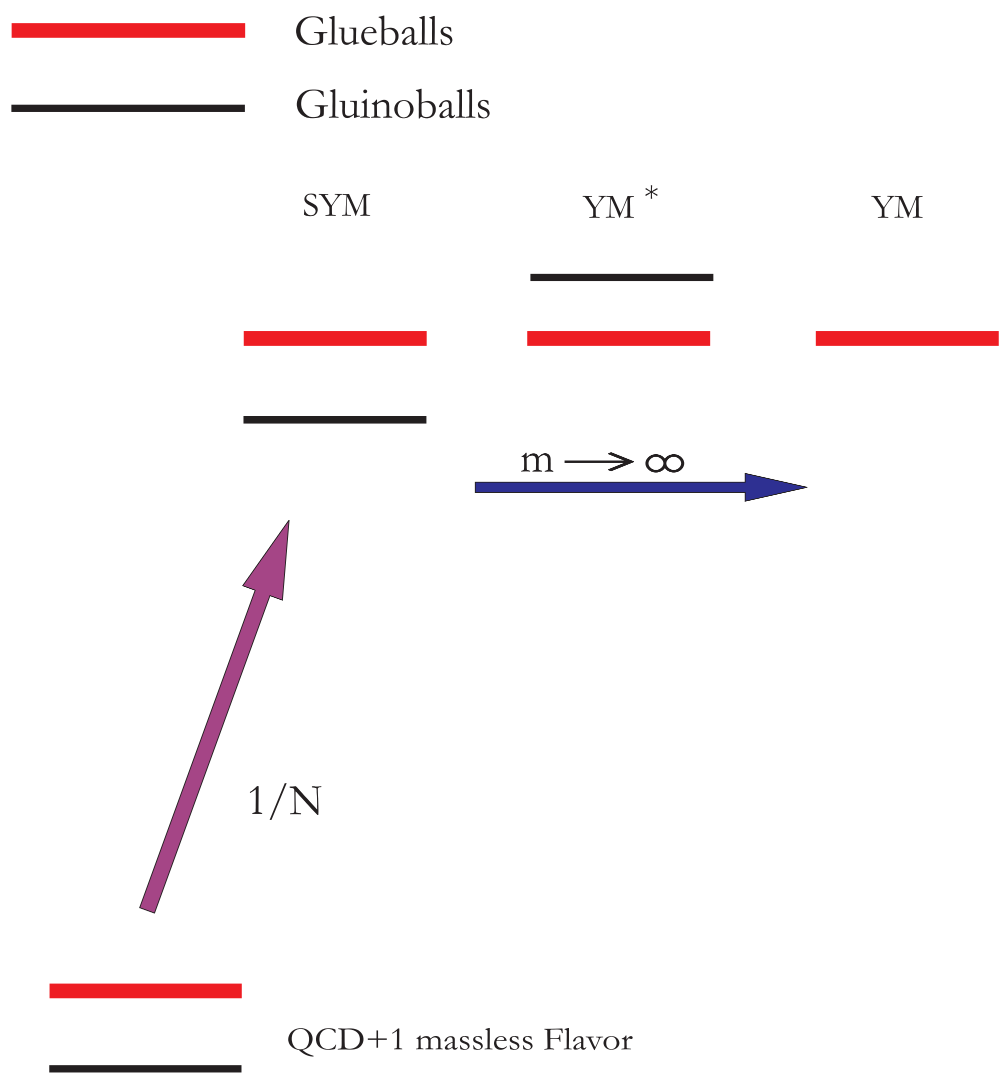
Information on the Super Yang-Mills Spectrum

It can be demonstrated that with an extended Lagrangian approach one can accomodate either the possibility in which the glueballs are heavier or lighter than the gluinoball fields.

If one use information about ordinary QCD one can deduce that the lightest states in super Yang-Mills are the gluinoballs.

Merlatti and Sannino, hep-th/0404251
Feo, Merlatti and Sannino, hep-th/0408214


2
1.75
1.5
1.25
1
0.75
0.5
0.25
Heavy
Light
0
0.5
1
1.5
2
γ
70
60
50
40
30
20
10
0
ϑ
0
0.5
1
1.5
2
2.5
γ


Glueballs
Gluinoballs
SYM
YM *
YM
m ⟶ ∞
1/N
QCD+1 massless Flavor

Wilson fermions

Propose to give up manifest SUSY on the lattice and restore it in the continuum limit.

Curci & Veneziano '87

SUSY is broken by the lattice, by the Wilson term and a a soft breaking due to the gluino mass is present.

- SUSY is recovered in the continuum limit by tuning the bare parameters g and gluino mass $m_{\tilde{g}}$ to the SUSY point.
- The chiral and SUSY limit can be recovered simultaneously at $m_{\tilde{g}} = 0$.

The inclusion of a gluino mass breaks supersymmetry softly and the bare gluino mass has to be tuned numerically to its critical value which is the chiral – SUSY limit – $m_{\tilde{g}} = 0$

Wilson fermions

The Curci and Veneziano action reads

$$S = S_G + S_F,$$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x) \right),$$

and $\beta \equiv 2N_c/g_0^2$ correspond to the bare gauge coupling.

$$\begin{aligned} S_F \;=\; \mathrm{Tr} \Bigg\{ \frac{1}{2a} \Big(& \bar{\lambda}(x)(\gamma_\mu - r)U_\mu^\dagger(x)\lambda(x + a\hat{\mu})U_\mu(x) \\ & - \bar{\lambda}(x + a\hat{\mu})(\gamma_\mu + r)U_\mu(x)\lambda(x)U_\mu^\dagger(x) \Big) + \left(m_0 + \frac{4r}{a} \right) \bar{\lambda}(x)\lambda(x) \Bigg\} . \end{aligned}$$

The Grassmann variables λ and $\bar{\lambda}$ are not independent

$$\bar{\lambda} = \lambda^T C\,, \qquad \lambda = C\bar{\lambda}^T\,.$$

Monte Carlo simulations : A different parametrization is used. The hopping parameter k is

$$k = \frac{1}{2(4 + m_0 a)}$$

The lattice Wilson fermion action

$$S_F \equiv \frac{1}{2}\bar{\lambda}Q\lambda \equiv \frac{1}{2}\sum_x \left\{ \bar{\lambda}^a_x \lambda^a_x - k \sum_{\mu=1}^{4} \left[\bar{\lambda}^a_{x+\hat{\mu}} V_{ab,x\mu}(1+\gamma_\mu)\lambda^b_x + \bar{\lambda}^a V^T_{ab,x\mu}(1-\gamma_\mu)\lambda^b_{x+\hat{\mu}} \right] \right\}$$

Montvay '98

with the adjoint link $V_{ab,x\mu}(x)$ in the adjoint representation

$$\begin{aligned} V_{ab,x\mu} &\equiv V_{ab,x\mu}[U] \equiv \\ &\equiv 2\mathrm{Tr}(U^\dagger_{x\mu} T_a U_{x\mu} T_b) = V^*_{ab,x\mu} = V^{-1T}_{ab,x\mu}. \end{aligned}$$

The path integral over the Majorana fermions gives the Pfaffian

$$\int [d\lambda] e^{-\frac{1}{2}\bar{\lambda} Q \lambda} = \int [d\lambda] e^{-\frac{1}{2}\lambda^T C Q \lambda} = Pf(M) \;\; = \;\; \underbrace{\pm}_{\text{sign}} \sqrt{det\, Q}\,.$$

where $M \equiv CQ = -M^T$ is an antisymmetric matrix.

