Duality in the Geometrically Exact Analysis of Three-Dimensional Framed Structures

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Outline

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- Reissner-Simo Beam Theory - BVP
- Primal Variational Problem
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- Dual FE Formulation
- Dual Analysis Method - *A Posteriori* Error Estimation
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- Closure
Displacement-based finite element formulations

- are on the basis of most of the finite element models used in computer analysis of structures;
- assume the configuration variables as primary unknowns;
- lead to approximate *kinematically admissible solutions* in which *stress discontinuities may occur across element boundaries*; stress ‘averaging’ procedures are required;
- are well developed for both linear and nonlinear analyses.
Equilibrium-based finite element formulations

- are less common than the displacement-based finite element formulations;
- lead to approximate *statically admissible solutions*;
- have a special appeal for practical design engineers due to the exact transmission of stresses across interelement boundaries, thus avoiding the need for ‘averaging’ procedures;
- are not well studied in the context of the geometrically nonlinear analysis of framed structures.
Objectives and Scope

To present, in the framework of the quasi-static linear elastic analysis of geometrically exact framed structures modeled using the three-dimensional Reissner-Simo beam theory:

- Two dual energy-based variational formulations: one (Primal) derived from the well known Principle of Stationary Total Potential Energy, and the other (Dual) resulting from the Principle of Stationary Total Complementary Energy;

- An equilibrium-based (hybrid-mixed) finite element formulation relying on a modified Principle of Complementary Energy;

- A duality based method in which both primal and dual variational problems are studied in conjunction.
Kinematical Considerations

- The deformed geometry of a beam is described by the centroidal axis and the set of orientations of cross-sections;

- Only initially straight beam configurations and initially undistorted cross-sections are assumed;

- The geometric shape of the cross-sections is assumed to be arbitrary and constant along the beam;

- The cross-sections are assumed to suffer only rigid body motions during deformation;

- The beam theory is valid for arbitrarily large displacements and rotations - Geometrically Exact Beam Theory.
The deformed configuration of a beam is described by the position of the line of centroids of the cross-sections and also the rotations of the cross-sections;

The rotations of the cross-sections are described using the Euler-Rodrigues formula, which is assumed to be parameterized through the total rotation vector as follows:

\[ Q = I + \frac{\sin \theta}{\theta} \Theta + \frac{1 - \cos \theta}{\theta^2} \Theta^2 \]

where \( \Theta = Skew(\theta) \) and \( \theta = \| \theta \| \)
Boundary-Value Problem

Reissner-Simo Beam BVP (Material Form)

- **Differential Equations**

  - **Equilibrium**
    \[
    T^r_e(d)\sigma^r + q = 0, \quad \text{in } \Omega
    \]

  - **Elasticity**
    \[
    \sigma^r = \frac{\partial W(\varepsilon^r(d))}{\partial \varepsilon^r}, \quad \text{in } \Omega
    \]

  - **Compatibility**
    \[
    \varepsilon^r = \varepsilon^r(d), \quad \text{in } \Omega
    \]

- **Neumann Boundary Conditions**: \( nH\sigma^r = \bar{q}, \quad \text{on } \Gamma_N \)

- **Dirichlet Boundary Conditions**: \( d = \bar{d}, \quad \text{on } \Gamma_D \)

**Remark**: If the strain energy \( W(\varepsilon^r) \) is differentiable and convex, by means of the Legendre transformation, the constitutive relations can be alternatively established using the format

\[
\varepsilon^r = \frac{\partial W_c(\sigma^r)}{\partial \sigma^r}, \quad \text{in } \Omega
\]
Principle of Stationary Total Potential Energy

Let $\mathcal{U}_k$ and $\mathcal{V}_k$ be the kinematically admissible function spaces

$$\mathcal{U}_k = \{ d \in H^1(\Omega) \mid d = \bar{d} \text{ on } \Gamma_D \}$$

$$\mathcal{V}_k = \{ \delta d \in H^1(\Omega) \mid \delta d = 0 \text{ on } \Gamma_D \}$$

