Elementi di Meccanica Computazionale
Corso di Laurea in Ingegneria Civile
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An introduction to the course

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Some references I

- O.C.Zienkiewicz and R.L.Taylor, *The Finite Element Method*, vol.1 & vol. 2, Butterworth-Heinemann (2005)
- T.J.R.Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Dover (2000)

- E.Onate, *Structural Analysis with the Finite Element Method. Linear Statics: Volume 1: Basis and Solids*, Springer (2013)
- E.Onate, *Structural Analysis with the Finite Element Method. Linear Statics: Volume 2: Beams, Plates and Shells*, Springer (2013)

- N.S.Ottosen and H.Petersson, *Introduction to the Finite Element Methods*, Prentice Hall (1992)
- J.Fish and T.Belytschko, *A first course in Finite Elements*, Wiley (2007)

- Course notes

Some computational tools I

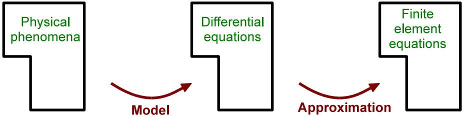
- **Sage**
  - Free open-source mathematics software system licensed under the GPL
  - Combines the power of many existing open-source packages into a common Python-based interface
  - Mission: Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab.
  - http://www.sagemath.org/
  - http://www.sagemath.org/pdf/SageTutorial.pdf

- **Matlab, Maple**
  - Non open-source software for some initial developments and computations

- **FEAPpv** (Finite Element Analysis Program - Personal Version)
  - Fortran-based computer analysis system designed for:
    - use in an instructional program to illustrate performance of different types of elements and modeling methods
    - in a research, and/or application environment which requires frequent modifications to address new problem areas or analysis requirements
  - Able to solve a wide variety of problem in linear and nonlinear continuum mechanics
  - Open code, download from web, possible to extend and modify it

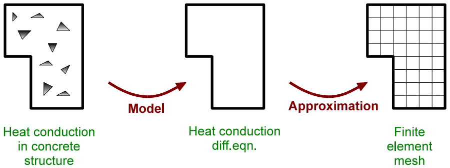
An introduction to FEM (from Ottosen and Petersson!) I

- All physical phenomena encountered in engineering are modeled by differential equations
- Differential equations, describing a physical problem, hold over a either 1D, 2D or 3D region
- In general, addressed problems are too complicated to be solved by classical analytical methods
- **Finite element method** (**FEM**) is a numerical approach, by which general differential equations are solved in an approximate manner

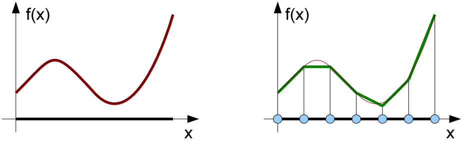


An introduction to FEM (from Ottosen and Petersson!) II

- **FEM characteristic feature:** instead of looking for a approximation holding over entire region, divide region into smaller parts (**finite-elements**) & introduce a rather simple approximation for each element
- **Finite element mesh**: collection of all elements
- **Nodal points**: nodes defining domain subdivision into elements



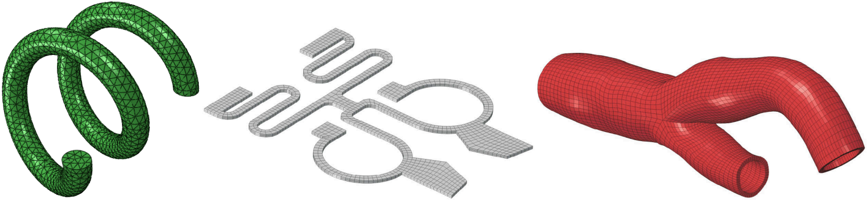
- Even if a variable varies in highly non-linear manner over the entire region, fair to assume that it varies in a linear fashion over each (small) element



An introduction to FEM (from Ottosen and Petersson!) III

- Finite element approximation depends on a limited discrete number of unknowns (**degrees of freedoms [dofs]**)
- A system with a finite number of dofs is known as discrete system, in contrast to the original continuous system which has an infinite number of dofs