Algorithm for numerical simulations

The effective action is

$$S_{CV} = \beta \sum_{pl} \left(1 - \frac{1}{2} \mathrm{Tr} U_{pl} \right) - \frac{1}{2} \log \det Q[U] \,.$$

Curci & Veneziano '87

The ommited sign of the Pfaffian can be taken into account by the reweighting formula

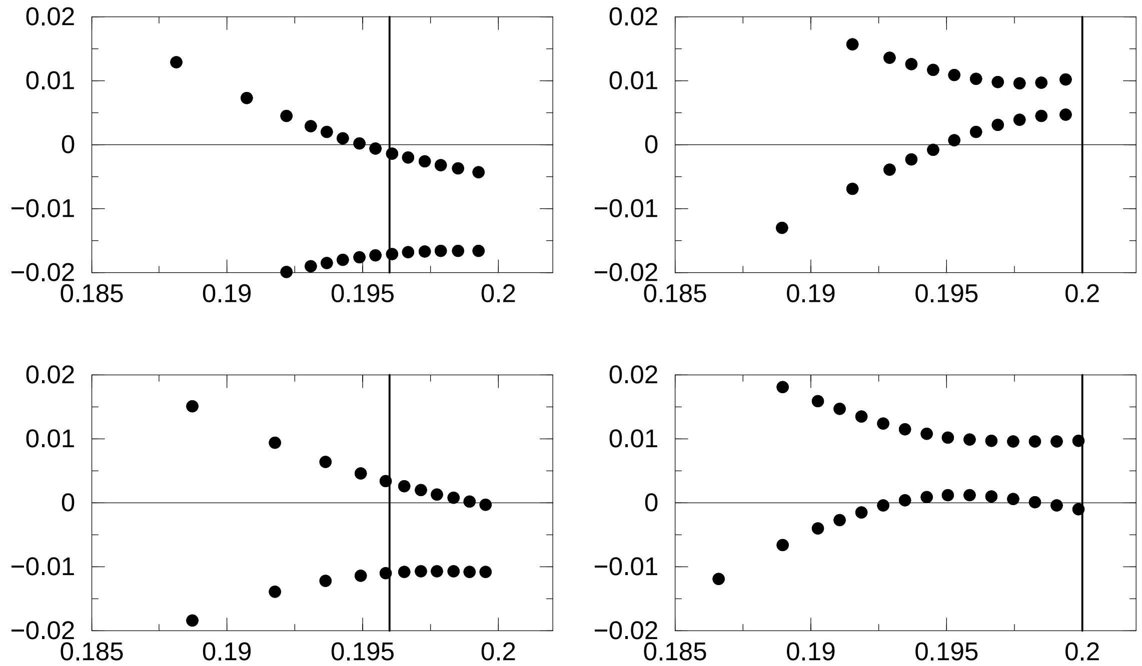
$$\left\langle \mathcal{O} \right\rangle = \frac{\left\langle \mathcal{O} \, \mathrm{sign} Pf(M) \right\rangle_{CV}}{\left\langle \mathrm{sign} Pf(M) \right\rangle_{CV}}$$

may rise to the **sign problem**

Spectral flow method: for the sign.

$$|Pf(M)| = \prod_{i=1}^{\Omega/2} |\tilde{\lambda}_i| \,, \quad \Longrightarrow Pf(M) = \prod_{i=1}^{\Omega/2} \tilde{\lambda}_i \,.$$

If the value of an eigenvalue $\tilde{\lambda}_i$ changes sign, the sign of $Pf(M)$ has to change too.



The spectral flow of the hermitian fermion matrix $\tilde{Q}$ for a configuration on $6^3 \times 12$ at $\beta = 2.3$. The value of k in the simulation correspond to the vertical line. Montvay et al. '99. For $k < k_c \rightarrow$ no serious sign problems!

The simulation has been develop in the *Two-step multibosonic algorithm* (TSMB)

Montvay '96,'98

To represent the fermion determinant ones uses a first polynomial $\mathcal{P}^{(1)}_{n_1}(x)$ for a crude approximation realizing a fine correction by another polynomial $\mathcal{P}^{(2)}_{n_2}(x)$

$$\mathcal{P}^{(1)}_{n_1}(x)\mathcal{P}^{(2)}_{n_2}(x) \approx x^{-N_f/2} \;\; x \in \;\; [\varepsilon, \lambda] \, .$$

The fermion determinant is approximated as

$$\det(Q^\dagger Q)^{N_f} \simeq \frac{1}{\det P^{(1)}_{n_1}(Q^\dagger Q) \det P^{(2)}_{n_2}(Q^\dagger Q)} \, .$$

Domain wall fermions

A new lattice fermion regulator. Very nice innovation. Application of DWF in SUSY theories

Neuberger '98
Kaplan & Schmaltz '00

Monte Carlo simulation for $N = 1$ $SU(2)$ SYM with DWF

Fleming, Kogut & Vranas '01

DWF were introduced by

Kaplan '92,'93

and further developed in

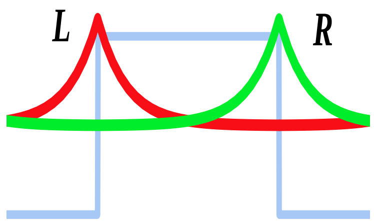
Narayanan & Neuberger '93,'94,'95
Shamir '93
Furman & Shamir '95

Difficulties in using Wilson fermions.

- Need to fine tuning. The Wilson term breaks chiral symmetry
- The Pfaffian. Is not positive definite at finite lattice spacing.

DWF are defined extending space-time to five dimensions.

L_s is the size of the fifth dimension.



In the limit $L_s \to \infty$ chiral symmetry is exact, even at finite lattice spacing.

- There is not need for fine tuning.

The domain wall action is

$$S = S_G(U) + S_F(\Psi, U) + S_{PV}(\Phi, U)$$

$$S_F = -\sum_{x,x',s,s'} \bar{\Psi}_{x,s} (D_F)_{x,s;x',s'} \Psi_{x',s'}$$

The effective action is

$$
\begin{aligned}
S_{KS} &= \beta \sum_{pl} \left(1 - \frac{1}{2} \mathrm{Tr} U_{pl} \right) - \frac{1}{2} \log \det D_F[U] \\
&\quad + \frac{1}{2} \log \det D_F[m_f = 1; U] \, .
\end{aligned}
$$

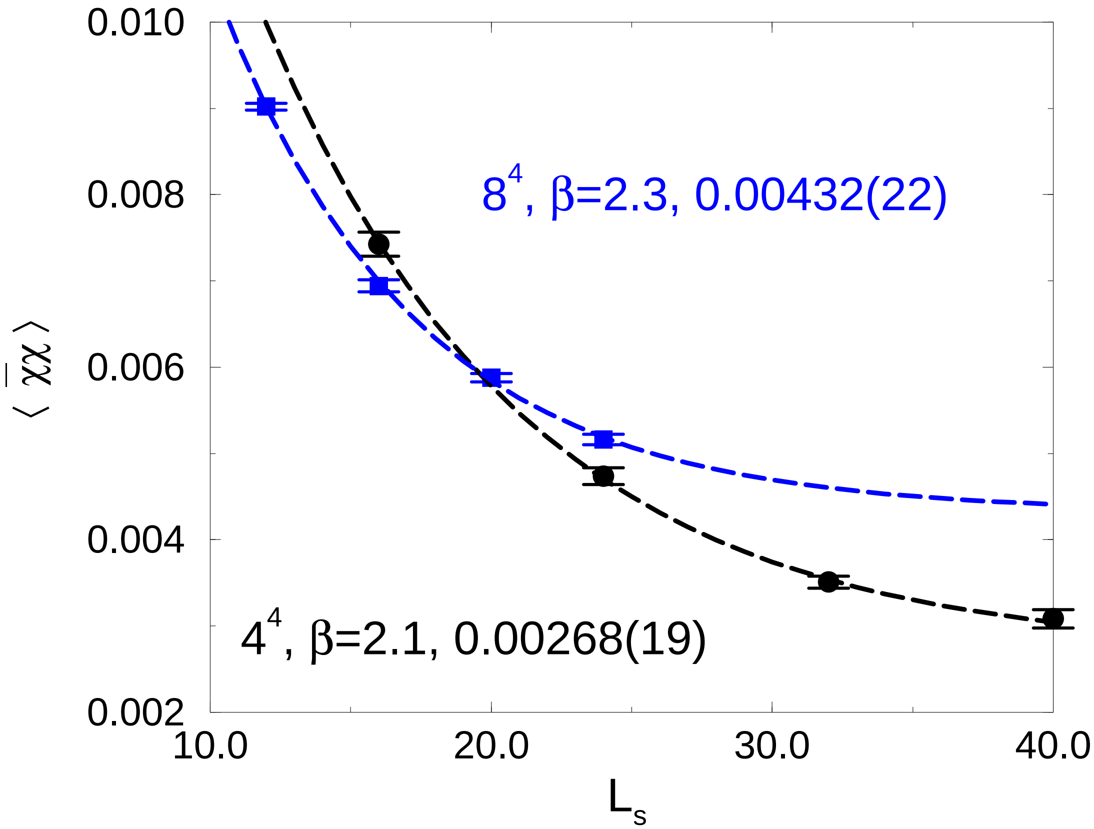
Difficulties in using DWF.