The total potential energy associated with vector $d$ is the one-field functional $\Pi_p(d) : \mathcal{U}_k(\Omega) \to \mathbb{R}$ given by

$$\Pi_p(d) = \int_{\Omega} \left[ W(\varepsilon^r(d)) - q \cdot d \right] dS - [\bar{q} \cdot d]_{\Gamma_N}$$

**Principle of Stationary Total Potential Energy (PSTPE):**
vector $d \in \mathcal{U}_k$ is a solution of the BVP iff $\delta \Pi_p = 0 \ \forall \delta d \in \mathcal{V}_k$, i.e., a beam is in equilibrium iff its total potential energy takes a stationary value for all kinematically admissible displacement fields.
Hybrid-Multi-Field Variational Principles

The PSTPE can be generalized by means of the Lagrangian multiplier method leading to a **Generalized Variational Principle (GVP)**;

The GVP can afterwards be particularized into different **Hybrid-Multi-Field Principles**, e.g.:

- Principles of Hu-Washizu;
- Principles of Hellinger-Reissner;
- Principle of Total Complementary Energy, etc.
Principle of Stationary Total Complementary Energy

Let $\mathcal{U}_s$ and $\mathcal{V}_s$ be the statically admissible function spaces:

$$\mathcal{U}_s = \{(\sigma^r, d) \in (\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)) \mid T^r_e(d)\sigma^r + q = 0 \text{ in } \Omega \text{ and } nH\sigma^r - \bar{q} = 0 \text{ on } \Gamma_N\}$$

$$\mathcal{V}_s = \{ (\delta\sigma^r, d) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \mid T^r_e(d)\delta\sigma^r = 0 \text{ in } \Omega \text{ and } nH\delta\sigma^r = 0 \text{ on } \Gamma_N\}$$

The complementary energy associated with $(\sigma^r, d)$ is the 2-field functional $\Pi_c : \mathcal{U}_s(\Omega) \to \mathcal{R}$ given by

$$\Pi_c(\sigma^r, d) = \int_0^L [W_c(\sigma^r) - \sigma^r \cdot \varepsilon^r(d) + \sigma^r \cdot T^r_c(d) d] dS - [nH\sigma^r \cdot \bar{d}]_{\Gamma_D}$$

**Principle of Stationary Total Complementary Energy (PSTCE):** the pair $(\sigma^r, d) \in \mathcal{U}_s$ is a solution of the BVP iff $\delta\Pi_c = 0 \forall (\delta\sigma^r, d) \in \mathcal{V}_s$. 
Hybrid-Mixed Complementary Energy

- If the equilibrium equations are assumed to be relaxed within the framework of the PSTCE, the following hybrid-mixed complementary energy \( \Pi^g_C : \chi(\Omega) \to \mathcal{R} \) can be obtained

\[
\Pi^g_C(\sigma^r, d, d^\Gamma) = \sum_{b=1}^{B} \int_{\Omega_b} \left[ W_c(\sigma^r_b) - \sigma^r_b \cdot \varepsilon^r_b(d_b) + q_b \cdot d_b \right] d\Omega_b
\]

\[
+ [\bar{q} \cdot d^\Gamma]_{\Gamma_N \cup \Gamma_{int}} + [nH\sigma^r \cdot (d - J_N \bar{d})]_{\Gamma_N \cup \Gamma_{int}} + [nH\sigma^r \cdot (d - J_D \bar{d})]_{\Gamma_D}
\]

- \( J_N \) and \( J_D \) represent transformation matrices mapping global vectors (matrices) onto local element vectors (matrices) defined on \( \Gamma_N \cup \Gamma_{int} \) and \( \Gamma_D \), respectively;

