Some Advances on Isogeometric Analysis at KAUST

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Joint work with:
N.O. Collier, L. Demkowicz, J. Gopalakrishnan,
K. Kuznik, I. Muga, A.H. Niemi, D. Pardo, M. Paszynski,
H. Radwan, G. Stenchikov, W. Tao, and J. Zitelli
The Cost of Continuity

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Joint work with:
N.O. Collier, D. Pardo, M.Paszynski, H. Radwan,
G. Stenchikov, and W. Tao
Claim: $C^{p-1}$ space economical $p$-refinement
Problem:
True w.r.t. DOFs, but ignores solution cost
**Goal:** Understand solution cost

Canonical Laplace problem on unit cube

\[ u = 1 \]

\[ u = 0 \]

\[ \hat{n} \cdot \nabla u = 0 \] everywhere

Record for direct solver (MUMPS):

- Solve time
- Required memory while varying polynomial order and continuity of discretization
Isogeometric finite elements

Partition mesh into elements

There are $p+1$ functions of order $p$ assigned to an element $K = \begin{bmatrix} \xi_k & \xi_{k+1} \end{bmatrix}$
MUMPS: Multi-frontal direct solver

Solution strategy split into 2 parts:
1. Static condensation of fully assembled dofs
2. LU-factorization of remaining problem (skeleton problem)

\[ \begin{bmatrix}
\ldots & 
\end{bmatrix} \]

\[ C^0 \text{ 1D finite element matrices} \]

Resulting problem size reduced!

Element Local Matrix

Removed by static condensation
$C^0$ B-spline spaces

Increasing polynomial order increases static condensation work, but skeleton problem is limiting cost and remains fixed!

$p=7$ commensurate to $p=1$
Computational estimates of cost for $C^0$ spaces

- Bubbles in ea. element: $\left(p - 1\right)^3 \Rightarrow$ Elimination time $O\left(\left(p - 1\right)^3\right)^3 \approx O\left(p^9\right)$
  - $N_e$ (\# elements) $\approx O\left(N_{dof} / p^3\right) \Rightarrow$ Static condensation time $O\left(N_e p^9\right)$
- Squeleton problem:
  - $N_e$ \# elements $\Rightarrow N_f$ (\# faces) $= 3 N_e$ $\Rightarrow$ Size: $N_f p^2 = 3 N_e p^2$
  - Average bandwidth $O\left(\left(p - 1\right)^2\right)$
    $\Rightarrow$ Elimination time: $O\left(\left(3 N_e p^2\right)^2 \left(p - 1\right)^2\right) \approx O\left(N_e^2 p^6\right)$
- Total problem complexity estimate:
  - $time = O\left(N_e p^9\right) + O\left(N_e^2 p^6\right) + \text{lower order terms}$
    $\Rightarrow$ for $N_{dof}$ squeleton problem cost is $O\left(N_{dof}^2\right)$
    $\Rightarrow$ for $N_{dof} \gg O\left(p^6\right)$ then static condensation time $\ll$ squeleton problem
- Total problem memory usage estimate: $O\left(N p^3\right) + O\left(N^{4/3}\right) + \text{lower order terms}$
The high price of continuity

(a) 10k degrees of freedom

(b) 30k degrees of freedom

(c) 100k degrees of freedom
Computational estimates of cost for $C^{p-1}$ spaces

- Elimination problem:
  - $N_e$ # elements $\Rightarrow$ Size: $N_e(p + 1)^3$
  - Average bandwidth $O\left((p + 1)^3\right)$
    $\Rightarrow$ Elimination time: $O\left(\left(N_e(p + 1)^3\right)^2(p + 1)^3\right) \approx O\left(N_e^2p^9\right)$

- Total problem complexity estimate:
  \[\text{time} = O\left(N_e^2p^9\right) + \text{lower order terms}\approx O\left(N^2p^6\right)\]

- Total problem memory usage estimate: $O\left(N^{4/3}p^2\right) + \text{lower order terms}$
The high price of continuity

Higher continuous basis result in element stiffness matrix blocks overlapping, causes performance loss of multi-frontal algorithm.
Goal: Understand value of continuity