- 2 extra parameters in DWF: L_s and m_0 (m_0 is the domain wall height or five-dimensional mass that controls the number of flavors).

$$m_{eff} = m_0(2 - m_0)[m_f + (1 - m_0)^{L_s}] , \quad 0 < m_0 < 2$$

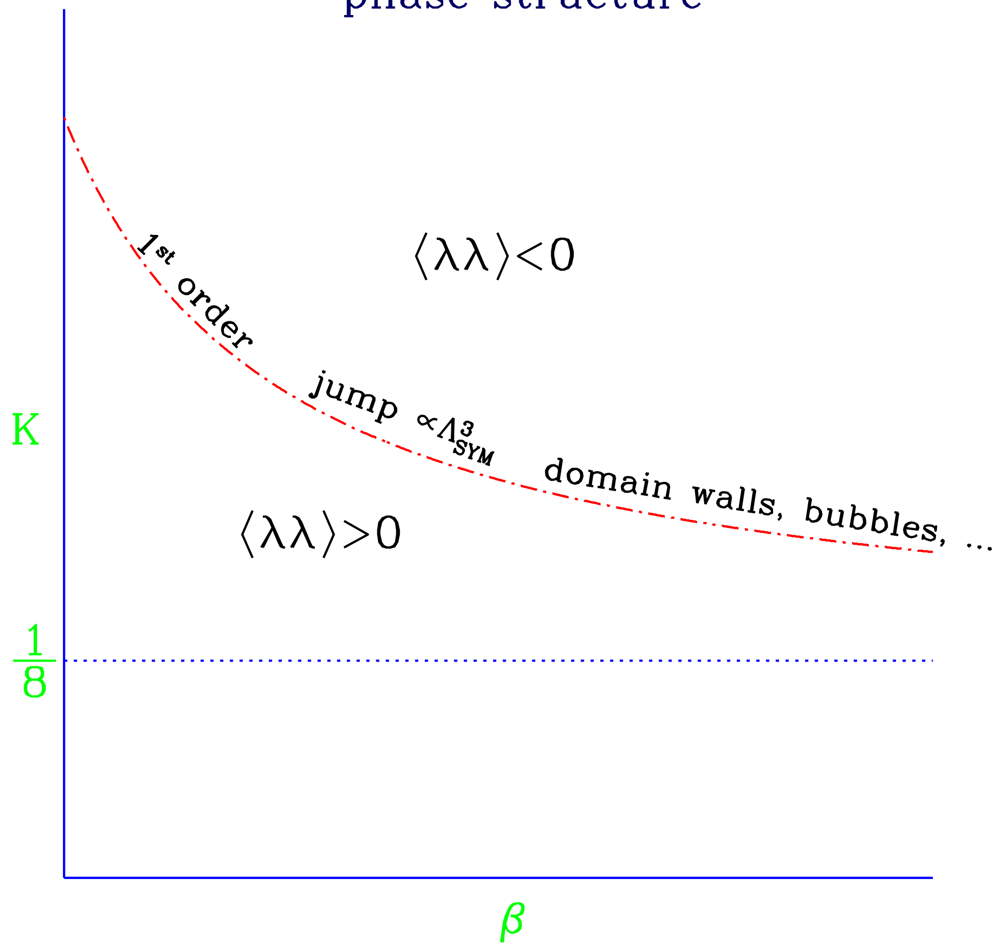
- The two chiralities do not decouple $\rightarrow$ no restoration of chiral symmetry. (Need large values of L_s)

- Harder to simulate than QCD (with Wilson fermions easier to simulate than QCD)

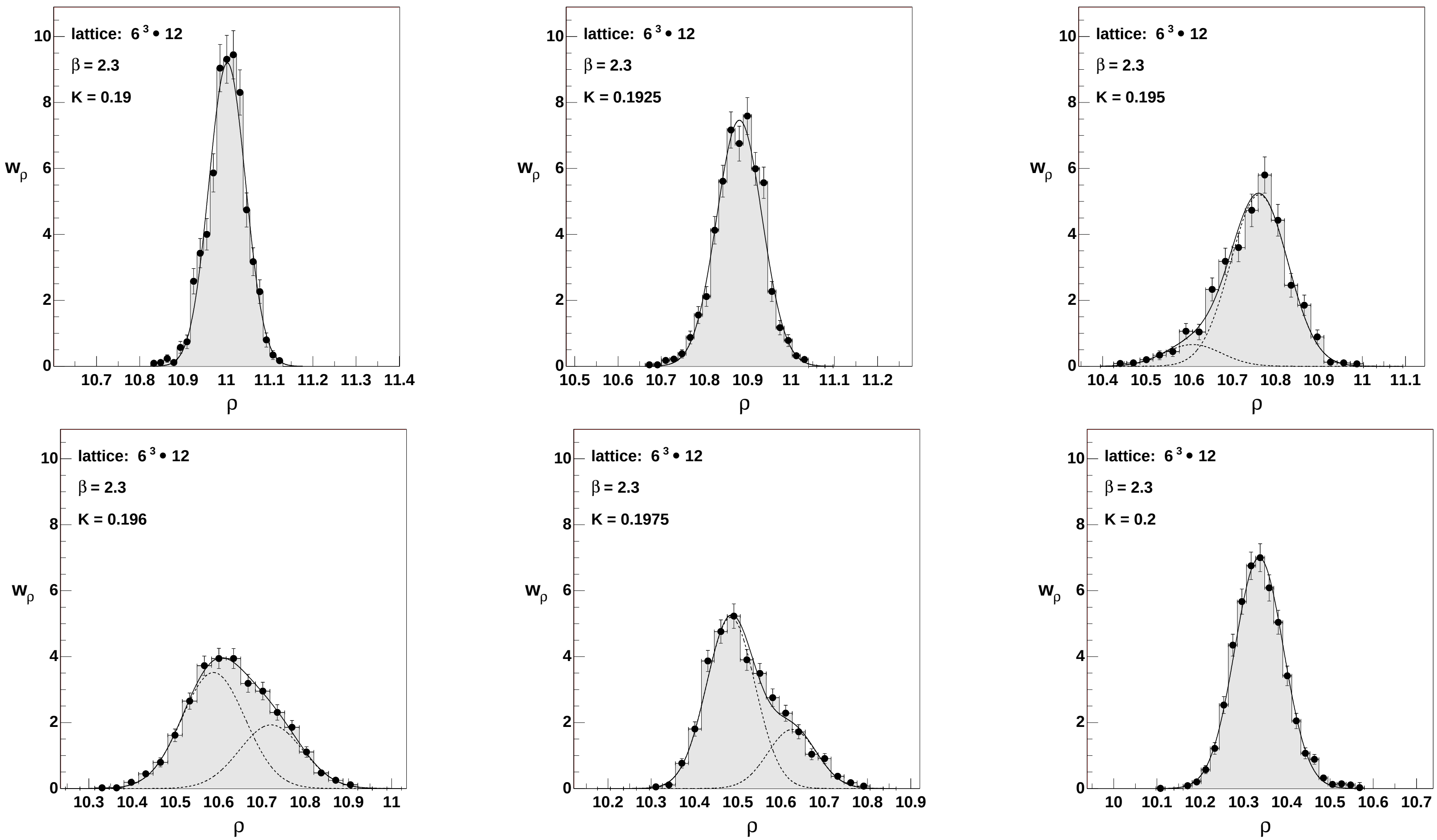


Dynamical gluino condensate at $m_f = 0$ vs L_s on two different lattices. (Fleming, Kogut, Vranas '00)

