Elementi di Meccanica Computazionale
Corso di Laurea in Ingegneria Civile
Pavia, 2014

Linear beams: extension to dynamics and finite element formulation

Ferdinando Auricchio [1] [2]

[1]DICAR – Dipartimento di Ingegneria Civile e Architettura, Università di Pavia, Italy
[2]IMATI – Istituto di Matematica Applicata e Tecnologie Informatiche, CNR, Italy

March 13, 2014

Some references I

- A. Chopra, "Dynamics of Structures: Theory and Applications to Earthquake Engineering (2nd Edition) ", Prentice Hall, 2000

Beam: extension to dynamics I

GOAL: extend beam formulation to include dynamics

1. **Weak formulation of the problem**
   Differential form $\rightarrow$ Integro-differential form
2. **Introduction of approximation fields**
   Integro-differential form $\rightarrow$ Algebraic-differential form
3. **Introduction of time-marching scheme**
   Algebraic-differential form $\rightarrow$ Algebraic form
4. **Solution of algebraic problem**

Beam: extension to dynamics at the continuous level I

- **Strong form for static conditions**
  - Assume $v = v(x)$ with x longitudinal abscissa
  - Distinct strong equation

$$\begin{cases} v' = \theta \\ \chi = \theta' \\ M = EI\chi \\ M' - S = 0 \\ S' - q = 0 \end{cases}$$

  - Compact strong form

$$\boxed{EIv^{IV} - q = 0}$$

Beam: extension to dynamics at the continuous level II

- **Strong form for dynamic conditions**
  - Assume $v = v(x,t)$ with t indicating time
  - All previous equations valid except for last one (transverse equilibrium) which should be modified as follow:

$$S' - q^{dyn} = 0$$

setting now

$$q^{dyn} = q^{stat} - \rho_v \ddot{v}$$

where

  - q^{dyn}: equivalent load under dynamical conditions
  - q^{stat}: externally applied load (static load)
  - ρ_v: transverse mass density (per unit length) given by

$$\rho_v = \int_{-h/2}^{h/2} \rho b dy = \rho A$$

with ρ mass density per unit volume, b cross-section depth and A cross-section area

  - Compact strong form

$$\boxed{EIv^{IV} - q^{dyn} = 0} \quad \text{or} \quad \boxed{EIv^{IV} + \rho_v \ddot{v} - q^{stat} = 0}$$

- Approach known as d'Alembert principle

Beam: dynamical weak form I

- **Weak form**
  - Assume to work with weight function $w = w(x)$
  - ★ **Note:** weight function independent of time !!!!
  - ★ **Note:** integral form to be valid for all times !!!!
  - Multiply differential form by w and integrate over domain $[a, b]$

$$\int_a^b \left[w \left(EIv^{IV} + \rho_v \ddot{v} - q^{stat} \right) \right] dx = 0$$

  - Integrate twice by parts the first term

$$\boxed{\int_a^b \left[w'' EIv'' \right] dx + \int_a^b \left[w \rho_v \ddot{v} \right] dx - \int_a^b \left[wq \right] dx + \left. Sw \right|_a^b - \left. Mw' \right|_a^b = 0}$$

Beam: dynamical weak form II

- In compact form, weak form can be written as

$$\boxed{f = f^{int} + f^{inert} - f^{ext} = 0}$$

with

$$\begin{cases} f^{int} = \int_a^b \left[w'' EIv'' \right] dx \\ f^{inert} = \int_a^b \left[w \rho_v \ddot{v} \right] dx \\ f^{ext} = \int_a^b \left[wq \right] dx - Sw|_a^b + Mw'|_a^b \end{cases}$$

- Interesting to observe that

$$f = f^{int}(v) + f^{inert}(\ddot{v}) - f^{ext}(v) = 0$$

therefore

$$f(v, \ddot{v}) = 0$$

Beam: extension to dynamics at the continuous level I

- **Approximated weak form**
  - Introduce an approximation for weight function and unknown function

$$\begin{cases} w(x) = \sum_{i=1}^{n} N_i(x)\hat{w}_i \\ v(x,t) = \sum_{j=1}^{n} N_j(x)\hat{v}_j(t) \end{cases}$$

    - ⋆ N_i: shape functions depending only on spatial coordinate x
    - ⋆ $\hat{w}_i$: arbitrary parameters independent of time t
    - ⋆ $\hat{v}_i$: unknown parameters depending on time t
  - As a consequence

$$\ddot{v}(x,t) = \sum_{j=1}^{n} N_j(x)\ddot{\hat{v}}_j(t)$$

  - In matrix form

$$\begin{cases} w(x) = \mathbf{N}(x)\hat{\mathbf{w}} \\ v(x,t) = \mathbf{N}(x)\hat{\mathbf{v}}(t) \\ \ddot{v}(x,t) = \mathbf{N}(x)\ddot{\hat{\mathbf{v}}}(t) \end{cases} \quad \text{as well as} \quad \begin{cases} w''(x) = \mathbf{B}(x)\hat{\mathbf{w}} \\ v''(x,t) = \mathbf{B}(x)\hat{\mathbf{v}}(t) \end{cases}$$