Canonical Laplace problem on unit cube

\[-\nabla \cdot (\nabla u) = f \quad \text{on } \Omega\]
\[u = 0 \quad \text{on } \Gamma_D\]
\[(\nabla u) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N\]

\[f(x, y, z) = \frac{3C^2 \pi^2}{4} \left[ \sin \left( \frac{C\pi}{2} x \right) \sin \left( \frac{C\pi}{2} y \right) \sin \left( \frac{C\pi}{2} z \right) \right]\]

\[u(x, y, z) = \sin \left( \frac{C\pi}{2} x \right) \sin \left( \frac{C\pi}{2} y \right) \sin \left( \frac{C\pi}{2} z \right)\]

\[\Omega = [0, 1]^3, \quad \Gamma_D = (0, :, :) \cup (:, 0, :) \cup (:, :, 0), \quad \Gamma_N = (1, :, :) \cup (:, 1, :) \cup (:, :, 1)\]
The value of continuity?

Table I. Results for Problem 1, $C = 5$, $\|E\|_{H_1} \approx 10^{-2}$

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<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
<th>$p$</th>
<th>$k$</th>
<th>$N_{dof}$</th>
<th>$|E|_{L_2}$</th>
<th>$|E|_{H_1}$</th>
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</table>
The value of continuity?

Table II. Results for Problem 1, $C = 3$, constant elements

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</tbody>
</table>
Isogeometric-Specific Multi-Frontal Solver

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Joint work with:
K. Kuznik and M. Paszynski
### Multi-frontal algorithm

Construct multiple frontal matrices s.t. they sum up to the full matrix.

Variables must be split into parts.

\[
X_i = X_i^I + X_i^{II}
\]
First all frontal matrices are constructed
• Assemble frontal matrices 1 and 2 into new 3x3 frontal matrix
• Rows 1 and 2 are fully assembled
Multi-frontal algorithm

\[
\begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
0 & \frac{1}{h^2} & -\frac{1}{h^2}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Column 1 eliminated using first row

\[\Rightarrow\quad 2^{nd} \text{ row} = 2^{nd} \text{ row} - \frac{1}{h^2} \times 1^{st} \text{ row}\]
Multi-frontal algorithm

\[
\begin{align*}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -2/h^2 & 1/h^2 \\
0 & 1/h^2 & -1/h^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
&= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

2\textsuperscript{nd} row = 2\textsuperscript{nd} row - \left[1/h^2\right] \ast 1\textsuperscript{st} row
Assemble frontal matrices 3 and 4 into new frontal matrix

Only row 2 is fully assembled
Multi-frontal algorithm

\[
\begin{bmatrix}
-\frac{2}{h^2} & 1/h^2 & 1/h^2 \\
1/h^2 & -1/h^2 & 0 \\
1/h^2 & 0 & -1/h^2 \\
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Change of the ordering
Multi-frontal algorithm

Eliminate entries in column 1 below row 1

2\textsuperscript{nd} row = 2\textsuperscript{nd} row − [1/h\textsuperscript{2}] / [-2/h\textsuperscript{2}] \times 1\textsuperscript{st} row

3\textsuperscript{rd} row = 3\textsuperscript{rd} row − [1/h\textsuperscript{2}] / [-2/h\textsuperscript{2}] \times 1\textsuperscript{st} row
Eliminate entries below the diagonal

2\textsuperscript{nd} row = 2\textsuperscript{nd} row \ − \ [1/h^2] / [-2/h^2] \ * \ 1\textsuperscript{st} row

3\textsuperscript{rd} row = 3\textsuperscript{rd} row \ − \ [1/h^2] / [-2/h^2] \ * \ 1\textsuperscript{st} row
Multi-frontal algorithm

\[
\begin{bmatrix}
-\frac{2}{h^2} & 1/h^2 \\
1/h^2 & -\frac{1}{h^2}
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1/h^2 & 1/2h^2 \\
1/2h^2 & -1/h^2
\end{bmatrix}
\begin{bmatrix}
x_{1/2} \\
x_{3/2}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1/h^2 & 1/h^2 & 0 \\
1/h^2 & -2/h^2 & 1/h^2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_5 \\
x_6 \\
x_7
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
20
\end{bmatrix}
\]