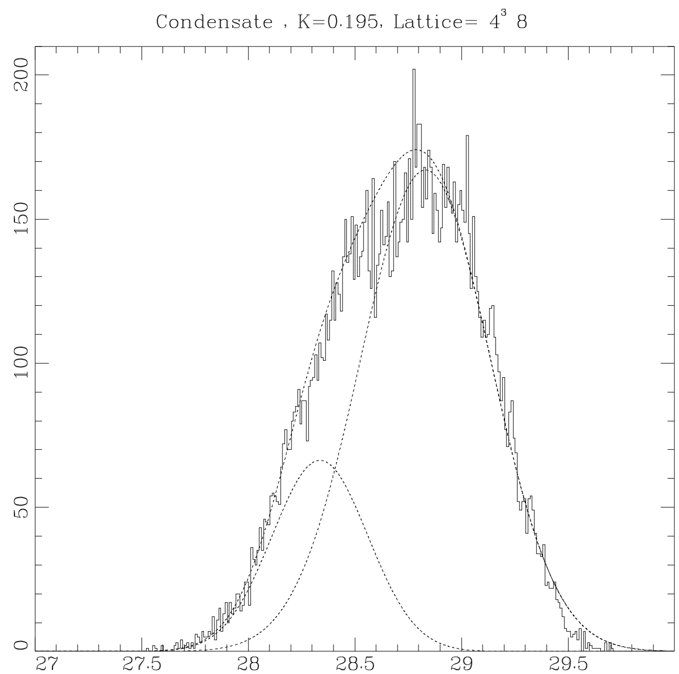
phase structure

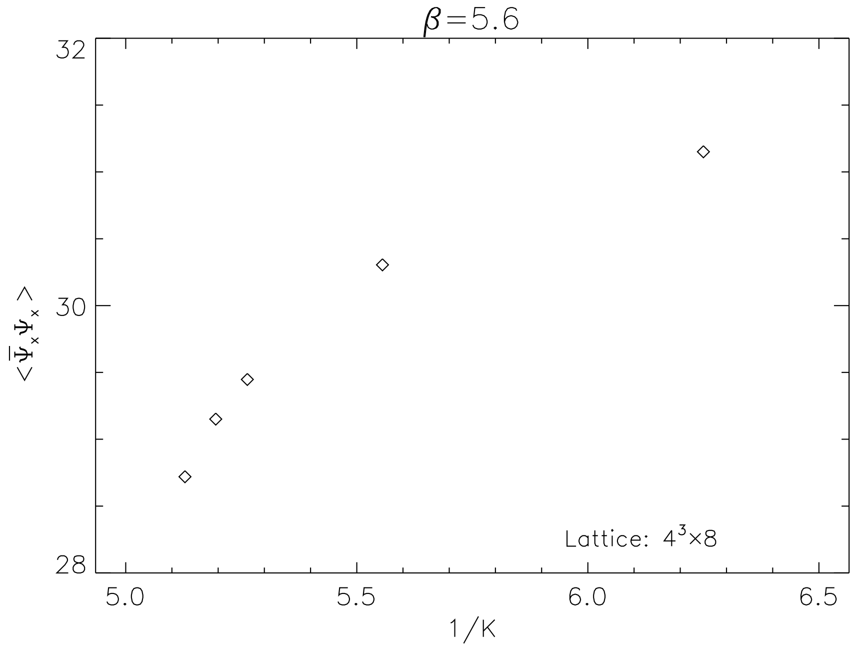


Expected phase structure of SYM in the (β, k) plane. Dashed line $k = k_c(\beta)$ is a first-order phase transition (or cross-over) at $m_{\tilde{g}} = 0$.



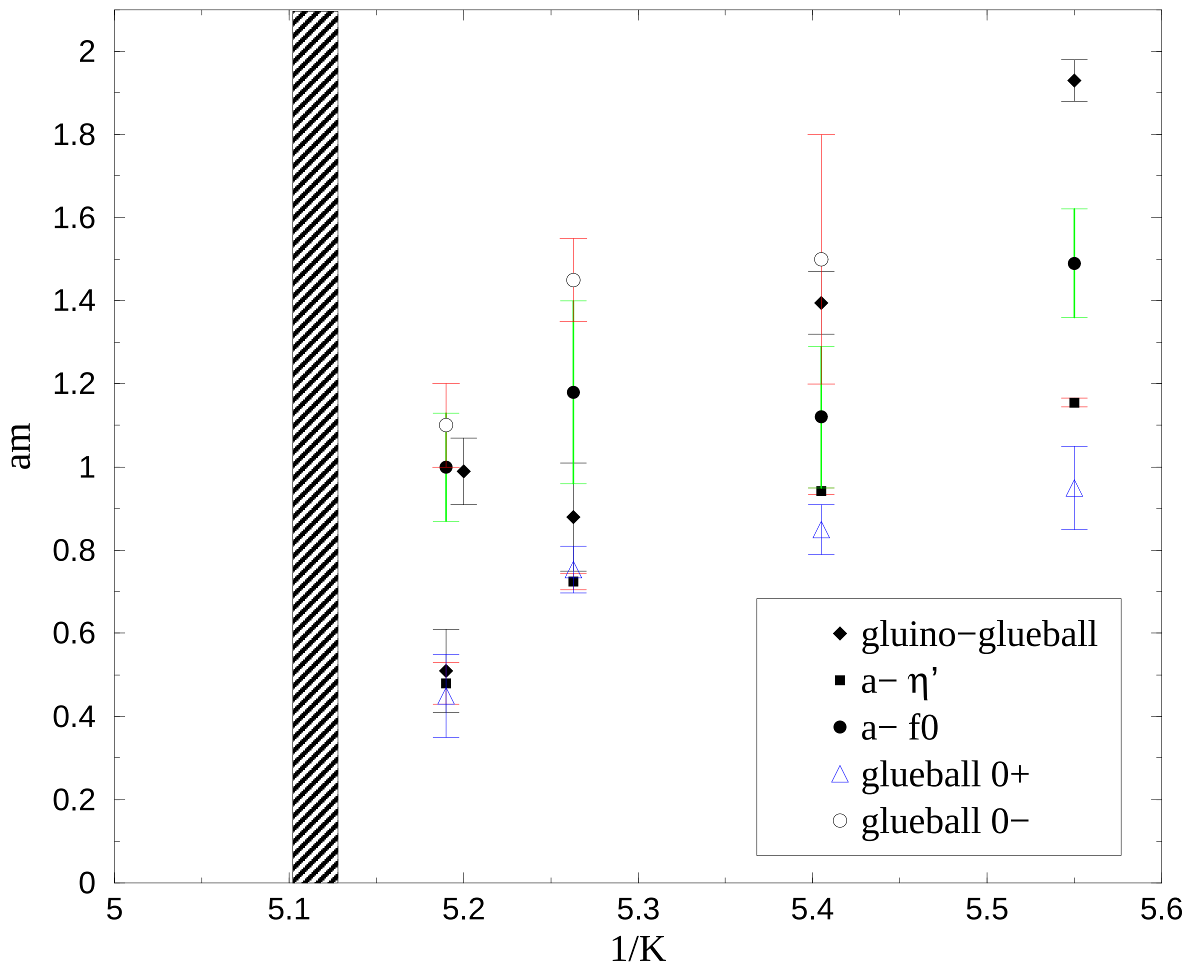
Distribution of the gluino condensate for different k values at $\beta = 2.3$ and lattice size $6^3 \times 12$, $k_c = (0.1955 \pm 0.0005)$ and $N_c = 2$. (Montvay et al., '99)





Distribution of the gluino condensate for $k = 0.195$ at $\beta = 5.6$ and lattice size $4^3 \times 8$ and $N_c = 3$. (Montvay et al., '00)

Light hadron spectrum in lattice units (Montvay et al., '00, '01). $k_c = (0.1955 \pm 0.0005)$.
Lattice size $12^3 \times 24$.



