Mixed Isogeometric Finite Cell Methods for the Stokes problem

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Highlights

- The application of the Isogeometric Finite Cell Method to mixed formulations is studied.
- The performance of four families of isogeometric mixed finite elements is compared.
- For all considered elements the inf–sup stability is tested using a generic Stokes test case.
- A detailed mesh convergence study is performed to assess the optimality of all elements.

Abstract

We study the application of the Isogeometric Finite Cell Method (IGA-FCM) to mixed formulations in the context of the Stokes problem. We investigate the performance of the IGA-FCM when utilizing some isogeometric mixed finite elements, namely: Taylor–Hood, Sub-grid, Raviart–Thomas, and Nédélec elements. These element families have been demonstrated to perform well in the case of conforming meshes, but their applicability in the cut-cell context is still unclear. Dirichlet boundary conditions are imposed by Nitsche’s method. Numerical test problems are performed, with a detailed study of the discrete inf–sup stability constants and of the convergence behavior under uniform mesh refinement.

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1. Introduction

Isogeometric analysis (IGA) was proposed in [1] as a framework to reduce the gap between Computer Aided Design (CAD) and Finite Element Analysis (FEA). The fundamental idea of IGA is to employ the same basis functions to describe both the geometry of the domain of interest and the field variables. In contrast to conventional FEA which typically uses Lagrange polynomials as basis functions, IGA utilizes basis functions inherited from...
CAD modeling such as B-splines and NURBS. For analysis-suitable CAD models, geometrically exact meshing procedures can seamlessly be performed on the coarsest level of the CAD geometry. Splines provide a flexible way for refinement, de-refinement, and degree elevation. Furthermore, splines allow one to achieve higher-order continuity, this in contrast to the \(C^0\)-continuity provided by the traditional FEM. Isogeometric analysis has been applied in a wide range of application areas, from solid and structures, to fluids, and multi-physics modeling; see [2] for an overview of established IGA developments. More recent contributions to the field of IGA include research on T-splines [3,4], collocation methods [5–7], multi-patch coupling techniques [8–11], local refinement strategies [12–14], and many more. A review of the mathematical foundation of isogeometric methods can be found in [15].

Recently, the advantages of the high order basis functions in isogeometric analysis have been combined with the topological flexibility of the finite cell method (FCM). The FCM in its original form was introduced by Rank and co-workers [16,17], and belongs to a larger class of methods for which the domain boundaries do not align with the meshes (e.g., embedded domain methods, immersed boundary methods, fictitious domain methods, see [18–20]). The main idea is to extend the physical domain of interest with complexly-shaped boundaries into a larger embedding domain of simple/regular geometry, where a mesh and approximation space can be built more easily. The exploitation of this concept in the context of isogeometric analysis was first considered by Schillinger et al. [21–23]. This approach has been successfully applied to various problems in solid and structural mechanics (see [24] for a review), in image-based analysis [25,26], in fluid–structure interaction problems [27,28], and in many other application areas.

In this work, we investigate the capability of the isogeometric-based finite cell method (IGA-FCM) for solving Stokes-flow problems. When discretizing the Stokes problem, the velocity and pressure spaces cannot be chosen arbitrarily. In order to obtain a discretization which is free of locking and spurious oscillations, this pair of spaces needs to satisfy the inf–sup (or LBB) condition [29–32]. In the context of IGA, the flexibility of B-splines on structured meshes allows one to construct inf–sup stable velocity and pressure spaces with arbitrary orders, and with different regularities. Herein we study the performance of the IGA-FCM in the context of mixed formulations utilizing four important families of isogeometric mixed elements, viz. Taylor–Hood [33–35], Sub-grid [35,36], Nédélec [34], and Raviart–Thomas [34,37] elements. These isogeometric element families have been demonstrated to perform well in the case of conforming meshes. However, in the cut-cell context their applicability is still not clear. Therefore, we present a detailed numerical study and comparison of these element families in terms of: (i) discrete inf–sup constants, and (ii) convergence behavior of the errors under uniform mesh refinements. This investigation provides valuable insights into the capabilities of these element families for mixed form FCM problems in general, and complements the recent advances on the application of (IGA-)FCM to flow problems [27,38–40].

The structure of this paper is as follows: Section 2 states the Stokes problem with Nitsche’s method and discusses its well-posedness. Section 3 presents a concise introduction to the IGA-FCM with mixed formulations, and discusses in detail some of the related computational aspects. Section 4 presents the above-mentioned pairs of mixed spaces which are then numerically investigated in Section 5. Conclusions are finally drawn in Section 6.