Recursively eliminate remaining frontal matrices

...
Multi-frontal algorithm

All frontal matrices are generated at the same time
Multi-frontal algorithm

Assembly and elimination are executed concurrently over pairs of frontal matrices
Multi-frontal algorithm

Concurrent assembly and elimination executed over different pairs of frontal matrices
Algorithm recursively repeated until root of the tree is reached

It results in upper triangular matrix stored in distributed manner

Computational complexity = height of the tree = \( \log(N_{\text{dof}}) \)
Numerical results

• NVIDIA CUDA
• GeForce GTX 260
• Multiprocessors x Cores/MP =
• Cores: 24 (MP) x 8 (Cores/MP) = 192 (Cores)
• Memory: 896MB
Numerical results

\[
\text{\(p=1\)}
\]

- **Local matrix generation**
- **Equation solution**
- **Total**

Time [s] vs. no. of elements
Numerical results

p=2

![Graph showing numerical results for different operations as a function of the number of elements. The graph compares local matrix generation, equation solution, and total time.]
Numerical results

$p = 3$

Diagram showing the relationship between time [s] and the number of elements, with different curves for local matrix generation, equation solution, and total time.
Numerical results

$p=4$

- Local matrix generation
- Equation solution
- Total

The graph shows the time in seconds (y-axis) against the number of elements (x-axis) for different operations. It indicates how the time increases with the number of elements for each operation, with the total time being the sum of local matrix generation and equation solution times.
Numerical results

p=5

Graph showing time [s] vs. no. of elements for different processes:
- Local matrix generation
- Equation solution
- Total

Axes:
- x-axis: no. of elements
- y-axis: time [s]
Numerical results
Numerical results
Flood Modeling

Joint work with:
N.O. Collier, H. Radwan, G. Stenchikov, and W. Tao
Overview

• Motivation:
  – KAUST and Jeddah flood
  – Model Problem: Manning’s model

• Application to KAUST topographic data
  – Numerical results
Diffusive-Wave Approximation

• **Strong Form**

\[
\begin{aligned}
\dot{u} - \nabla \cdot (\kappa(u, \nabla u) \nabla u) &= f & \text{on } \Omega \times (0, T] \\
u &= u_0 & \text{on } \Omega \times \{t = 0\} \\
(\kappa(u, \nabla u) \nabla u) \cdot n &= B_N & \text{on } \Gamma_N \times (0, T] \\
u &= B_D & \text{on } \Gamma_D \times (0, T]
\end{aligned}
\]

where

\[
\kappa(u, \nabla u) = \frac{(u - z)^{\alpha_M}}{C_f |\nabla u|^{1-\gamma_M}}
\]

• **Weak Form**

\[
B(w, u) = \left( w, \frac{\partial u}{\partial t} \right)_\Omega + (\nabla w, \kappa(u, \nabla u) \nabla u)_\Omega + (w, f)_\Omega = 0
\]
1D Results

(a) $t = 0.4, \ step = 25$
(b) $t = 3.9, \ step = 90$
(c) $t = 7.5, \ step = 145$
(d) $t = 13.8, \ step = 163$
(e) $t = 23.8, \ step = 184$
(f) $t = 26.0, \ step = 195$
(g) $t = 35.5, \ step = 235$
(h) $t = 48.0, \ step = 275$
Relevant Topography
Relevant Topography (Zoom in)
2D Results
Discontinuous Petrov-Galerkin Method

Joint work with:
N.O. Collier, L. Demkowicz, J. Gopalakrishnan,
I. Muga, A.H. Niemi, D. Pardo, and J. Zitelli
Towards Discretization without Numerical Dispersion

- Motivation
  - Model Problem
  - Dispersion/Phase error - Pollution effect
- DPG framework
  - Features and Characteristics
- Application to the Helmholtz Equation
  - Numerical results
- DPG formulation details
Wave Propagation

- Equation governing wave propagation (at speed $c$): $\Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$

- Assuming time-harmonic dependance: $p(\mathbf{x}, t) = \exp(i\omega t)p(\mathbf{x})$

- $\omega$ is the angular frequency

- Helmholtz equation (second order formalism):
  \[
  \begin{cases}
  \Delta p + k^2 p = -ikf & \text{in } \Omega \\
  \partial_n p + ikp = 0 & \text{on } \partial\Omega
  \end{cases}
  \]  

- $k = \frac{\omega}{c}$ is the wave number

- $p(\mathbf{x}) = \exp(ik\mathbf{v} \cdot \mathbf{x})$ are particular solutions to \[\text{(\textsection)}\]

  (time-harmonic wave trains)
Dispersion Analysis

After discretization and numerical solution of (∆), wave trains with discrete wave-number $k^h (\neq k)$ are obtained.