- The functions in class \( \chi(\Omega) \) consist of pairs \( (\sigma^r_b, d_b) \in \mathcal{H}^0(\Omega_b) \times \mathcal{H}^1(\Omega_b) \), with \( 1 \leq b \leq B \), and a real-valued vector \( d^\Gamma \) defined on \( \Gamma_N \cup \Gamma_{int} \).
The variational (weak) problem
\[ \delta \Pi^g_c = 0, \; \forall (\delta \sigma^r, \delta d, \delta d^\Gamma) \in \chi(\Omega) \] is formally equivalent to the following system of Euler-Lagrange equations

\[ T^r_e (d_b) \sigma^r_b + q_b = 0 \; \text{in} \; \Omega_b \]
\[ \varepsilon^r_b (\sigma^r_b) - \varepsilon^r_b (d_b) = 0 \; \text{in} \; \Omega_b \]
\[ \bar{q} - n J^T_N H \sigma^r = 0 \; \text{in} \; \Gamma_N \cup \Gamma_{int} \]
\[ d - J_N d^\Gamma = 0 \; \text{in} \; \Gamma_N \cup \Gamma_{int} \]
\[ d - J_D \bar{d} = 0 \; \text{on} \; \Gamma_D \]

with \( 1 \leq b \leq B \).
Approximations

- **Element variables:**

\[
\sigma^{rh} = \begin{bmatrix}
\mathbf{n}^r \\
\mathbf{m}_i^r + (\mathbf{m}_j^r - \mathbf{m}_i^r) \frac{S}{L}
\end{bmatrix}, \quad \mathbf{d}^h = \begin{bmatrix}
\mathbf{u}_i + (\mathbf{u}_j - \mathbf{u}_i) \frac{S}{L} \\
\theta
\end{bmatrix}
\]

- **Nodal variables:** \(\mathbf{d}^\Gamma\) (generalized displacements)

**Remarks:**

- As the approximate displacements are one degree greater than the approximate rotations, this formulation is capable of representing zero shear solutions and is, thus, completely free from shear locking;

- Using these approximations, the formulation can provide solutions that satisfy the equilibrium differential equations in strong form, as well as the stress continuity conditions (when assuming zero distributed loads);

- Furthermore, the necessary and sufficient condition for solvability of the discrete linearized system of equations is fulfilled either for a single element or a patch of elements with appropriate boundary conditions \((n_{\sigma r} \geq n_d - n_r)\).
The linearized global system of equations can be stated as

\[ r(p) + T(p)\Delta p = 0, \]

\[ p = \begin{bmatrix} p_{\sigma r} \\ p_d \end{bmatrix}, \quad T = \begin{bmatrix} F & A^T \\ A & K_c \end{bmatrix} \]

\[ K_{eq} = AF^{-1}A^T - K_c \]

(for the classification of the stability of the equilibrium)

- \( F = \frac{\partial^2 \Pi^g_c}{\partial p_{\sigma r} \partial p_{\sigma r}} \) - flexibility matrix
- \( A = \frac{\partial^2 \Pi^g_c}{\partial p_d \partial p_{\sigma r}} \) - equilibrium matrix; \( A^T \) - compatibility matrix
- \( K_c = \frac{\partial^2 \Pi^g_c}{\partial p_d \partial p_d} \) - stiffness matrix
Fully Linear Case (FLC) vs Geom. Nonlinear Case (GNC)

<table>
<thead>
<tr>
<th>FLC</th>
<th>GNC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_p(d)$ is convex</td>
<td>$\Pi_p(d)$ is nonconvex</td>
</tr>
<tr>
<td>$\Pi_c(\sigma')$ is concave</td>
<td>$\Pi_c(\sigma',d)$ is a saddle functional</td>
</tr>
</tbody>
</table>

* Extremum conditions of $\Pi_p$ and $\Pi_c$ are required (Nobel and Sewell 1972, Gao and Strang 1989)

$$\epsilon = |\overline{\Pi}_p - \overline{\Pi}_c|, \quad \epsilon_k = |\overline{\Pi}_p - \Pi_p|, \quad \epsilon_s = |\Pi_c - \overline{\Pi}_c|$$

$$\overline{\Pi}_p = \inf_{d \in U_k} \Pi_p(d), \quad \overline{\Pi}_c = \sup_{\sigma' \in U_s} \Pi_c(\sigma')$$
Cantilever beam subject to an end force