Exact supersymmetry on the lattice

Four dimensional lattice Wess-Zumino model with GW fermions

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at a fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the naive continuum limit $a \to 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, which is a necessary ingredient to guarantee the request of supersymmetry.

Bonini and Feo, hep-lat/0402034, hep-lat/0504010
Feo, hep-lat/0512028 and in preparation.

The Ginsparg-Wilson (GW) relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry (Lüscher '98).

A solution was given by Neuberger '97,'98

$$D = \frac{1}{a}\left(1 - X\frac{1}{\sqrt{X^\dagger X}}\right), \qquad X = 1 - aD_w\,,$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu$$

In terms of the component fields the lattice Wess-Zumino action reads (Fujikawa 2002) and (Fujikawa and Ishibashi 2002)

$$S_{WZ} = S_0 + S_{int}\,,$$

with

$$S_0 = \sum_x \left\{ \frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2\chi - \frac{1}{a}(AD_1A + BD_1B) + \frac{1}{2}F(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}G(1 - \frac{a}{2}D_1)^{-1}G \right\},$$

$$S_{int} = \sum_x \left\{ \frac{1}{2}m\bar{\chi}\chi + m(FA + GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A + i\gamma_5 B)\chi + \frac{1}{\sqrt{2}}g\left[F(A^2 - B^2) + 2G(AB)\right] \right\}.$$

where A, B, F and G are real scalar fields and χ is a Majorana fermion which satisfies the Majorana condition.

The supersymmetric transformations

The total action is invariant under the transformations

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta\chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi\,,
\end{aligned}$$

$$R = R^{(1)} + gR^{(2)} + \cdots$$

where

$$R^{(1)} = ((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\Delta L$$

with

$$\Delta L \equiv \frac{1}{\sqrt{2}}\Big\{2(AD_2A - BD_2B) - D_2(A^2 - B^2) + 2i\gamma_5\Big[(AD_2B + BD_2A) - D_2(AB)\Big]\Big\}$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}(A + i\gamma_5 B)R^{(n-1)}\,.$$

$$\Big[(1 - \frac{a}{2}D_1)^{-1}D_2 + m + \sqrt{2}g(A + i\gamma_5 B)\Big]R = \Delta L\,.$$

It has been shown that the algebra associated to the lattice supersymmetry transformation closes.

The existence of this exact symmetry should be responsible for the restoration of supersymmetry in the continuum limit without fine tuning. We prove this using the WTi to order g, g^2, g^3.

Two-point Ward-Takahashi identity

Order g^2

$$\langle \chi_y \bar{\chi}_x \rangle - \langle D_{2yz}(A_z - i\gamma_5 B_z) A_x \rangle - \langle (F_y - i\gamma_5 G_y) A_x \rangle + g \langle R_y A_x \rangle = 0\,.$$

This WTi is satisfied at fixed lattice spacing and in the continuum limit.

This WTi determines the finite part of the scalar and fermion renormalization wave functions which automatically leads to restoration of susy in the continuum limit. In particular these wave functions coincide in this limit.

The two dimensional $N = 2$ Super Yang-Mills theory

The two dimensional continuum theory

The two dimensional $N = 2$ Super Yang-Mills theory can be written as a topological field theory form or Q-exact form [Witten, 1988]

$$S_{SYM}^{N=2,d=2} = Q\frac{1}{2g_0^2}\int d^2x \mathrm{Tr}\left[\frac{1}{4}\eta[\phi,\bar{\phi}] - i\chi_{12}\Phi + \chi_{12}B_{12} - i\psi_\mu D_\mu\bar{\phi}\right],$$

where μ is the index for the two dimensional space-time

The bosonic fields are represented by two scalar fields ϕ and $\bar{\phi}$, a vector field A_μ and another commuting field B_{12}, which is an auxiliary field

The fermionic fields are represented by a vector ψ_μ, an anticommuting scalar field η and a field χ_{12} conjugate to B_{12}

Φ is a function of the field strength $F_{\mu\nu}$ and for two dimensions is given by

$$\Phi \equiv 2F_{12}\,.$$

Q is one of the supercharges of $N = 2$ Super Yang-Mills theory and its transformation rule over the fields is given by the following rule,

$$
\begin{aligned}
&QA_\mu = \psi_\mu \\
&Q\psi_\mu = iD_\mu\phi \\
&Q\phi = 0 \\
&Q\chi_{12} = B_{12} \\
&QB_{12} = [\phi, \chi_{12}] \\
&Q\bar{\phi} = \eta \\
&Q\eta = [\phi, \bar{\phi}]\,.
\end{aligned}
$$

Q is nilpotent up to infinitesimal gauge transformations with parameter ϕ, i.e., the square of Q yields an infinitesimal gauge transformations, $Q^2 = \delta_G^\phi$, with parameter ϕ. Carrying out the Q-variation leads to the more explicit form for the $N = 2$ Super Yang-Mills action,

$$
\begin{aligned}
S_{SYM}^{N=2,d=2} = \frac{1}{2g_0^2}\int d^2x \mathrm{Tr}\bigg[&\frac{1}{4}[\phi,\bar{\phi}]^2 + B_{12}^2 - iB_{12}\Phi \\
&+ D_\mu\phi D_\mu\bar{\phi} - \frac{1}{4}\eta[\phi,\eta] - \chi_{12}[\phi,\chi_{12}] + \psi_\mu[\bar{\phi},\psi_\mu] \\
&+ i\chi_{12}Q\Phi + i\psi_\mu D_\mu\eta\bigg]\,.
\end{aligned}
$$

and integrate out the field B_{12} gives

$$S_{SYM}^{N=2,d=2} = \frac{1}{2g_0^2}\int d^2x \mathrm{Tr}\left[\frac{1}{4}[\phi,\bar{\phi}]^2 + F_{12}^2 + D_\mu\phi D_\mu\bar{\phi} - \frac{1}{4}\eta[\phi,\eta] - \chi_{12}[\phi,\chi_{12}] + \psi_\mu[\bar{\phi},\psi_\mu] + i\chi_{12}Q\Phi + i\psi_\mu D_\mu\eta\right].$$

Lattice Formulation with One Exact Supercharge

F.Sugino, hep-lat/0311021, hep-lat/0401017, hep-lat/0410035;
S. Catterall, hep-lat/0410052, hep-lat/0503036, hep-lat/0602004;
D'Adda, et al., hep-lat/0406029, hep-lat/0507029.

Start with a formulation of the theory on a two dimensional hypercubic lattice where the gauge field $A_\mu(x)$ is represented by the unitary link variable $U_\mu(x) = e^{iaA_\mu(x)}$ [Sugino]

the Q-transformation can be generalized on the lattice preserving the property that $Q^2 =$ (is an infinitesimal gauge transformation with the parameter ϕ)

A possible solution is

$$QU_\mu(x) = i\psi_\mu(x)U_\mu(x)$$

$$Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\Big(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu^\dagger(x)\Big)$$

$$Q\phi(x) = 0$$
$$Q\chi_{12}(x) = B_{12}(x)$$
$$QB_{12}(x) = [\phi(x), \chi_{12}(x)]$$
$$Q\bar{\phi}(x) = \eta(x)$$
$$Q\eta(x) = [\phi(x), \bar{\phi}(x)]$$

where the dependence on the lattice spacing for each field variable is the following,

$$\begin{aligned}
&Q = O(a^{1/2})\,,\\
&\psi_\mu(x), \chi_{12}(x), \eta(x) = O(a^{3/2})\,,\\
&\phi(x), \bar{\phi}(x) = O(a)\,,\\
&B_{12}(x) = O(a^2)\,,
\end{aligned}$$

All transformations are the same in the continuum except for $QU_\mu(x)$ and $Q\psi_\mu(x)$.