Dispersion analysis: analysis of wave-number error

Spectrum analysis is equivalent to dispersion analysis in the regime where $k^h$ is real

Duality principle

Linear approximation case
**Pollution Effect**

- Mathematical characterization of dispersion
- Given exact solution \( p \in U \) and its discrete approximation \( p^h \in U^h \subset U \)

\[
\frac{\|p - p^h\|}{\|p\|} \leq C(k) \inf_{w^h \in U^h} \frac{\|p - w^h\|}{\|p\|}
\]

where \( C(k) = C_1 + C_2 k^\beta (kh)^\gamma \)

2D numerical evidence

\[
p(x) = \exp\left(ik\left(x_1 \cos \theta + x_2 \sin \theta\right)\right)
\]
Discontinuous Petrov-Galerkin Method

Objective
- Eliminate pollution error in multi-dimensions

Features
- Hermitian positive definite algebraic systems
- Unconditional stability

Characteristics
- Discontinuous Galerkin (DG) method
- Petrov-Galerkin method
- Least-squares-type Galerkin method
Discontinuous Petrov-Galerkin Method

- Optimal convergence rate in “energy” norm of problem irrespective of physical parameters
  - Constructed discrete weighting space guarantees optimal convergence in energy norm
    - Uses weighting test function space fully discontinuous
    - Discrete variational problem is local to each element
    - Local problems are symmetric and are solved approximately using standard Galerkin method
  - Attractive for problem with strong dependence on physical parameters
    - Parametric dependence complicates computation of optimal basis functions
- Discrete approximation of field variables, traces, and fluxes
Six linear elements per wavelength

\[ p(x) = \exp(ikx) \]
1D Result

Ratio between error and best approximation error as a function of $k$
\[ p(x) = \exp \left( ik \left( x_1 \cos \frac{\pi}{4} + x_2 \sin \frac{\pi}{4} \right) \right) \]
2D Result

4 bilinear elements per wavelength, pure impedance boundary cond.

\[ p(x) = \exp \left( i k \left( x_1 \cos \theta + x_2 \sin \theta \right) \right) \]

\[ \Delta p = 2, \text{ flux degree } = 2, \text{ trace degree } = 2 \]

Ratio between error and best approximation error as a function of \( \frac{k}{2\pi} \)
Steady Transport Problem

• Motivation for DPG framework
  – Design discretization such that
    • Trial space grants approximability
    • Weighting space grants stability
    • Parameter-free convergence rate

• Application to the Transport Equations
  – Numerical results
Advection-Diffusion-Reaction in 1D

- Conservative second order form:

\[
\begin{cases}
  s(x)u(x) + \left( a(x)u(x) - \kappa u(x) \right)' = f(x) \quad \text{in } ]a,b[ \\
  u(a) = g_a, \quad u(b) = g_a
\end{cases}
\]

where

- \( \kappa > 0 \) is the constant diffusion coefficient
- \( a(x), s(x) \) smoothly varying coefficients (advection velocity, reaction term)
- \( f(x) \) source term

- Equivalent first order form:

\[
\begin{cases}
  \sigma(x) - \left( a(x)u(x) - \kappa u(x) \right)' = 0 \quad \text{in } ]a,b[ \\
  s(x)u(x) + \sigma(x)' = f(x) \quad \text{in } ]a,b[ 
\end{cases}
\]
Ultra Weak Formulation