- Some examples of realistic problems



A FEM introduction: 1D spring problem I

★ **Finite element key ingredient:**

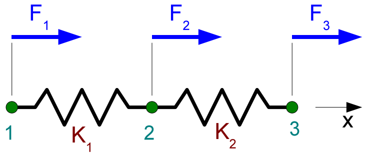
- divide body into small regions (elements)
- definition of simple behavior occurring on each element
- technique to patch together informations relative to single elements

patching together element behavior,
enable to predict response of the whole body

- Later on we discuss how it is possible to obtain a FE formulation starting from a general differential equation
- However, in some situations the behavior of single elements is very simple (spring & truss & beam)
- In the following consider the response of structures made of elastic springs

A FEM introduction: 1D spring problem II

- Consider a simple structure made of two springs, respectively of k_1 and k_2 stiffness



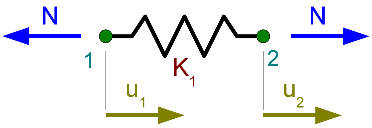
- Structure loaded by three forces, supposed to be applied to nodes

- In the finite-element spirit, natural to consider the above structure as made of:
  - Two elements and three nodes
  - Element 1 delimited by nodes 1-2, element 2 delimited by nodes 2-3
  - System has three dofs, identified with the horizontal nodal displacements

$$u_1\,,\, u_2\,,\, u_3$$

A FEM introduction: 1D spring problem III

- Focus on a specific element and try to characterize its equilibrium

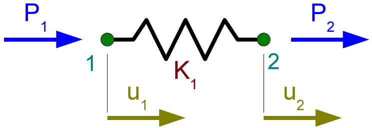


  - Very simple considerations tell that force N within the element is given by

$$N = k_1(u_2 - u_1)$$

    assuming positive displacements in the x-axis direction and positive internal force if tensile

- Consider now the same element but with a different approach



  - P_1 and P_2 are element forces, i.e. forces acting on the element, clearly possibly different than external applied loads
  - Spring equilibrium requires

$$P_1 + P_2 = 0$$

A FEM introduction: 1D spring problem IV

- We clearly have

$$P_2 = \quad N = k_1(u_2 - u_1)$$
$$P_1 = -N = k_1(u_1 - u_2)$$

- Possible to combine previous equation into

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} P_1 \\ P_2 \end{array} \right\} \tag{1}$$

or in matrix form:

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e$$

with

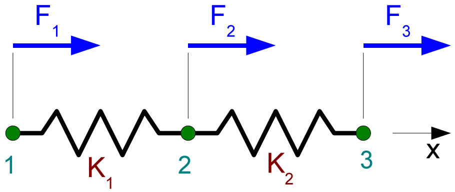
$$\begin{cases} \mathbf{K}^e = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \text{element stiffness} \quad (\textit{symmetric}!!) \\ \mathbf{u}^e = \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} & \text{element displacement vector} \\ \mathbf{f}^e = \left\{ \begin{array}{c} P_1 \\ P_2 \end{array} \right\} & \text{element force vector} \end{cases}$$

- Equation **??** fully characterizes response of spring 1
- Equation **??** represents equilibrium of spring 1
- Same considerations for spring 2

A FEM introduction: 1D spring problem V

- We many now consider **global equilibrium of nodal points**
- To do so, we need to distinguish between
  - actions of elements on nodes
  - actions of each single element ($e = 1$ & $e = 2$)
- Equilibrium equations for single nodes:

$$\begin{cases} F_1 - P_1^{e=1} = 0 \\ F_2 - P_2^{e=1} - P_1^{e=2} = 0 \\ F_3 - P_2^{e=2} = 0 \end{cases}$$



which can be rewritten as:

$$\begin{Bmatrix} P_1^{e=1} \\ P_2^{e=1} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ P_1^{e=2} \\ P_2^{e=2} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

A FEM introduction: 1D spring problem VI

- Making explicit the internal force actions

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad (2)$$

  which can be written in matrix form as

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

  where

$$\begin{cases} \mathbf{K} & \text{global stiffness matrix} \\ \mathbf{u} & \text{global displacement vector} \\ \mathbf{f} & \text{global force vector} \end{cases}$$