In fact,

$$\begin{aligned}
Q^2 U_\mu(x) &= Q(i\psi_\mu(x)U_\mu(x))\\
&= (\phi(x)U_\mu(x) - U_\mu(x)\phi(x+\widehat{\mu}))
\end{aligned}$$

then we have

$$\begin{aligned}
Q^2\psi_\mu(x) &= Q[i\psi_\mu(x)\psi_\mu(x) - i(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu(x)^\dagger)]\\
&= [\phi(x), \psi_\mu(x)]
\end{aligned}$$

Once one have the Q-transformation rule closed among all the lattice variables, it is easy to write the lattice action with the exact supersymmetry Q,

$$S_{SYM}^{N=2} = Q\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Big[\frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\chi_{12}(x)\Phi(x) + \chi_{12}(x)B_{12}(x) + i\sum_{\mu=1}^{4}\psi_\mu(x)\Big(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x)\Big)\Big]$$

where $\Phi(x) \equiv -i(P_{12}(x) - P_{21}(x))$ and $P_{12}(x) = U_1(x)U_2(x+1)U_1^\dagger(x+2)U_2^\dagger(x)$ and

$$\lim_{a\to 0}\Phi(x) = 2F_{12}(x)\,.$$

that originates the lattice $N=2$ SYM action

$$S_{SYM}^{N=2} = \frac{1}{g_0^2}\int \mathrm{Tr}\Big[\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x)$$
$$+\sum_\mu \Big(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu(x)^\dagger\Big)\Big(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu(x)^\dagger\Big)$$
$$-\frac{1}{4}\eta(x)[\phi(x),\eta(x)] - \chi_{12}(x)[\phi(x),\chi_{12}(x)] + i\chi_{12}(x)Q\Phi(x)$$
$$-i\sum_\mu \psi_\mu(x)\Big(\eta(x) - U_\mu(x)\eta(x+\widehat{\mu})U_\mu(x)^\dagger\Big)$$
$$-\sum_\mu \psi_\mu(x)\psi_\mu(x)\Big(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu(x)^\dagger\Big)\Big]$$

and reduces to the continuum $N=2$ SYM action in the limit $a \to 0$ without any fine tuning of the parameters of the action.

In fact, the fermionic kinetic term,

$$i\chi_{12}(x)Q\Phi(x) - i\sum_{\mu}\psi_\mu(x)\left(\eta(x) - U_\mu(x)\eta(x+\widehat{\mu})U_\mu(x)^\dagger\right)$$

has the correct continuum naive limit and contains no doublers. The continuum naive limit is

$$\lim_{a\to 0} i\mathrm{Tr}(\chi_{12}Q\Phi(x)) = 2i\mathrm{Tr}[\chi_{12}(D_1\psi_2(x) - D_2\psi_1(x))]$$

is of order $O(1)$ and is exactly the continuum value, while the second term gives $i\psi_\mu(x)D_\mu\eta(x)$. Moreover, the second term of the lattice action gives $D_\mu\phi(x)D_\mu\bar{\phi}(x)$, while the last term gives $\psi_\mu(x)[\bar{\phi}(x), \psi_\mu(x)]$.

After integrating out the auxiliary field $B_{12}(x)$, one is left with a gauge kinetic term of the form

$$\frac{1}{2g_0^2}\sum_x \sum_{\mu<\nu} \mathrm{Tr}\Big[-(U_{\mu\nu}(x)-U_{\nu\mu}(x))^2\Big]$$

which is slightly different to the one corresponding to the Wilson action

$$\frac{1}{2g_0^2}\sum_x \sum_{\mu<\nu} \mathrm{Tr}\Big[2-U_{\mu\nu}(x)-U_{\nu\mu}(x)\Big].$$

As has been discussed in [Sugino], while the term here gives a unique minimun $U_{\mu\nu}(x)=1$, the piece above contains many classical vacua ± 1. This problem was resolved later on where and admissibility condition on the plaquette variable was included, similar to the one used for the Ginsparg-Wilson operator without spoiling the exact supersymmetry on the lattice.

Lattice Action for the Other Three Supercharges

We extended Sugino's formulation [A. Feo, in preparation] and showed that it is possible to construct other three supercharges that are nilpotent up to infinitesimal gauge transformations and we write the lattice action as an exact
$\tilde{Q}, Q_1, Q_2$-form.

The continuum $\tilde{Q}, Q_1, Q_2$ supercharges are given in
Kato, Kawamoto, Miyake, hep-th/0502119

Supersymmetry $\tilde{Q}$

Using the same naive discretization for the derivative,

$$
\begin{aligned}
&\tilde{Q}\psi_\mu(x) = i\epsilon_{\mu\nu}\Big(\phi(x) - U_\nu(x)\phi(x+\widehat{\nu})U_\nu^\dagger(x)\Big) - i\epsilon_{\mu\nu}\psi_\mu(x)\psi_\mu(x) \\
&\tilde{Q}U_\mu(x) = i\epsilon_{\mu\nu}\psi_\nu(x)U_\mu(x) \\
&\tilde{Q}\phi(x) = 0 \\
&\tilde{Q}\bar{\phi}(x) = 2\chi_{12}(x) \\
&\tilde{Q}B_{12}(x) = \frac{1}{2}[\phi(x), \eta(x)] \\
&\tilde{Q}\eta(x) = -2B_{12}(x) \\
&\tilde{Q}\chi_{12}(x) = \frac{1}{2}[\phi(x), \bar{\phi}(x)]
\end{aligned}
$$

$\tilde{Q}$ is nilpotent up to infinitesimal gauge transformations, in fact,

$$
\begin{aligned}
\tilde{Q}^2 U_\mu(x) &= \tilde{Q}(i\epsilon_{\mu\nu}\psi_\nu(x)U_\mu(x)) \\
&= i\epsilon_{\mu\nu}[i\epsilon_{\nu\rho}(\phi(x) - U_\rho(x)\phi(x+\widehat{\rho})U_\rho^\dagger(x)) - i\epsilon_{\nu\rho}\psi_\nu(x)\psi_\nu(x)]U_\mu(x) \\
&\quad - i\epsilon_{\mu\nu}\psi_\nu(x)(i\epsilon_{\mu\rho}\psi_\rho(x)U_\mu(x)) \\
&= (\phi(x)U_\mu(x) - U_\mu(x)\phi(x+\widehat{\mu}))
\end{aligned}
$$

and

$$\tilde{Q}^2\psi_\mu(x) = \tilde{Q}[i\epsilon_{\mu\nu}(\phi(x) - U_\nu(x)\phi(x+\widehat{\nu})U_\nu^\dagger(x)) - i\epsilon_{\mu\nu}\psi_\mu(x)\psi_\mu(x)]$$
$$= [\phi(x), \psi_\mu(x)]$$

The action can be written as a $\tilde{Q}$-variation of

$$S_{SYM}^{N=2} = \tilde{Q}\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Bigg[\frac{1}{2}\chi_{12}(x)[\phi(x), \bar{\phi}(x)] - \frac{1}{2}\eta(x)B_{12}(x) + \frac{i}{2}\eta(x)\Phi(x) + i\sum_{\mu,\rho}\epsilon_{\mu\rho}\psi_\rho(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\right)\Bigg]$$

and is $\tilde{Q}$-invariant since it is a $\tilde{Q}$-exact form.