Beam: extension to dynamics at the continuous level II

- Assume from now only trivial boundary conditions
- Neglect to explicitly indicate spatial and time dependence
- Plugging approximations into weak form

$$\int_a^b \left[(\mathbf{B}\hat{\mathbf{w}})\, EI\, (\mathbf{B}\hat{\mathbf{v}})\right] dx + \int_a^b \left[(\mathbf{N}\hat{\mathbf{w}})\, \rho_v \left(\mathbf{N}\ddot{\hat{\mathbf{v}}}\right)\right] dx - \int_a^b \left[(\mathbf{N}\hat{\mathbf{w}})\, q\right] dx = 0$$

which can be rewritten as

$$\hat{\mathbf{w}}^T \left\{ \int_a^b \left[\mathbf{B}^T EI\, (\mathbf{B}\hat{\mathbf{v}})\right] dx + \int_a^b \left[\mathbf{N}^T \rho_v \left(\mathbf{N}\ddot{\hat{\mathbf{v}}}\right)\right] dx - \int_a^b \left[\mathbf{N}^T q\right] dx \right\} = 0$$

- Recalling the arbitrariness of $\hat{\mathbf{w}}$, we have

$$\int_a^b \left[\mathbf{B}^T EI\, (\mathbf{B}\hat{\mathbf{v}})\right] dx + \int_a^b \left[\mathbf{N}^T \rho_v \left(\mathbf{N}\ddot{\hat{\mathbf{v}}}\right)\right] dx - \int_a^b \left[\mathbf{N}^T q\right] dx = \mathbf{0}$$

Beam: extension to dynamics at the continuous level III

- In compact form dynamical equilibrium can be written as

$$\boxed{\mathbf{f} = \mathbf{f}^{int} + \mathbf{f}^{inert} - \mathbf{f}^{ext} = \mathbf{0}}$$

where

$$\begin{cases} \mathbf{f}^{int} = \int_a^b \left[\mathbf{B}^T EI \left(\mathbf{B}\hat{\mathbf{v}} \right) \right] dx \\ \mathbf{f}^{inert} = \int_a^b \left[\mathbf{N}^T \rho_v \left(\mathbf{N}\ddot{\hat{\mathbf{v}}} \right) \right] dx \\ \mathbf{f}^{ext} = \int_a^b \left[\mathbf{N}^T q \right] dx \end{cases}$$

- In a more standard form dynamical equilibrium can be written as

$$\boxed{\mathbf{K}\hat{\mathbf{v}} + \mathbf{M}\ddot{\hat{\mathbf{v}}} = \mathbf{f}^{ext}}$$

with

$$\begin{cases} \mathbf{K} = \int_a^b \left[\mathbf{B}^T EI \mathbf{B} \right] dx & \text{stiffness matrix} \\ \mathbf{M} = \int_a^b \left[\mathbf{N}^T \rho_v \mathbf{N} \right] dx & \text{“consistent” mass matrix} \end{cases}$$

Beam: extension to dynamics at the continuous level IV

- The terminology consistent is introduced to stress that the mass matrix is obtained using a consistent variational formulation
- The terminology consistent is often used in contrast with the terminology lumped which is in general non-consistent and diagonal!!

- Using previously introduced shape functions

$$\mathbf{M} = \rho_v \begin{bmatrix} \frac{13}{35}l & \frac{11}{210}l^2 & \frac{9}{70}l & -\frac{13}{420}l^2 \\ \frac{11}{210}l^2 & \frac{1}{105}l^3 & \frac{13}{420}l^2 & -\frac{1}{140}l^3 \\ \frac{9}{70}l & \frac{13}{420}l^2 & \frac{13}{35}l & -\frac{11}{210}l^2 \\ -\frac{13}{420}l^2 & -\frac{1}{140}l^3 & -\frac{11}{210}l^2 & \frac{1}{105}l^3 \end{bmatrix}$$

☐ **Exercise.** Using a symbolic code verify expression for the Euler-Bernoulli beam mass matrix

☐ **Exercise.** Compute the determinants of the element mass matrix and of the element stiffness matrix. Comment on the fact that the element mass matrix is non-singular while the element stiffness matrix is singular.

Beam: extension to dynamics at the continuous level I

- Extend dynamical strong equilibrium to include also a linear viscous term

$$\boxed{EIv^{IV} + \rho_v \ddot{v} + \mu \dot{v} - q^{stat} = 0}$$

- Corresponding weak form produces the following system

$$\boxed{\mathbf{M}\ddot{\hat{\mathbf{v}}} + \mathbf{C}\dot{\hat{\mathbf{v}}} + \mathbf{K}\hat{\mathbf{v}} = \mathbf{f}^{ext}} \qquad (1)$$

with

$$\begin{cases} \mathbf{M} = \displaystyle\int_a^b \left[\mathbf{N}^T \rho_v \mathbf{N}\right] dx & \text{mass matrix} \\ \mathbf{C} = \displaystyle\int_a^b \left[\mathbf{N}^T \mu \mathbf{N}\right] dx & \text{damping matrix} \\ \mathbf{K} = \displaystyle\int_a^b \left[\mathbf{B}^T EI \mathbf{B}\right] dx & \text{stiffness matrix} \end{cases}$$