Problem Definition

\[ P = 2.5 \times 10^{-5} \]
\[ E = 1 \times 10^4 \]
\[ \nu = 0.2 \]
\[ \alpha = \frac{5}{6} \]
\[ L = 1 \]
\[ h = 0.01 \]
\[ b = 0.005 \]
Cantilever beam subject to an end force

Deformed Configurations

- Primal 2FE
- Primal 8FE
- Dual 2FE
- Dual 8FE
Cantilever beam subject to an end force

Diagrams of moments for $P = 2.5 \times 10^{-5}$ (16FE)
Cantilever beam subject to an end force

Energies

\[ \log(n) \times 10^5 \]

\[ \log(n) \times 10^1 \]

\[ \log(n) \times 10^2 \]
Lee Frame

Problem Definition

\[ P = 50000 \]

\[ EI = 1.44 \times 10^7 \]

\[ EA = 4.32 \times 10^7 \]

\[ GA' = 1.66 \times 10^7 \]

\[ L = 120 \]
Deformed Configurations (5FE per leg)

- $P = 5000$
- $P = 10000$
- $P = 15000$
- $P = 20000$
Diagrams of moments for $P = 10000$ (5FE per leg)
Energies for $P = 15000$
Problem Definition

\[ \alpha = \frac{5}{6} \]
\[ P = 2 \]
\[ E = 71240 \]
\[ \nu = 0.31 \]
\[ L = 240 \]
\[ h_1 = 30 \]
\[ h_2 = 0.6 \]
Deformed Configurations

Lateral Torsion Buckling
Right-Angle Cantilever Frame

Equilibrium Paths

Displacement in the $Z$ direction at the tip of the cantilever

Primal 4FE

Dual 4FE
Problem Definition

\[ M = 700 \]
\[ E = 71240 \]
\[ \nu = 0.31 \]
\[ \alpha = \frac{5}{6} \]
\[ L = 240 \]
\[ h_1 = 30 \]
\[ h_2 = 0.6 \]
Equilibrium Paths

Right-Angle Simply-Supported Frame under End Moments

Displacement in the Z direction at the mid-span of the frame
Energies (3FE per leg)
Problem Definition

\[ T = 270 \]
\[ E = 71240 \]
\[ \nu = 0.31 \]
\[ L = 240 \]
\[ A = 1 \]
\[ I_1 = I_2 = 0.0833 \]
\[ J = 2.16 \]
Equilibrium Paths

Rotation in the X direction at point B of the cable

Primal 10FE
Dual 10FE
Energies for $T = 210$
Conclusions

The present hybrid-mixed FE formulation, established within the framework of the geometrically exact (Reissner-Simo) analysis of 3D framed structures, is:

- variationally consistent;
- completely free from shear locking;
- capable of producing statically admissible approximate solutions;

The present duality based method opens a new way on a posteriori error estimation and on possible bounding aspects within the framework of geometrically nonlinear analysis of framed structures.
Future Developments

- Consider higher-order polynomial sets of approximate functions within the dual FE formulation ($p$-type refinement schemes);

- Consider initially curved beam elements within the framework of the dual formulation;

- Incorporate general cross-sectional in-plane changes and out-of-plane warping phenomena within the framework of the dual formulation;

- Include physical nonlinearities within the framework of the dual formulation;
Future Developments (Cont.)

- Extend the dual formulation to shells and membranes;

- Derive (hybrid-) mixed FE formulations from other (hybrid-) multi-field variational principles;

- Investigate, from a mathematical point of view, the numerical stability of the present dual FE formulation;

- Investigate alternative error estimation methods which can provide guaranteed upper bounds of the exact error of the approximate solutions (considering both global and local quantities of interest).
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