Applying the $\tilde{Q}$-variation over the different pieces we get

$$S_{SYM}^{N=2} = \frac{1}{g_0^2}\int \mathrm{Tr}\Big[\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x)$$
$$- \frac{1}{4}\eta(x)[\phi(x),\eta(x)] + \chi_{12}(x)[\phi(x),\chi_{12}(x)] - \frac{i}{2}\eta(x)\tilde{Q}\Phi(x)$$
$$+ \sum_{\mu}\left(\phi(x) - U_\nu(x)\phi(x+\hat{\nu})U_\nu(x)^\dagger\right)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)$$
$$- 2i\sum_{\mu,\rho}\epsilon_{\mu\rho}\psi_\rho(x)\left(\chi_{12}(x) - U_\mu(x)\chi_{12}(x+\hat{\mu})U_\mu(x)^\dagger\right)$$
$$- \sum_{\mu,\rho}\psi_\rho(x)\psi_\rho(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)(1-\delta_{\mu\rho})\Big]$$

which is exactly the lattice $N = 2$ Super Yang-Mills with the change of variables

$$\psi_1(x) \to -\psi_2(x)\,,\psi_2(x) \to \psi_1(x)\,,\chi_{12}(x) \to \frac{1}{2}\eta(x)\,,\frac{1}{2}\eta(x) \to -\chi_{12}(x)$$

which corresponds to a transformation $\Psi \to \sigma_1\sigma_2\Psi$

where the fermionic fields components can be combined in a two-components Dirac spinor as

$$\Psi = -i \begin{pmatrix} \psi_1 + i\psi_2 \\ \chi_{12} + i\frac{\eta}{2} \end{pmatrix},$$

After applying these change of variables we get

$$\lim_{a \to 0} \left[-\frac{i}{2} \mathrm{Tr}(\eta(x)\tilde{Q}\Phi(x)) \right] = 2i\mathrm{Tr}[\chi_{12}(D_1\psi_2(x) - D_2\psi_1(x))],$$

and the action reduces to the continuum $N = 2$ SYM without fine tuning of any parameters of the action.

Supersymmetry Q_μ

We now show the algebra associated with the supercharge Q_μ, which can be naively discretized as,

$$
\begin{aligned}
&Q_\mu U_\nu(x) = i\varepsilon_{\mu\nu}\chi_{12}(x)U_\nu(x) - \frac{i}{2}\delta_{\mu\nu}\eta(x)U_\nu(x) \\
&Q_\mu \eta(x) = -2i\Big(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\Big) - \frac{1}{2}i\delta_{\mu\nu}\eta^2(x) \\
&Q_\mu \chi_{12}(x) = i\varepsilon_{\mu\nu}\Big(\bar{\phi}(x) - U_\nu(x)\bar{\phi}(x+\widehat{\nu})U_\nu^\dagger(x)\Big) + i\varepsilon_{\mu\nu}\chi_{12}^2(x) \\
&Q_\mu \psi_\nu(x) = \varepsilon_{\mu\nu}B_{12} + \frac{1}{2}\delta_{\mu\nu}[\phi(x), \bar{\phi}(x)] \\
&Q_\mu B_{12}(x) = [\varepsilon_{\mu\nu}\psi_\nu(x), \bar{\phi}(x)] \\
&Q_\mu \bar{\phi}(x) = 0 \\
&Q_\mu \phi(x) = 2\psi_\mu(x)\,.
\end{aligned}
$$

The terms $\frac{1}{2}\eta^2$ and χ_{12}^2 are $O(a)$ improved respect to the other ones thus, in the continuum limit they disappear and these lattice transformation goes to the continuum one.

One can close the algebra associated with Q_1 and Q_2, separately,

$$
\begin{aligned}
&Q_1U_1(x) = -\frac{i}{2}\eta(x)U_1(x)\,, \qquad Q_1U_2(x) = i\chi_{12}(x)U_2(x)\\
&Q_1\eta(x) = -2i\Big(\bar\phi(x) - U_1(x)\bar\phi(x+1)U_1^\dagger(x)\Big) - \frac{1}{2}i\eta^2(x)\\
&Q_1\chi_{12}(x) = i\Big(\bar\phi(x) - U_2(x)\bar\phi(x+2)U_2^\dagger(x)\Big) + i\chi_{12}^2(x)\\
&Q_1\psi_1(x) = \frac{1}{2}[\phi(x),\bar\phi(x)]\,, \qquad Q_1\psi_2(x) = B_{12}(x)\\
&Q_1B_{12}(x) = [\psi_2(x),\bar\phi(x)]\\
&Q_1\bar\phi(x) = 0\\
&Q_1\phi(x) = 2\psi_1(x)\,,
\end{aligned}
$$

where the following rules for Q_1^2 are satisfied,

$$
\begin{aligned}
Q_1^2\eta(x) &= Q_1\Big[-2i(\bar\phi(x) - (U_1(x)\bar\phi(x+1)U_1^\dagger(x)) - \frac{1}{2}i\eta^2(x)\Big] \\
&= [\eta(x), \bar\phi(x)]
\end{aligned}
$$

and

$$
\begin{aligned}
Q_1^2\chi_{12}(x) &= Q_1\Big[i(\bar\phi(x) - (U_2(x)\bar\phi(x+2)U_2^\dagger(x)) + i\chi_{12}^2(x)\Big] \\
&= [\chi_{12}(x), \bar\phi(x)]\,.
\end{aligned}
$$

Then we also have,

$$
\begin{aligned}
Q_1^2 U_1(x) &= Q_1(-\frac{1}{2}i\eta(x)U_1(x)) \\
&= -(\bar\phi(x)U_1(x) - U_1(x)\bar\phi(x+1))
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
Q_1^2 U_2(x) &= Q_1(i\chi_{12}(x)U_2(x)) \\
&= -(\bar\phi(x)U_2(x) - U_2(x)\bar\phi(x+2))\,.
\end{aligned}
$$

Since Q_1 is nilpotent up to infinitesimal gauge transformations, we can write the action as a Q_1-variation of,

$$S_{SYM}^{N=2} = Q_1 \frac{1}{2g_0^2} \sum_x \mathrm{Tr}\Big[\frac{1}{2}\psi_1(x)[\phi(x), \bar{\phi}(x)] + \psi_2(x) B_{12}(x) - i\psi_2(x)\Phi(x) + \frac{i}{2}\eta(x)\Big(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\Big) - i\chi_{12}(x)\Big(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\Big)\Big] .$$

Applying the Q_1-variation over the different fields we obtain

$$
\begin{aligned}
S_{SYM}^{N=2} = \frac{1}{2g_0^2} \sum_x \mathrm{Tr}\Bigg[& \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x) + i\psi_2(x) Q_1 \Phi(x) \\
& - \psi_1(x)[\psi_1(x), \bar{\phi}(x)] - \psi_2(x)[\psi_2(x), \bar{\phi}(x)] \\
& + \sum_\mu \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x) \right) \left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x) \right) \\
& + \frac{1}{4}\eta^2(x) \left(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x) \right) \\
& + \chi_{12}^2(x) \left(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x) \right) \\
& - i\eta(x) \left(\psi_1(x) - U_1(x)\psi(x+1)U_1^\dagger(x) \right) \\
& + 2i\chi_{12}(x) \left(\psi_1(x) - U_2(x)\psi_1(x+2)U_2^\dagger(x) \right) \\
& + \frac{1}{2}\eta^2(x)U_1(x)\phi(x+1)U_1^\dagger(x) \\
& + 2\chi_{12}(x)U_2(x)\phi(x+2)U_2^\dagger(x) \Bigg].
\end{aligned}
$$

This expression is the lattice $N = 2$ SYM action after a change variables,

$$\psi_1 \rightarrow \frac{1}{2}\eta\,, \psi_2 \rightarrow -\chi_{12}\,, \chi_{12} \rightarrow -\psi_2\,, \frac{1}{2}\eta \rightarrow \psi_1\,,$$

that corresponds a transformation $\Psi \rightarrow \sigma_2 \Psi$ if simultaneously change

$$\phi \leftrightarrow -\bar{\phi}$$

It reduces to the continuum supersymmetric limit without any fine tuning of the parameters of the action.