- In structural analysis often **C** is assumed to be in the form

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}$$

with α and β to be determined experimentally (Rayleigh damping)

Dynamic problem: problem classification I

- As a result of semi-discretization in space, time-dependent problem is reduced to a set of ordinary differential equations

- In the following discuss three different possible situations:
  - Determination of free-response (i.e. $\mathbf{f}^{ext} = \mathbf{0}$)
  - Determination of steady-state periodic response (i.e. $\mathbf{f}^{ext}$ periodic in time)
  - Determination of transient response (i.e. $\mathbf{f}^{ext}$ arbitrary in time)

Dynamic problem: free-response / eigenvalue problem I

- From now on consider always discrete problems (i.e. simplify notation)
- If no damping or forcing terms exist, dynamical problem 1 reduces as

$$\boxed{\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}} \tag{2}$$

- A general solution can be written in the form

$$\mathbf{u} = \bar{\mathbf{u}} \exp(i\omega t)$$

with the corresponding real part representing an harmonic response since

$$\exp(i\omega t) = \cos(\omega t) + i \sin(\omega t)$$

- Back-substitution into equation 2 returns

$$\left(-\omega^2 \mathbf{M} + \mathbf{K}\right) \bar{\mathbf{u}} = \mathbf{0} \tag{3}$$

i.e. a general linear eigenvalue or characteristic value problem and for non-zero solutions the determinant of the above coefficient matrix must be zero

$$\boxed{\det\left[-\omega^2 \mathbf{M} + \mathbf{K}\right] = 0} \tag{4}$$

Dynamic problem: free-response / eigenvalue problem II

- Provided that $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$ symmetric-definite matrices, condition 4 gives n positive roots or eigenvalues ω_i with $i = 1..n$
- For each eigenvalue, using equation 3, possible to compute system eigenvectors or normal modes $\bar{\mathbf{u}}_i$, in general requiring a proper normalization, i.e. s.t.

$$\boxed{\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_i = 1 \qquad \text{for} \quad i = 1..n}$$

- At this stage also possible to introduce classical modal orthogonality condition

$$\boxed{\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_j = \bar{\mathbf{u}}_j^T \mathbf{M} \bar{\mathbf{u}}_i = 0 \quad \text{for} \quad i \neq j}$$

as well as to prove the result

$$\bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_i = \omega_i^2$$

Dynamic problem: free-response / eigenvalue problem III

- **Exercise.** Using a symbolic code prove that using Euler-Bernoulli shape functions a beam element is characterized by the following eigenvalues

$$\omega_1 = \frac{720}{l^4} \quad , \quad \omega_2 = \omega_3 = 0 \quad , \quad \omega_4 = \frac{8400}{l^4}$$

  and by the following eigenvectors

$$\bar{\mathbf{u}}_1 = \left\{ \begin{array}{r} l/6 \\ -1 \\ l/6 \\ 1 \end{array} \right\} \quad , \quad \bar{\mathbf{u}}_2 = \left\{ \begin{array}{r} -l \\ 1 \\ 0 \\ 1 \end{array} \right\} \quad , \quad \bar{\mathbf{u}}_3 = \left\{ \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right\} \quad , \quad \bar{\mathbf{u}}_4 = \left\{ \begin{array}{r} -l/12 \\ 1 \\ l/12 \\ 1 \end{array} \right\}$$

- **Exercise.** Verify normality and orthogonality conditions on the given eigenvectors.
- **Exercise.** Comment on the dimensionality of the terms appearing in each single eigenvalue as well as on the shape of each single eigenvector.

Dynamic problem: free-response / eigenvalue problem IV

- To preserve methods based on the invertibility of stiffness matrix, possible to use an artifice
- Modify problem

$$\left[-\omega^2 \mathbf{M} + \mathbf{K} \right] \bar{\mathbf{u}} = \mathbf{0}$$

  into

$$\left[-\left(\omega^2 + \alpha \right) \mathbf{M} + (\mathbf{K} + \alpha \mathbf{M}) \right] \bar{\mathbf{u}} = \mathbf{0}$$

  where α is an arbitrary constant of the same order of the typical ω^2 sought
- The new matrix $\mathbf{K} + \alpha \mathbf{M}$ is no more singular

Dynamic problem: damped free-response / eigenvalue problem I

- If damping exists but no no forcing terms exist , dynamical problem 1 reduces as

$$\boxed{\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}} \tag{5}$$

- A general solution can again be written in the form

$$\mathbf{u} = \bar{\mathbf{u}} \exp(\alpha t)$$

- Back-substitution into equation 5 returns

$$\left(\alpha^2 \mathbf{M} + \alpha \mathbf{C} + \mathbf{K}\right) \bar{\mathbf{u}} = \mathbf{0} \tag{6}$$

  with α and $\bar{\mathbf{u}}$ possibly complex quantities, with a real part representing a decaying vibration