- Discontinuous Petrov-Galerkin formalism (ultra weak form):
  - Integration by parts over $K = (x_k, x_{k-1})$,
    where $a = x_0 < x_1 < ... < x_N = b$
    \[
    0 = \int_{x_{k-1}}^{x_k} \tau (\sigma - au) \, dx - \int_{x_{k-1}}^{x_k} \tau' \kappa u \, dx + \kappa \tau u \Big|_{x_{k-1}}^{x_k}
    \]
    \[
    + \int_{x_{k-1}}^{x_k} v (su - f) \, dx - \int_{x_{k-1}}^{x_k} v' \sigma \, dx + v \sigma \Big|_{x_{k-1}}^{x_k}
    \]
    for all test functions $v, \tau \in H^1(x_k, x_{k-1})$
  - Let $\hat{\sigma}, \hat{u}$ denote traces of $\sigma, u$, which are independent variables
Ultra Weak Formulation

• The resulting abstract ultra weak form is:
  Find $u = \{\sigma, u, \hat{\sigma}, \hat{u}\} \in U$ s.t. $b(v, u) = l(v) \quad \forall v \in V$

where

$$b(v, u) = \sum_{k=1}^{N} \left\{ \int_{x_{k-1}}^{x_{k}} (\tau - \nu') \sigma \, dx + \int_{x_{k-1}}^{x_{k}} (\nu' \kappa - a \tau + sv) \, dx + (v \sigma - \kappa \tau \hat{u}) \right\}_{x_{k-1}}^{x_{k}}$$

$$l(v) = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_{k}} vf \, dx$$

and

$$U = L_2(a, b) \times L_2(a, b) \times R^{N+1} \times R^{N+1}$$

$$V = W_N \times W_N$$

$$W_N = \left\{ w : w_{(x_k, x_{k-1})} \in H^1(x_k, x_{k-1}), \ k = 1, 2, ..., N \right\}$$
Computational Considerations

- Fields \( \{\sigma,u\} \) approximated using Bernstein polynomials of order \( p \)
  - Optimal weighting space computed on an enriched polynomial space
    - Enrichment: uniform \( \Delta p \) degree elevation to \( p_e = p + \Delta p \)
  - Discrete variational problem develops exponential boundary layers
    - Using p-FEM recipes, boundary layer mesh of size \( p_e \kappa \) is added
    - \( C^0 \) and \( C^{p_e-1} \) B-splines are used

\[
\begin{align*}
x_{k-1} & \quad x_{k-1} + p_e \kappa \quad x_k + p_e \kappa \quad x_k \\
\end{align*}
\]

Subgrid within each element (boundary layer mesh)
Model Transport Problem

\[
\begin{aligned}
\left\{
\begin{aligned}
&\left(u(x) - \kappa u(x)\right)' = 0 \quad \text{in } ]0,1[ \\
u(0) = 1, \quad u(1) = 0
\end{aligned}
\right.
\Rightarrow u(x) = \frac{1 - \exp\left(\frac{x - 1}{\kappa}\right)}{1 - \exp\left(-\frac{1}{\kappa}\right)}
\end{aligned}
\]

\[
\begin{array}{ll}
p = 1 & \text{DPM solution} \\
p = 2 & \text{Exact solution}
\end{array}
\]

Four elements, \( \kappa = 10^{-1}, \ \Delta p = 2 \)
Model Transport Problem

Four elements, $\kappa = 10^{-3}$, $\Delta p = 2$

Four elements, $\kappa = 10^{-6}$, $\Delta p = 2$
Hemker’s Transport Problem

\[
\begin{aligned}
\left( xu(x) - \kappa u'(x) \right)' &= \kappa \pi^2 \cos(\pi x) + \pi x \sin(\pi x) & \text{in }]0,1[ \\
\end{aligned}
\]

\[
\begin{cases}
u(-1) = -2, & \quad \nu(1) = 0 \\
\end{cases}
\]

\[
\Rightarrow \quad \nu(x) = \cos(\pi x) + \frac{\text{erf}\left( \frac{x}{\sqrt{2\kappa}} \right)}{\text{erf}\left( \frac{1}{\sqrt{2\kappa}} \right)}
\]

Eight elements, \( \kappa = 10^{-3}, \Delta p = 2 \)
Hemker’s Transport Problem

Eight elements, \( \kappa = 10^{-3}, \ \Delta p = 2 \)

Eight elements, \( \kappa = 10^{-7}, \ \Delta p = 2 \)
Advection-Diffusion-Reaction in 2D