2. Problem formulation

Our investigation of the properties of IGA-FCM for mixed problems will be presented in the context of the Stokes equations. The Stokes equations are the archetypal model problems for mixed formulations, representative of incompressible creeping flow and incompressible linear elasticity [29]. In this work, we will restrict ourselves to two-dimensional problems. However, most results extend mutatis mutandis to three dimensions.

2.1. Stokes problem

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega \). We assume that \( \partial \Omega \) is composed of two complementary open subsets \( \Gamma_D \) and \( \Gamma_N \), i.e., \( \partial \Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \) and \( \Gamma_D \cap \Gamma_N = \emptyset \). The steady Stokes problem is given by

\[
\begin{align*}
-\nabla \cdot (2\mu \nabla \mathbf{u}) + \nabla p &= \mathbf{f} \quad \text{in} \; \Omega \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in} \; \Omega \\
\mathbf{u} &= \mathbf{g} \quad \text{on} \; \Gamma_D \\
2\mu \nabla^2 \mathbf{u} \cdot \mathbf{n} - \beta \mathbf{n} &= \mathbf{h} \quad \text{on} \; \Gamma_N
\end{align*}
\]
where the body force $\mathbf{f} : \Omega \to \mathbb{R}^2$, the Dirichlet data $\mathbf{g} : \Gamma_D \to \mathbb{R}^2$, and the Neumann data $\mathbf{h} : \Gamma_N \to \mathbb{R}^2$ are exogenous data. The exterior unit normal vector to $\partial \Omega$ is denoted by $\mathbf{n}$, and $\mathbf{V}^T \mathbf{u} := \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the symmetric gradient of $\mathbf{u}$.

In a creeping-flow context, $\mu$ represents the kinematic viscosity, and $\mathbf{u}$ and $p$ indicate fluid velocity and pressure, respectively. In the context of incompressible linear elasticity, $\mu$ stands for the shear modulus, and $\mathbf{u}$ and $p$ respectively represent the displacement and pressure-like fields.

The weak formulation of (1) reads:

\[
\begin{align*}
\text{Find } (\mathbf{u}, p) & \in \mathbf{V}_{g, \Gamma_D} \times Q \text{ such that } \\
2\mu \int_{\Omega} \nabla^s \mathbf{u} : \nabla^s \mathbf{w} d\Omega & - \int_{\Omega} p \text{ div } \mathbf{w} d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} d\Omega + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{n} d\Gamma & \forall \mathbf{w} \in \mathbf{V}_{0, \Gamma_D} \\
- \int_{\Omega} q \text{ div } \mathbf{u} d\Omega & = 0 & \forall q \in Q
\end{align*}
\]

where the function spaces are defined as

\[
\mathbf{V}_{g, \Gamma_D} := \left\{ \mathbf{u} \in [H^1(\Omega)]^2 : \mathbf{u} = \mathbf{g} \text{ on } \Gamma_D \right\}, \quad Q := L^2(\Omega).
\]

Here $H^1(\Omega)$ denotes the usual Sobolev space of square-integrable functions with square integrable weak derivatives. In the case of pure Dirichlet boundary conditions, i.e., if $\Gamma_D$ coincides with all of $\partial \Omega$, the pressure is determined up to a constant. Therefore, in that case, we will supplement the system with the zero average pressure condition

\[
Q := L^2_0(\Omega) \equiv \left\{ q \in L^2(\Omega) : \int_{\Omega} q d\Omega = 0 \right\}.
\]

### 2.2. Nitsche’s method for Dirichlet boundary conditions

In (2), Dirichlet boundary conditions are imposed strongly, i.e., they are incorporated into the function space as $\mathbf{V}_{g, \Gamma_D}$. Standard finite-element approximations of (1) are based on conforming subspace approximations, which implies that the finite-element approximation space is subject to the same Dirichlet boundary conditions as $\mathbf{V}_{g, \Gamma_D}$. In the immersed-boundary setting considered in this work, such a strong enforcement of boundary conditions is intractable.

One way to bypass this issue is to impose Dirichlet boundary conditions in a weak sense. Common approaches are (see, e.g., [41]): penalty methods, Lagrange multipliers, or Nitsche’s method. In this work, Nitsche’s method [42] is favored because it preserves consistency, symmetry, ellipticity, and it extends directly to high-order finite-element approximations.