- Equation 6 corresponds again to an eigenvalue problem, which is more complex than the one previously considered

Dynamic problem: damped free-response / eigenvalue problem II

- Usually the problem is solved by splitting it into two first-order problems, defining

$$\dot{\mathbf{u}} = \mathbf{v}$$

s.t. we get

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \left\{ \begin{array}{c} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{array} \right\} + \begin{bmatrix} \mathbf{C} & \mathbf{K} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{v} \\ \mathbf{u} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \end{array} \right\}$$

- Setting now

$$\mathbf{u} = \bar{\mathbf{u}} \exp(\alpha t) \quad , \quad \mathbf{v} = \bar{\mathbf{v}} \exp(\alpha t)$$

and substituting gives the general linear eigenvalue

$$\boxed{\left(\alpha \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{K} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \right) \left\{ \begin{array}{c} \bar{\mathbf{v}} \\ \bar{\mathbf{u}} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \end{array} \right\}} \qquad (7)$$

Dynamic problem: forced periodic response I

- If in equation 1 the forcing term is periodical, i.e.

$$\mathbf{f} = \bar{\mathbf{f}} \exp(\alpha t)$$

with $\alpha = \alpha_1 + i\alpha_2$, then general solution can be again written as

$$\mathbf{u} = \bar{\mathbf{u}} \exp(\alpha t)$$

which plugged into 1 gives

$$\left(\alpha^2 \mathbf{M} + \alpha \mathbf{C} + \mathbf{K}\right) \bar{\mathbf{u}} = \bar{\mathbf{f}}$$

- Obtained problem is no more an eigenvalue problem and it can simply solved as

$$\bar{\mathbf{u}} = \tilde{\mathbf{K}}^{-1} \bar{\mathbf{f}}$$

where

$$\tilde{\mathbf{K}} = \alpha^2 \mathbf{M} + \alpha \mathbf{C} + \mathbf{K}$$

Dynamic problem: transient response procedures I

- To treat the response of a linear dynamical system to a general force we can adopt two approaches:
  - frequency response procedures
  - numerical time-integration

★ Start considering **frequency response procedures**

- In general a completely arbitrary forcing function can be represented approximately by a Fourier series or in the limit, exactly, as a Fourier integral
- Accordingly, response to a completely arbitrary forcing function can be obtained properly combining the response to a series of periodic forcing functions
- The frequency response technique is readily adapted to problems where the matrix $\mathbf{C}$ is of an arbitrary form

Dynamic problem: modal decomposition analysis I

- For free response the general solution is of the form

$$\mathbf{u} = \sum_i^n \bar{\mathbf{u}}_i \exp(\alpha_i t)$$

where α_i are the complex eigenvalues and $\mathbf{u}_i$ are the complex eigenvectors.

- For forced response and linear problem, the solution can be again expressed as a linear combination of the modes

$$\mathbf{u} = \sum_i^n \bar{\mathbf{u}}_i y_i(t) \tag{8}$$

with y_i **scalar modal partecipation factors**, assumed to be function of time

- Position 8 presents no restriction as all the modes are linearly indipendent

Dynamic problem: modal decomposition analysis II

- Using position 8 in dynamical equation and premultiplying by the complex conjugate eigenvectors $\bar{\mathbf{u}}_i^T$, the result is simply a set of scalar, indipendent equations

$$\boxed{m_i \ddot{y} + c_i \dot{y} + k_i y_i = f_i} \tag{9}$$

where

$$m_i = \bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_i \quad , \quad c_i = \bar{\mathbf{u}}_i^T \mathbf{C} \bar{\mathbf{u}}_i \quad , \quad k_i = \bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_i \quad , \quad f_i = \bar{\mathbf{u}}_i^T \mathbf{f}$$

since we have

$$\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_j = \bar{\mathbf{u}}_i^T \mathbf{C} \bar{\mathbf{u}}_j = \bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_j = 0 \qquad \text{when} \qquad i \neq j$$

- Each scalar equation 9 can be solved indipendently by elementary procedures
- Total motion is obtained by **superposition** using 8

Dynamic problem: modal decomposition analysis III

- Most classical approach is to work with real eigenvalues obtained as solution of

$$\left[-\omega^2\mathbf{M} + \mathbf{K}\right]\bar{\mathbf{u}} = \mathbf{0}$$

- With this position decoupled equations only in the case

$$\bar{\mathbf{u}}_i\mathbf{C}\bar{\mathbf{u}}_j = 0 \quad \text{if} \quad i \neq j$$

  which is a non-trivial condition since eigenvectors in general guarantee orthogonality wrt **M** and **K** and not necessarily wrt **C**

- If using damping matrix in the form of Rayleigh damping, then trivial.