- Conservative second order form:
  \[
  \begin{cases}
    s u + \nabla \cdot (a u - \kappa \nabla u) = f, \quad \text{in } \Omega \\
    u = g, \quad \text{on } \partial \Omega
  \end{cases}
  \]

  where
  - \( \kappa > 0 \) is the constant diffusion coefficient
  - \( a(x), s(x) \) smoothly varying coefficients (adv. velocity, reaction term)
  - \( f(x) \) source term

- Equivalent first order formalism:
  \[
  \begin{cases}
    \sigma - (a u - \kappa \nabla u) = 0 \quad \text{in } \Omega \\
    s u + \nabla \cdot \sigma = f \quad \text{in } \Omega
  \end{cases}
  \]
Ultra Weak Formulation

- Discontinuous Petrov-Galerkin formalism:
  - Integration by parts over a single element \( K \) in a mesh \( \Omega_h \)
    \[
    0 = \int_{K} \tau \cdot (\sigma - au) \, dx - \int_{K} \nabla \cdot \tau \kappa u \, dx + \int_{\partial K} \kappa \mathbf{n} \cdot \tau u \\
    + \int_{K} \nu \left( su - f \right) \, dx - \int_{K} \nabla \nu \cdot \sigma \, dx + \int_{\partial K} \nu \mathbf{n} \cdot \sigma 
    \]
    for all test functions \( \nu \in H^1(K), \tau \in H(\text{div}, K) \)
  - Let \( \hat{\sigma}_n, \hat{u} \) denote traces of \( \mathbf{n} \cdot \sigma, u \), which are independent variables
    - The functional setting is not trivial
      \[
      \hat{u} \in H^{1/2}_0(\partial \Omega^h), \quad \hat{\sigma}_n \in H^{-1/2}_0(\partial \Omega^h) 
      \]
Variational Crimes

- Discrete Trial-to-Test operator:
  \[
  \begin{cases}
  \text{Find } v_A^h \in V^{h^+} \text{ s.t. } \\
  \left(v_A^h, w^h\right)_V = b\left(N_A, w^h\right) \quad \forall w^h \in V^{h^+}
  \end{cases}
  \]

  where:
  - \( V^{h^+} \) is an enriched (\( h \) and \( p \)) finite element space, not in infinite dimensional (continuous) \( V \)
    - \( 3 \times 3 \)-spline grid constructed analogously to the 1D case
  - \( N_A \) is each trial basis function
    - \( \sigma, u \) are approximated using \( L_2 \)-conforming piecewise Bernstein polynomials of degree \( p \)
    - \( \hat{\sigma}_n, \hat{u} \) are approximated using \( H^{1/2} \) - and \( H^{-1/2} \)-conforming piecewise Bernstein polynomials of degree \( p + 1 \) on wireframe (continuous/discontinuous)
Advection Skew to Mesh

$5 \times 5$ mesh, $p = 1$, $\kappa = 10^{-4}$, $\Delta p = 2$
Advection Skew to Mesh

20 × 20 mesh, $p = 2$, $\kappa = 10^{-4}$, $\Delta\rho = 3$
Advection Skew to Mesh

5 × 5 mesh, \( p = 1 \), \( \kappa = 10^{-6} \), \( \Delta p = 2 \)
Advection Skew to Mesh

$5 \times 5$ mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 3$
Advection Skew to Mesh

$5 \times 5$ mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 4$
Advection Skew to Mesh

$5 \times 5$ mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 5$
Advection Skew to Mesh

5 × 5 mesh, \( p = 1, \ \kappa = 10^{-6}, \ \Delta p = 6 \)
Advection Skew to Mesh

5 × 5 mesh, \( p = 1 \), \( \kappa = 10^{-6} \), \( \Delta p = 7 \)
Conclusions

**Helmholtz**
- DPG for multiD Helmholtz has best approximation property
  - Pollution/Dispersion effects are significantly reduced for discrete approximation (numerical evidence)
- Robust methodology for small perturbation problems
  - Advection-diffusion, Helmholtz, thin structures, etc.

**Steady Transport**
- Studied numerically the approximation of advection-diffusion-reaction problems using the discontinuous Petrov-Galerkin method with Bernstein polynomials and B-splines
- Constructed robust algorithm based on optimal test space norm and element subgrid enrichment for resolving weighting functions
- Smooth B-splines yield some computational saving when computing weighting function space