To provide a setting for the Galerkin finite-element approximation of (2) with weakly-enforced boundary conditions via Nitsche’s method, we first introduce a rectangular ambient domain $\mathcal{A} \supset \Omega$ that encompasses the physical domain $\Omega$, see Fig. 1. We cover $\mathcal{A}$ with a uniform mesh $\mathcal{T}_h^A$ composed of rectangular open element domains with diameter $h > 0$. We denote by $\mathcal{T}_h$ the corresponding mesh on $\Omega$,

\[
\mathcal{T}_h = \left\{ \kappa \subset \Omega : \kappa = k \cap \Omega, \ k \in \mathcal{T}_h^A \right\}
\]

and by $\mathcal{E}_h$ the corresponding set of boundary edges,

\[
\mathcal{E}_h = \left\{ e \subset \partial \Omega : e = \text{int}(\partial \kappa \cap \partial \Omega), \ \kappa \in \mathcal{T}_h \right\}
\]

where int(·) denotes the interior of set (·). In particular, $\mathcal{E}_h$ designates the boundary mesh that covers the Dirichlet boundary, $\Gamma_D$. The mesh $\mathcal{T}_h^A$ serves as a support structure for a pair of regular finite-element or isogeometric approximation spaces. The restrictions of these approximation spaces to the physical domain in turn provide the approximation spaces $\mathbf{V}_h \subset [H^1(\Omega)]^2$ for the velocity approximation and $Q_h \subset Q$ for the pressure approximation.

Weak enforcement of Dirichlet boundary conditions via Nitsche’s method relies on a stabilization term that is proportional to the reciprocal length of boundary edges. To define the Nitsche stabilization term, for each boundary edge $e \in \mathcal{E}_h$ we denote by $h_e$ its length. Alternatively, $h_e$ can be defined as $h_e := \sqrt{\text{area}(\kappa_e)}$ where $\kappa_e$ is the element connected to the boundary edge $e$, and area($\kappa_e$) is its area. The two definitions of $h_e$ are equivalent for
Fig. 1. Setup of the FCM: The physical domain $\Omega$ is embedded into an ambient domain $\mathcal{A}$. The ambient domain is divided into cells, which serve as a support structure for a FEM or IGA approximation space. Cells that do not intersect the physical domain are discarded (white). Cells that do intersect the physical domain form a discrete embedding domain $\Omega^h$ (gray).

shape-regular meshes, in the sense that for shape-regular meshes there exist (moderate) constants $\bar{c} \geq c > 0$ such that
\[ c \text{ length}(e) \leq \sqrt{\text{area}(\kappa_e)} \leq \bar{c} \text{ length}(e). \]

The discretized problem with Nitsche’s weak enforcement of Dirichlet boundary conditions can be cast into the form:

\[
\text{Find } (u^h, p^h) \in V^h \times Q^h \text{ such that } \\
a^h(u^h, v^h) + b(p^h, v^h) = l^1_h(v^h), \quad \forall v^h \in V^h \\
b(q^h, u^h) = l^2(q^h), \quad \forall q^h \in Q^h
\]

where

\[
a^h(u^h, v^h) = 2\mu \left( \int_{\Omega} \nabla^s u^h : \nabla^s v^h \, d\Omega - \int_{\Gamma_D} (\nabla^s u^h \cdot n) \cdot v^h \, d\Gamma \right) \\
\quad - \int_{\Gamma_D} (\nabla^s v^h \cdot n) \cdot u^h d\Gamma \right) + \mu \sum_{e \in E_D} \int_{e} \frac{\beta}{h_e} u^h \cdot v^h \, d\Gamma
\]

\[
b(q^h, v^h) = -\int_{\Omega} q^h \text{ div } v^h \, d\Omega + \int_{\Gamma_D} q^h \cdot n \cdot v^h \, d\Gamma
\]

\[
l^1_h(v^h) = \int_{\Omega_u} f \cdot v^h \, d\Omega + \int_{\Gamma_N} h \cdot v^h \, d\Gamma - 2\mu \int_{\Gamma_D} (\nabla^s v^h \cdot n) \cdot g \, d\Gamma + \mu \sum_{e \in E_D} \int_{e} \frac{\beta}{h_e} g \cdot v^h \, d\Gamma
\]

\[
l^2(q^h) = \int_{\Gamma_D} q^h \cdot n \cdot g \, d\Gamma
\]

in which $\beta > 0$ is a suitable stabilization parameter. In the bilinear form $a^h$, the second term in parenthesis is the consistency term. The third term is the symmetric consistency term, which is added to preserve the symmetry of $a^h$ and, correspondingly, to retain consistency of the dual bilinear form. The ultimate term associated with the $\beta$ parameter is referred to as the stabilization term and serves to ensure stability.