Exercises I

- **Exercise.** Modify the frame Matlab code to compute also mass matrix, structural eigenvalues and eigenmodes.
- **Exercise.** For some simple beam problems (i.e. simply supported or cantilever beams), compute eigenvalue and eigenvector solutions in close form and compare with numerical solutions obtained for an increasing number of elements. Plot convergence diagrams in a log-log-scale.
- **Exercise.** Choose a structure and compute free response comparing it with a standard commercial software.
- **Exercise.** Develop mass matrix for a Timoshenko beam model

MDOF numerical solution I

- Analytical solutions previously described provide insights into the behaviour patterns of linear systems
  - Not computationally economical for the solution of transient problems
  - Cannot be extended to nonlinear cases

- We now investigate discretization methods directly applicable to time-domain
- Since time domain is infinite, refer to a finite time increment Δt between t_n and $t_{n+1} = t_n + \Delta t$, obtaining the so-called recurrence relations

MDOF numerical solution II

- **GOAL:** solve equation of motion for a general load condition and possibly for general internal force response
- Time interval of interest: $[0, T]$
- Discretize time interval in N time sub-intervals of size Δt, such that

$$t_0 = 0, \; t_1 = \Delta t, \; t_2 = 2\Delta t, \; .... \; t_N = N\Delta t = T$$

  or assume variable time sub-intervals

- Resort to time-marching numerical scheme to compute motion in time, i.e. assuming to know solutions up to t_n, compute solution at t_{n+1}

- Look for numerical solutions with a **bounded error** wrt exact solution, i.e.

$$\mathbf{u}_{n+1} \approx \mathbf{u}(t_{n+1}) \qquad \text{such that} \qquad \mathbf{u}_{n+1} = \mathbf{u}(t_{n+1}) + O(\Delta t^k)$$

  then time integration method is **k-order accurate**

- Second order accurate methods are desirable !!!!

MDOF numerical solution: central difference method I

- Equation of motion at time t_n

$$\mathbf{M}\ddot{\mathbf{u}}_n + \mathbf{C}\dot{\mathbf{u}}_n + \mathbf{K}\mathbf{u}_n = \mathbf{f}_n \tag{10}$$

- **Central difference approximation** of first derivative

$$\dot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - \mathbf{u}_{n-1}}{2\Delta t}$$

- Central difference approximation of second derivative

$$\ddot{\mathbf{u}}_n = \frac{\dot{\mathbf{u}}_{n+\frac{1}{2}} - \dot{\mathbf{u}}_{n-\frac{1}{2}}}{\Delta t} = \frac{\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \frac{\mathbf{u}_n-\mathbf{u}_{n-1}}{\Delta t}}{\Delta t}$$
$$= \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{(\Delta t)^2}$$

MDOF numerical solution: central difference method II

- Substituting positions 28 and 28 into equation of motion 10, we obtain:

$$\left[\frac{1}{(\Delta t)^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n+1} = \mathbf{f}_n - \left[\frac{1}{(\Delta t)^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n-1} - \left[\mathbf{K} - \frac{2}{(\Delta t)^2}\mathbf{M}\right]\mathbf{u}_n \tag{11}$$

i.e.

$$\bar{\mathbf{K}}\mathbf{u}_{n+1} = \bar{\mathbf{f}}_n$$

where

$$\begin{cases} \bar{\mathbf{K}} = \left[\frac{1}{(\Delta t)^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right] \\ \bar{\mathbf{f}}_n = p_n - \left[\frac{1}{(\Delta t)^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n-1} - \left[\mathbf{K} - \frac{2}{(\Delta t)^2}\mathbf{M}\right]\mathbf{u}_n \end{cases}$$

- **Note: explicit method** since restoring force based only on quantities evaluated at t_n
- Easy to extend to non-linear problems

MDOF numerical solution: central difference method III

- **Starting procedure**
  - Required $\mathbf{u}_0, \mathbf{u}_{-1}$ to solve for $\mathbf{u}_1$!!
  - Compute acceleration at $t = 0$

$$\ddot{\mathbf{u}}_0 = \mathbf{M}^{-1} \left(\mathbf{f}_0 - \mathbf{C}\dot{\mathbf{u}}_0 - \mathbf{K}\mathbf{u}_0 \right)$$

  - Assume that acceleration $\ddot{\mathbf{u}}_0$ at $t = 0$ acts for whole time-interval $[t_{-1}, t_0]$
  - Estimate $\mathbf{u}_{-1}$ backward from t_0 to t_{-1} and initial conditions at t_0

$$\mathbf{u}_{-1} = \mathbf{u}_0 + (-\Delta t)\, \dot{\mathbf{u}}_0 + \frac{1}{2} (-\Delta t)^2\, \ddot{\mathbf{u}}_0$$

i.e.

$$\mathbf{u}_{-1} = \mathbf{u}_0 - \Delta t \dot{\mathbf{u}}_0 + \frac{1}{2} (\Delta t)^2\, \ddot{\mathbf{u}}_0$$

MDOF numerical solution: central difference method IV

Properties of numerical scheme

- **Consistency** with order of accuracy k:
  - gives informations on the relation between numerical approximated scheme and original differential equation based on time-interval discretization Δt
  - "Approximated" numerical scheme is said consistent if it reproduces "exact" differential equation when $\Delta t \to 0$
- **Stability**:
  - Local truncation error in numerical approximation does not grow in time