- **Note:** global stiffness matrix is symmetric

A FEM introduction: 1D spring problem VII

- **Note:** $\mathbf{K}$ is constructed from $\mathbf{K}^{e=1}$ and $\mathbf{K}^{e=2}$, however

$$\mathbf{K} \neq \mathbf{K}^{e=1} + \mathbf{K}^{e=2}$$

  but

$$\boxed{\mathbf{K} = \mathbf{A}\left(\mathbf{K}^{e=1}, \mathbf{K}^{e=2}\right)}$$

  with $\mathbf{A}$ a special patch operator (**assembly**) for finite elements

- In fact

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^{e=1} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{K}^{e=2} \end{bmatrix}$$

- Global relations are explicitly constructed enforcing equilibrium of each single dof
- In the assembly process we have also implicitly used compatibility conditions between elements, i.e. elements are connected and share the nodes

A FEM introduction: 1D spring problem VIII

- ★ If we assign nodal displacements, Equation **??** can be used to compute forces that needed to be imposed to guarantee equilibrium
- ★ If we assign nodal forces, Equation **??** can be used to compute unknown displacements, solving the given linear system

- However, it can easily seen that

$$\det(\mathbf{K}) = 0$$

- This condition is clear since so far we have only enforced global equilibrium of the system; therefore, if we satisfy Equation **??**, the system will be in equilibrium.
- However, to an equilibrium condition there corresponds a family of infinite possible displacements. Any two elements of such a family differ by a rigid body motion.

- To remove, such indeterminacy in terms of displacements, we may fix a **boundary condition**, i.e. we may specify a value to a nodal displacement.

A FEM introduction: 1D spring problem IX

- Assume for example

$$u_1 = 0$$

- We now have only two unknowns (u_2 and u_3) and the need of writing only two equilibrium equations, extracted from **??**

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \tag{3}$$

  - The determinant associated to this new system is different than zero

$$det \left(\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \right) = (k_1 + k_2)k_2 - k_2 k_2 = k_1 k_2 \neq 0$$

  - The matrix is also definite positive

$$\left\{ \; x_1 x_2 \; \right\} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = k_1 x_1^2 + k_2 (x_1 - x_2)^2 > 0$$

$$\text{for all} \qquad x_1 \;\&\; x_2$$

A FEM introduction: 1D spring problem X

- From Equation **??** it is possible to compute the solution

$$\begin{cases} u_2 = \dfrac{1}{k_1}(F_2 + F_3) \\ u_3 = \dfrac{F_2}{k_1} + \left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) F_3 \end{cases}$$

- Once computed the solution, it is also possible to re-use the first equation of **??**

$$k_1 u_1 - k_1 u_2 = F_1$$

enforcing equilibrium of node 1, to compute the reaction force:

$$\text{reaction force} \qquad F_1 = -k_1 u_2 = -(F_2 + F_3)!!$$

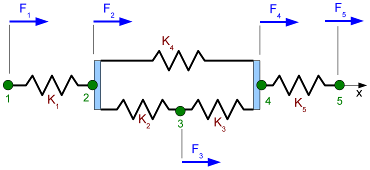
- To guarantee global equilibrium, reaction force on node 1 should be equal and opposite to the sum of the forces acting on nodes 2 and 3 !!

A FEM introduction: 1D spring problem XI

- Approach adopted for the spring system is an example of matrix structural analysis
- Often also indicated as displacement method since the displacements are chosen as unknown or stiffness method since it is mainly based on the construction of the stiffness of the single elements
- Apart from the construction of the stiffness relation for the single elements (extremely simple for the case under investigation), the followed steps are key steps of the finite-element method
  - Establish stiffness relation for each element
  - Enforce compatibility, i.e. connect elements
  - Enforce equilibrium (assembling)
  - Enforce boundary conditions
  - Solve system of equations
- Clearly, the finite element method is very general and the way in which it is presented here is simplified and relative to a specific application
- Later on we present a more flexible and general way for deriving finite element approximations for differential equations

A FEM introduction: 1D spring problem XII

- **Exercise.** Consider the following structure consisting of five springs



Show that the equilibrium equations are in the following format:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2+k_4 & -k_2 & -k_4 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_4 & -k_3 & k_3+k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$