The stabilization parameter $\beta$ can be set uniformly for all edges or it can be determined locally for each edge by solving an associated local eigenvalue problem; see [43,44]. In our numerical experiments in Section 5, we select $\beta$ as a uniform global constant. We note that if the stabilization parameter is selected appropriately, i.e., large enough to retain stability but not so large to cause ill-conditioning, the observed differences in the results are negligible. The two
distinct definitions of $h_e$, i.e., as length($e$) or as $\sqrt{\text{area}(\kappa_e)}$, then generally also lead to negligible differences in the observations. Sensitivity of the results to the stabilization parameter $\beta$ and to the definition of $h_e$ can however emerge if irregular (e.g., sliver-type) cut elements occur.

It is to be noted that the condition $b(q^h, v^h) = 0$ in combination with $Q^h \supseteq \text{div} v^h$ does not generally imply $\text{div} v^h = 0$ on account of the additional term $\int_{\Gamma_D} q^h \mathbf{n} \cdot v^h \, d\Gamma$ in the bilinear form $b$. This additional term emerges from the weak imposition of Dirichlet boundary conditions via Nitsche’s method. Indeed, if the Dirichlet boundary conditions are strongly imposed and incorporated in $v^h$, then $\mathbf{n} \cdot v^h = 0$ on $\Gamma_D$ and the additional term vanishes.

2.3. Well-posedness: continuity and inf–sup conditions

To establish conditions for well-posedness of the saddle-point problem (3), we introduce the norms:

$$
\|v\|_{V^h}^2 := \|\nabla v\|_{L^2(\Omega)}^2 + \sum_{e \in E_D^h} \frac{\nu}{h_e} \|v\|_{L^2(e)}^2, \quad \|q\|_{Q^h}^2 := \|q\|_{L^2(\Omega)}^2
$$

for some positive constant $\nu \in \mathbb{R}_{>0}$. Problem (3) is well-posed if and only if the following conditions hold [29,30]:

- Continuity of $a^h$ and $b$:

  $$
  \exists C_a \in \mathbb{R}_{>0} : \quad a^h(u^h, v^h) \leq C_a \|u^h\|_{V^h} \|v^h\|_{V^h}, \quad \forall (u^h, v^h) \in V^h \times V^h
  $$

- Weak coercivity of $a^h$ on the kernel of $b$:

  $$
  \exists \alpha \in \mathbb{R}_{>0} : \inf_{u^h \in Z^h \setminus \{0\}} \sup_{v^h \in Z^h \setminus \{0\}} \frac{a^h(u^h, v^h)}{\|u^h\|_{V^h} \|v^h\|_{V^h}} \geq \alpha
  $$

  with $Z^h := \{v^h \in V^h : b(q^h, v^h) = 0, \forall q^h \in Q^h\}$.

- Inf–sup condition on $b$:

  $$
  \exists \gamma \in \mathbb{R}_{>0} : \inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(q^h, v^h)}{\|q^h\|_{Q^h} \|v^h\|_{V^h}} =: \gamma^h \geq \gamma.
  $$

The mesh-dependent term in the norm for the velocity space in (5) is required for continuity of the bilinear form $a^h$ according to (6) with respect to the $\| \cdot \|_{V^h}$-norm. Indeed, the Cauchy–Schwarz inequality in combination with a standard discrete trace inequality (see, e.g., [45, Lemma 1.44] or [46]) conveys:

$$
\left| \int_{\Gamma_D} (\nabla^e u^h \cdot \mathbf{n}) \cdot v^h \, d\Gamma \right| \leq \sum_{e \in E_D^h} \|\nabla^e u^h\|_{L^2(e)} \|v^h\|_{L^2(e)} \leq \sum_{e \in E_D^h} \frac{C_{tr.e}}{\sqrt{h_e}} \|\nabla u^h\|_{L^2(e)} \|v^h\|_{L^2(e)} \leq \frac{C_{tr}}{\sqrt{\nu}} \left( \sum_{e \in E_D^h} \|\nabla u^h\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in E_D^h} \frac{\nu}{h_e} \|v^h\|_{L^2(e)}^2 \right)^{1/2} \leq \frac{C_{tr}}{\sqrt{\nu}} \|u^h\|_{V^h} \|v^h\|_{V^h}
$$

for certain edge-wise positive constants $C_{tr.e} \in \mathbb{R}_{>0}$, dependent on the shape of $\kappa_e$ but independent of $h_e$, and $C_{tr} = \max\{C_{tr.e} : e \in E_D^h\}$. From (10) one can infer that $a^h$ is bounded and coercive on $V^h$ for appropriate choices of $\beta$ and $\nu$. Eq. (6) follows directly from (10) for appropriate $\beta$ and $\nu$. The continuity condition on $b$ (7) is also satisfied for the norms in (5). The coercivity of the bilinear form $a^h : V^h \times V^h \rightarrow \mathbb{R}$ transfers to the subspace $Z^h \subset V^h$, which implies (8).