Consistency + stability = convergence

- **Convergence**:
  - Numerical solution approaches exact solution when Δt goes to zero

SDOF central difference method: consistency I

- Consider equation

$$\ddot{\mathbf{u}} + \omega^2 \mathbf{u} = \mathbf{f}(t)$$

- Define **local truncation error** τ as

$$\tau = \text{approx. ODE} - \text{exact ODE}$$

- In the particular case under investigation

$$\tau = \left[\frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} + \omega \mathbf{u}_n - \mathbf{f}_n \right] - \left[\ddot{\mathbf{u}}_n + \omega^2 \mathbf{u}_n - \mathbf{f}_n \right]$$

and after simplyfication

$$\boxed{\tau = \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} - \ddot{\mathbf{u}}_n} \tag{12}$$

- Expand displacement in Taylor series

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \frac{1}{2} \ddot{\mathbf{u}}_n (\Delta t)^2 + \frac{1}{6} \dddot{\mathbf{u}} (\Delta t)^3 + \mathcal{O}\left(\Delta t^4\right) \\ \mathbf{u}_{n-1} = \mathbf{u}_n - \dot{\mathbf{u}}_n \Delta t + \frac{1}{2} \ddot{\mathbf{u}}_n (\Delta t)^2 - \frac{1}{6} \dddot{\mathbf{u}} (\Delta t)^3 + \mathcal{O}\left(\Delta t^4\right) \end{cases}$$

SDOF central difference method: consistency II

- Substituting Taylor expansion in 12 we obtain local truncation error as:

$$\tau = \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} - \ddot{\mathbf{u}}_n = \mathcal{O}\left(\Delta t^2\right) \tag{13}$$

- Central difference method is consistent with second order accuracy
- Damping term does not affect second order accuracy

MDOF central difference method: stability I

- We obtain a condition on time step $\Rightarrow$ numerical approximation of ODE is conditionally stable

$$\frac{\Delta t}{T} < \frac{1}{\pi}$$

- However, for accuracy a good guideline is

$$\frac{\Delta t}{T} < \frac{1}{10}$$

MDOF numerical solution: central difference method & nonlinear response I

- Equation of motion at time t_n with nonlinear internal force

$$\boxed{\mathbf{M}\ddot{\mathbf{u}}_n + \mathbf{C}\dot{\mathbf{u}}_n + f^{int}(\mathbf{u}_n) = \mathbf{f}_n}$$

- Substituting positions 28 and 28 into equation of motion 35, we obtain:

$$\left[\frac{1}{(\Delta t)^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n+1} = \mathbf{f}_n - \left[\frac{1}{(\Delta t)^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n-1} + \left[\frac{2}{(\Delta t)^2}\mathbf{M}\mathbf{u}_n + f^{int}(\mathbf{u}_n)\right] \tag{14}$$

i.e.

$$\bar{\mathbf{K}}\mathbf{u}_{n+1} = \bar{\mathbf{f}}_n$$

where

$$\begin{cases} \bar{\mathbf{K}} = \left[\dfrac{1}{(\Delta t)^2}\mathbf{M} + \dfrac{1}{2\Delta t}\mathbf{C}\right] \\ \bar{\mathbf{f}}_n = \mathbf{f}_n - \left[\dfrac{1}{(\Delta t)^2}\mathbf{M} - \dfrac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n-1} + \left[\dfrac{2}{(\Delta t)^2}\mathbf{M}\mathbf{u}_n + f^{int}(\mathbf{u}_n)\right] \end{cases}$$

- Nonlinear internal force can be produced for example by an elasto-plastic spring

MDOF numerical solution: Newmark's method I

- Start from the assumption on the form of acceleration during a time interval $[t_n, t_{n+1}]$

$$\begin{array}{ll} \textbf{constant acceleration, i.e.} & \ddot{\mathbf{u}}(\tau) = \frac{1}{2}\left(\ddot{\mathbf{u}}_n + \ddot{\mathbf{u}}_{n+1}\right) \\ \textbf{linear acceleration, i.e.} & \ddot{\mathbf{u}}(\tau) = \ddot{\mathbf{u}}_n + \left(\ddot{\mathbf{u}}_{n+1} - \ddot{\mathbf{u}}_n\right)\frac{\tau}{\Delta t} \end{array} \tag{15}$$

  with $\tau \in [0, \Delta t]$ and $\Delta t = t_{n+1} - t_n$

- Integrating twice to obtain velocity and displacement over the time interval we get:

$$\begin{array}{ll} \textbf{constant acceleration} & \begin{cases} \dot{\mathbf{u}}(\tau) = \dot{\mathbf{u}}_n + \frac{1}{2}\left(\ddot{\mathbf{u}}_n + \ddot{\mathbf{u}}_{n+1}\right)\tau \\ \mathbf{u}(\tau) = \mathbf{u}_n + \dot{\mathbf{u}}_n\tau + \frac{1}{4}\left(\ddot{\mathbf{u}}_n + \ddot{\mathbf{u}}_{n+1}\right)\tau^2 \end{cases} \\ \textbf{linear acceleration} & \begin{cases} \dot{\mathbf{u}}(\tau) = \dot{\mathbf{u}}_n + \ddot{\mathbf{u}}_n\tau + \frac{1}{2}\left(\ddot{\mathbf{u}}_{n+1} - \ddot{\mathbf{u}}_n\right)\frac{\tau^2}{\Delta t} \\ \mathbf{u}(\tau) = \mathbf{u}_n + \dot{\mathbf{u}}_n\tau + \frac{1}{2}\ddot{\mathbf{u}}_n\tau^2 + \frac{1}{6}\left(\ddot{\mathbf{u}}_{n+1} - \ddot{\mathbf{u}}_n\right)\frac{\tau^3}{\Delta t} \end{cases} \end{array} \tag{16}$$

MDOF numerical solution: Newmark's method II

- Evaluating at the end of time interval

$$
\begin{array}{ll}
\textbf{constant acceleration} & \begin{cases} \dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \dfrac{1}{2}\left(\ddot{\mathbf{u}}_{n+1} + \ddot{\mathbf{u}}_n\right)\Delta t \\ \mathbf{u}_{n+1} = \mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \dfrac{1}{4}\left(\ddot{\mathbf{u}}_{n+1} + \ddot{\mathbf{u}}_n\right)\Delta t^2 \end{cases} \\
\textbf{linear acceleration} & \begin{cases} \dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \dfrac{1}{2}\left(\ddot{\mathbf{u}}_{n+1} + \ddot{\mathbf{u}}_n\right)\Delta t \\ \mathbf{u}_{n+1} = \mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \left(\dfrac{1}{6}\ddot{\mathbf{u}}_{n+1} + \dfrac{1}{3}\ddot{\mathbf{u}}_n\right)\Delta t^2 \end{cases}
\end{array}
\tag{17}
$$

- Both the constant acceleration case and the linear acceleration case can be put in the same form in terms of two parameters β and γ defining the integration method:

$$
\boxed{\begin{cases} \dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \left[\left(1-\gamma\right)\ddot{\mathbf{u}}_n + \gamma\ddot{\mathbf{u}}_{n+1}\right]\Delta t \\ \mathbf{u}_{n+1} = \mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \left[\left(\dfrac{1}{2}-\beta\right)\ddot{\mathbf{u}}_n + \beta\ddot{\mathbf{u}}_{n+1}\right]\Delta t^2 \end{cases}}
\tag{18}
$$

such that

$$
\begin{array}{lll}
\textbf{constant acceleration} & \gamma = \dfrac{1}{2} \quad \text{and} \quad \beta = \dfrac{1}{4} \\
\textbf{linear acceleration} & \gamma = \dfrac{1}{2} \quad \text{and} \quad \beta = \dfrac{1}{6}
\end{array}
\tag{19}
$$

MDOF numerical solution: Newmark's method III

- It is interesting that both methods can be put in a predictor-corrector

$$\begin{cases} \dot{\mathbf{u}}_{n+1} = [\dot{\mathbf{u}}_n + (1-\gamma)\,\ddot{\mathbf{u}}_n \Delta t] + [\gamma \ddot{\mathbf{u}}_{n+1} \Delta t] \\ \mathbf{u}_{n+1} = \left[\mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \left(\frac{1}{2} - \beta\right) \ddot{\mathbf{u}}_n \Delta t^2\right] + \left[\beta \ddot{\mathbf{u}}_{n+1} \Delta t^2\right] \end{cases} \tag{20}$$

which can be rewritten as

$$\begin{cases} \dot{\mathbf{u}}_{n+1} = \tilde{\dot{\mathbf{u}}}_{n+1} + \gamma \Delta t \ddot{\mathbf{u}}_{n+1} \\ \mathbf{u}_{n+1} = \tilde{\mathbf{u}}_{n+1} + \beta \Delta t^2 \ddot{\mathbf{u}}_{n+1} \end{cases} \tag{21}$$

where

$$\begin{cases} \tilde{\dot{\mathbf{u}}}_{n+1} = \dot{\mathbf{u}}_n + (1-\gamma)\,\ddot{\mathbf{u}}_n \Delta t \\ \tilde{\mathbf{u}}_{n+1} = \mathbf{u}_n + \dot{\mathbf{u}}_n \Delta t + \left(\frac{1}{2} - \beta\right) \ddot{\mathbf{u}}_n \Delta t^2 \end{cases} \tag{22}$$

MDOF numerical solution: Newmark's method IV

- Assume to know solution at t_n, want to solve at t_{n+1}

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1} + \mathbf{C}\dot{\mathbf{u}}_{n+1} + \mathbf{K}\mathbf{u}_{n+1} = \mathbf{f}_{n+1}$$