Supersymmetry Q_2

$$Q_2 U_1(x) = -i\chi_{12}(x) U_1(x)\,, \qquad Q_2 U_2(x) = -\frac{i}{2}\eta(x) U_2(x)$$
$$Q_2 \eta(x) = -2i\left(\bar\phi(x) - U_2(x)\bar\phi(x+2)U_2^\dagger(x)\right) - \frac{1}{2} i\eta^2(x)$$
$$Q_2 \chi_{12}(x) = -i\left(\bar\phi(x) - U_1(x)\bar\phi(x+1)U_1^\dagger(x)\right) - i\chi_{12}^2(x)$$
$$Q_2 \psi_1(x) = -B_{12}(x)\,, \qquad Q_2\psi_2(x) = \frac{1}{2}[\phi(x), \bar\phi(x)]$$
$$Q_2 B_{12}(x) = -[\psi_1(x), \bar\phi(x)]$$
$$Q_2 \bar\phi(x) = 0$$
$$Q_2 \phi(x) = 2\psi_2(x)\,,$$

and close the algebra in the following way,

$$
\begin{aligned}
Q_2^2\eta(x) &= Q_2[-2i(\bar{\phi}(x) - (U_2(x)\bar{\phi}(x+2)U_2^\dagger(x)) - \frac{1}{2}i\eta^2(x)] \\
&= [\eta(x), \bar{\phi}(x)]
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2^2\chi_{12}(x) &= Q_2[-i(\bar{\phi}(x) - (U_1(x)\bar{\phi}(x+1)U_1^\dagger(x)) - i\chi_{12}^2(x)] \\
&= [\chi_{12}(x), \bar{\phi}(x)]\,.
\end{aligned}
$$

Then we also have,

$$
\begin{aligned}
Q_2^2U_1(x) &= Q_2(-i\chi_{12}(x)U_1(x)) \\
&= -(\bar{\phi}(x)U_1(x) - U_1(x)\bar{\phi}(x+1))
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2^2U_2(x) &= Q_2(-\frac{1}{2}i\eta(x)U_2(x)) \\
&= -(\bar{\phi}(x)U_2(x) - U_2(x)\bar{\phi}(x+2))\,.
\end{aligned}
$$

The action can be written as an exact Q_2-variation of

$$S_{SYM}^{N=2} = Q_2 \frac{1}{2g_0^2} \sum_x \mathrm{Tr} \Bigg[\frac{1}{2} \psi_2(x) [\phi(x), \bar{\phi}(x)] - \psi_1(x) B_{12}(x) + i\psi_1(x)\Phi(x) + \frac{i}{2} \eta(x) \Big(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x) \Big) + i\chi_{12}(x) \Big(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x) \Big) \Bigg] .$$

and applying the transformations rule we have,

$$
\begin{aligned}
S_{SYM}^{N=2} = \frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Bigg[&\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x) - i\psi_1(x)Q_2\Phi(x) \\
&- \psi_2(x)[\psi_2(x),\bar{\phi}(x)] - \psi_1(x)[\psi_1(x),\bar{\phi}(x)] \\
&+ \sum_\mu \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\right)\left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x)\right) \\
&+ \frac{1}{4}\eta^2(x)\left(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\right) \\
&+ \chi_{12}^2(x)\left(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\right) \\
&- i\eta(x)\left(\psi_2(x) - U_2(x)\psi_2(x+2)U_2^\dagger(x)\right) \\
&- 2i\chi_{12}(x)\left(\psi_2(x) - U_1(x)\psi_2(x+1)U_1^\dagger(x)\right) \\
&+ \frac{1}{2}\eta^2(x)U_2(x)\phi_2(x+2)U_2^\dagger(x) \\
&+ 2\chi_{12}^2(x)U_1(x)\phi_1(x+1)U_1^\dagger(x)\Bigg].
\end{aligned}
$$

This expression is again the lattice $N = 2$ super Yang-Mills action with the change of variables,

$$\psi_1 \rightarrow \chi_{12}\,, \psi_2 \rightarrow \frac{1}{2}\eta\,, \chi_{12} \rightarrow \psi_1\,, \frac{1}{2}\eta \rightarrow \psi_2$$

and simultaneously,

$$\phi \leftrightarrow -\bar{\phi}\,,$$

that corresponds to a transformation, $\Psi \rightarrow \sigma_1 \Psi$,

and reduces to the continuum supersymmetric action without any fine tuning of the parameters of the action.

Lattice Action as a $Q\tilde{Q}$-form

A natural question that can be analyzed is whether more than one supercharge can be preserved exactly on the lattice using this formulation.

It is possible to write the $N = 2$ Super Yang-Mills action as a product of two supercharges Q and $\tilde{Q}$, which are separately exact on the lattice,

$$S_{SYM}^{N=2} = Q\tilde{Q}\frac{1}{2g_0^2}\sum_x \text{Tr}\Big[-\frac{1}{2}\eta(x)\chi_{12}(x) - \frac{i}{2}\bar{\phi}(x)\Phi(x)\Big].$$

Applying $\tilde{Q}$ we get,

$$= Q\frac{1}{2g_0^2}\sum_x \text{Tr}\Big[B_{12}(x)\chi_{12}(x) + \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\chi_{12}(x)\Phi(x) - \frac{i}{2}\bar{\phi}(x)\tilde{Q}\Phi(x)\Big].$$

The first three pieces correspond are OK, while the last term should be investigated more carefully and gives,

$$\sum_x \text{Tr}\Big[-\frac{i}{2}\bar{\phi}(x)\tilde{Q}\Phi(x)\Big] =$$

$$-\frac{1}{2}\sum_x \text{Tr}\bar{\phi}(x)\tilde{Q}\Big[U_1(x)U_2(x+1)U_1^\dagger(x+2)U_2^\dagger(x)$$

$$-U_2(x)U_1(x+2)U_2^\dagger(x+1)U_1^\dagger(x)\Big]$$

$$\equiv -\frac{i}{2}\sum_x \text{Tr}\bar{\phi}(x)F_1(x)\,.$$

Now applying Q we have,

$$\sum_x \text{Tr}\Big[-\frac{i}{2}\bar{\phi}(x)F_1(x)\Big] = -\frac{i}{2}\sum_x \text{Tr}\Big[\eta(x)F_1(x) + \bar{\phi}(x)QF_1(x)\Big]\,.$$

Let us investigate its continuum limit:

In the limit $a \to 0$ the first piece gives,

$$-\frac{i}{2}\sum_x \mathrm{Tr}\Big[\eta(x)F_1(x)\Big] \approx_{a\to 0} i\sum_x \mathrm{Tr}\Big[\eta(x)D_\mu\psi_\mu(x)\Big]$$

which is order $O(1)$. Integrating by part we obtain

$$i\psi_\mu(x)D_\mu\eta(x)\,.$$

While the second piece

$$-\frac{i}{2}\sum_x \mathrm{Tr}\bar{\phi}(x)QF_1(x) \approx_{a\to 0}\,,$$

gives two pieces:

$$\sum_x \mathrm{Tr}\psi_\mu(x)[\bar{\phi}(x),\psi_\mu(x)]$$

and

$$a^2\sum_x \mathrm{Tr}(\partial_\mu\bar{\phi}(x)+i[A_\mu(x),\phi(x)])(\partial_\mu\bar{\phi}(x)+i[A_\mu(x),\phi(x)]) = a^2\sum_x \mathrm{Tr}D_\mu\bar{\phi}(x)D_\mu\phi(x)\,.$$

Collecting all terms we obtain the classical continuum action without fine tuning even if we used a $Q\tilde{Q}$-form where the Q and $\tilde{Q}$ do not satisfy on the lattice the condition,

$$\left\{Q, \tilde{Q}\right\} = 0\,.$$

(two dimensions?)

Conclusions & Perspectives

A big effort has been made it order to describe supersymmetry on the lattice.

Traditional Wilson fermions have been used in realistic computations with nice results.

Improved chiral fermions results are starting too.

Exact supersymmetry on the lattice have been achieved for 'simple' models as Wess-Zumino and, on the other side, for $N = 2$ two dimensional SYM theory.

$N = 1$ SYM in $d = 4$??

GRAZIE per la vostra attenzione!!