The inf–sup condition on $b$ in (9) is generally more subtle and requires a suitable choice of the approximation-space pair $(V^h, Q^h)$. In the context of IGA, stable pairs $(V^h, Q^h)$ have been studied in, e.g., [33–36]. These IGA velocity–pressure pairs will be discussed in Section 4.2 before considering their extension to the finite-cell setting. The discrete inf–sup constant $\gamma^h$ in (9) can be computed explicitly based on the algebraic representation of (3); see Section 3.4.
Remark 2.1. With a minor modification, system (3) is also representative of compressible elasticity problems:

\[
\begin{align*}
\text{Find } (u^h, p^h) &\in V^h \times Q^h \text{ such that } \\
\alpha^h(u^h, v^h) + b(p^h, v^h) &= l_1^h(v^h) \forall v^h \in V^h \\
b(q^h, u^h) - \frac{1}{2} c(q^h, p^h) &= l_2(q^h) \forall q^h \in Q^h.
\end{align*}
\]

(11)

where \( \mu, \lambda \) are the first and second Lamé parameters, respectively, \( u^h \) is the displacement field, \( p = -\lambda \text{div } u \) represents the pressure-like field, and \( c(p, q) = \int_{\Omega} pq \, d\Omega \). For well-posedness of the problem in the nearly incompressible limit \( \lambda/\mu \gg 1 \), it is important that the inf–sup condition of (9) holds also in this case; see, e.g., [29].

3. Mixed formulation of the Finite Cell Method

The Finite Cell Method (FCM) was introduced by Rank and coworkers in [17]. In its original form, the FCM combines three concepts: the fictitious domain method, the \( p \)-version FEM, and an adaptive integration technique for cut cells. In this section, we first discuss the fundamental ideas of the FCM along with its volume and boundary integration procedures. We then present the algebraic form of the FCM for mixed formulations.

3.1. Basic setup of the Finite Cell Method

In the FCM the physical domain is embedded into a fictitious (or embedding or ambient) domain \( A \) with simple – typically rectangular – geometry; cf. Fig. 1. This extended domain is covered by a collection \( T^h_A \) of cells of regular shape, where the affix \( h > 0 \) indicates a resolution parameter, e.g., \( h = \max(\text{diam}(\kappa) : \kappa \in T^h_A) \). Cells that do not intersect the physical domain are discarded, and the remaining cells serve as a support structure for basis functions in a similar manner as the elements in FEM. The remaining cells form a discrete embedding domain,

\[
\Omega^h := \{ \kappa \in T^h_A : \kappa \cap \Omega \neq \emptyset \};
\]

see Fig. 1. The FCM provides a formulation for the numerical approximation of the problem under consideration on the physical domain \( \Omega \) and its extension onto the discrete embedding domain \( \Omega^h \). Significance is assigned only to the discrete approximation on \( \Omega \). While the general computational scheme is similar to standard FEM, essential boundary conditions are typically weakly enforced in the FCM. This weak enforcement is most commonly based on Nitsche’s method [42], which circumvents the indeterminate behavior of the approximation space along the – in principle arbitrary shaped – domain boundary \( \partial \Omega \).

To mitigate conditioning problems related to the occurrence of cut cells with small volume fractions, in the FCM it is common practice to assign a (very small) virtual “stiffness” to the exterior of the domain [47]. This extension of the problem onto the exterior \( \Omega^h \setminus \Omega \) generally leads to inconsistency of the formulation, which in some cases can affect the behavior of mixed elements. Therefore, we do not use a virtual “stiffness” in this work, but instead we use a Jacobi preconditioner to avoid ill-conditioning problems [48].

3.2. Computation of volume integrals

By virtue of the fact that the FCM formulation is restricted to the physical domain, \( \Omega \), the integrands to be evaluated over cells that are intersected by the boundary, \( \partial \Omega \), are generally discontinuous. Accordingly, standard quadrature rules yield inadequate accuracy on cut cells. The FCM is therefore generally complemented with an adaptive numerical-integration technique (see Fig. 2).