- Two options:
  - Express $\mathbf{u}_{n+1}$ and $\dot{\mathbf{u}}_{n+1}$ in terms of $\ddot{\mathbf{u}}_{n+1}$ and solve for $\ddot{\mathbf{u}}_{n+1}$
  - Express $\ddot{\mathbf{u}}_{n+1}$ and $\dot{\mathbf{u}}_{n+1}$ in terms of $\mathbf{u}_{n+1}$ and solve for $\mathbf{u}_{n+1}$
- Second approach is particular interesting in case of nonlinear internal forces

MDOF Newmark's method in terms of displacements I

- From equation 21_2 we express

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\beta \Delta t^2} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \tag{23}$$

and using 21_1

$$\begin{aligned} \dot{\mathbf{u}}_{n+1} &= \tilde{\dot{\mathbf{u}}}_{n+1} + \gamma \Delta t \ddot{\mathbf{u}}_{n+1} \\ &= \tilde{\dot{\mathbf{u}}}_{n+1} + \frac{\gamma}{\beta \Delta t} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \end{aligned} \tag{24}$$

- Plugging back into equilibrium equation we obtain

$$\mathbf{M} \left[\frac{1}{\beta \Delta t^2} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \right] + \mathbf{C} \left[\tilde{\dot{\mathbf{u}}}_{n+1} + \frac{\gamma}{\beta \Delta t} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \right] + \mathbf{K} \mathbf{u}_{n+1} = \mathbf{f}_{n+1}$$

i.e.

$$\left[\frac{1}{\beta \Delta t^2} \mathbf{M} + \frac{\gamma}{\beta \Delta t} \mathbf{C} + \mathbf{K} \right] \mathbf{u}_{n+1} = \left[\mathbf{f}_{n+1} + \mathbf{M} \frac{1}{\beta \Delta t^2} \tilde{\mathbf{u}}_{n+1} + \mathbf{C} \frac{\gamma}{\beta \Delta t} \tilde{\mathbf{u}}_{n+1} \right]$$

MDOF Newmark's method in terms of displacements II

- Previous equation can be rewritten as:

$$\boxed{\tilde{\mathbf{K}}\mathbf{u}_{n+1} = \tilde{\mathbf{f}}_{n+1}}$$

where

$$\begin{cases} \tilde{\mathbf{K}} = \left[\dfrac{1}{\beta\Delta t^2}\mathbf{M} + \dfrac{\gamma}{\beta\Delta t}\mathbf{C} + \mathbf{K}\right] \\ \tilde{\mathbf{f}}_{n+1} = \left[\mathbf{f}_{n+1} + \mathbf{M}\dfrac{1}{\beta\Delta t^2}\tilde{\mathbf{u}}_{n+1} + \mathbf{C}\dfrac{\gamma}{\beta\Delta t}\tilde{\mathbf{u}}_{n+1}\right] \end{cases}$$

and where we recall that

$$\begin{cases} \tilde{\dot{\mathbf{u}}}_{n+1} = \dot{\mathbf{u}}_n + (1-\gamma)\,\ddot{\mathbf{u}}_n\Delta t \\ \tilde{\mathbf{u}}_{n+1} = u_n + \dot{\mathbf{u}}_n\Delta t + \left(\dfrac{1}{2} - \beta\right)\ddot{\mathbf{u}}_n\Delta t^2 \end{cases} \tag{25}$$

- Once solved for $\mathbf{u}_{n+1}$, possible to compute new velocities and accelerations as

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\beta\Delta t^2}\left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1}\right)$$

$$\dot{\mathbf{u}}_{n+1} = \tilde{\dot{\mathbf{u}}}_{n+1} + \gamma\Delta t\ddot{\mathbf{u}}_{n+1}$$

MDOF Newmark's method in terms of displacements III

- Simple starting procedure: only need is to compute acceleration $\ddot{\mathbf{u}}_0 = \ddot{\mathbf{u}}(0)$

$$\ddot{\mathbf{u}}_0 = \mathbf{M}^{-1} \left(p_0 - \mathbf{C}\dot{\mathbf{u}}_0 - \mathbf{K} u_0 \right)$$

SDOF Newmark's method: consistency I

- Evaluation of local truncation error of velocity and acceleration shows that:

$$\tau = \mathcal{O}(\Delta t^k)$$

with:

$$\begin{cases} k = 2 \quad \text{for} \quad \gamma = \dfrac{1}{2} & \text{i.e second order} \\ k = 1 \quad \text{for} \quad \gamma \neq \dfrac{1}{2} & \text{i.e first order} \end{cases}$$

Possible exercises I

- **Exercise.** Modify the frame Matlab code to include some time-integration scheme.
- **Exercise.** Michael A. Koplow, Abhijit Bhattacharyya, Brian P. Mann "Closed form solutions for the dynamic response of EulerBernoulli beams with step changes in cross section", Journal of Sound and Vibration 295 (2006) 214225
- **Exercise.** Seon M. Han, Haym Bebaroya and Timothy Wei "Dynamics of transversely vibrating beams using four engineering theories", Journal of Sound and Vibration (1999) 225(5), 935-988