Herein we employ the commonly used multi-level integration scheme based on recursive bi-sectioning [22]. While cells that are completely within the physical domain are integrated using standard quadrature rules, an adaptive integration technique is used for cut cells. This technique subdivides each cut cell into four uniform subcells and this subdivision process is continued recursively, i.e., the subcells which are intersected by the boundary are again partitioned into four uniform subcells. This process is repeated until a prescribed recursion depth \( k \in \mathbb{Z}_{\geq 0} \) has been reached. On this deepest level of recursion, quadrature points exterior to \( \Omega \) are discarded. The resulting recursive bi-sectioning process exhibits a quadtree structure in 2D and an octree structure in 3D. The detection of subcell boundary intersections is based on performing an inside-outside check of the subcell vertices on each level of the
recursion. We note that for the analytical geometries considered herein the employed top-down approach robustly controls the integration accuracy. For arbitrary geometries this top-down recursion can halt prematurely, leading to geometric inaccuracies [49]. A bottom-up approach can in such cases be employed to retain geometric precision.

In this work we use B-spline basis functions in combination with the aforementioned adaptive integration technique. The integrals in each (sub)cell are evaluated with $p + 1$ Gauss quadrature points in each direction. The recursion depth is selected such that the integration error is negligible compared to the discretization error. For higher-order approximations, this implies that the required integration depth generally increases rapidly as the mesh is refined, typically proportional to $h^{1-p}$ where $p$ indicates the order of accuracy of the considered FEM or IGA space (in the energy norm). Although no new degrees of freedom are introduced as the integration depth increases, the computational cost related to the integration of cut cells does increase significantly. This increase in computational effort can be ameliorated by means of recent, more advanced cut-cell integration techniques, e.g., [26,50–52].

3.3. Computation of boundary integrals

When the FCM is used in combination with IGA, the boundaries of the domain are usually represented by spline curves (in 2D) or spline surfaces (in 3D). These parameterized curves or surfaces can then be used directly for the evaluation of boundary integrals.

To determine the boundary integrals in (4), notably, the contribution of the Dirichlet boundary to all functionals and the contribution of the Neumann boundary to $l^h_1$, we assume that a parameterization of the boundary is available. As illustrated in Fig. 3 the intersections of the boundary with the background mesh $T^h_A$ are located. In this way, the boundary is partitioned into a set of (curved) edge elements, $E^h$. Each of these edge elements inherits its parameterization from the underlying parameterization of the boundary. For simple boundary geometries, such as lines, circles or conic sections, this step can be efficiently performed, especially in 2D, by virtue of the regularity of the background mesh. By assigning quadrature points to the edge elements, the boundary integrals can be evaluated. In this work we employ $p + 1$ Gauss points for the edge elements.

In case of complex geometries, one can linearize or approximate the boundary with a more simple/regular parameterization before computing the intersection with the mesh. Alternatively, following [26], an accurate approximation to the boundary can be obtained by connecting the intersection points of the boundary with the integration subgrid, viz.
the highest level of bi-sectioning used in the integration procedure. The latter procedure is also particularly useful if a parameterization of the boundary is not available, e.g., if the boundary is described by means of a level-set function.

3.4. Matrix form of the mixed finite cell method

Suppose that \( \{ N_u^i \}_{i=1}^{n_u} \) and \( \{ N_p^i \}_{i=1}^{n_p} \) are basis functions of the finite dimensional velocity space \( V^h \) and pressure space \( Q^h \), respectively, i.e., \( V^h = \text{span}(\{ N_u^i \}_{i=1}^{n_u}) \) and \( Q^h = \text{span}(\{ N_p^i \}_{i=1}^{n_p}) \). These basis functions can be constructed by using finite element technology, isogeometric analysis (e.g., B-spline, NURBS), or possibly with (local) adaptivity [23]. A detailed review can be found in [24].

The approximate velocity \( u^h \) and pressure \( p^h \) are then written as

\[
    u^h(x) = \sum_{i=1}^{n_u} N_u^i(x) \hat{u}_i, \quad p^h(x) = \sum_{i=1}^{n_p} N_p^i(x) \hat{p}_i
\]

where \( \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots)^T \) and \( \hat{p} = (\hat{p}_1, \hat{p}_2, \ldots)^T \) are vectors of degrees of freedom. The corresponding algebraic form of (3) reads

\[
\begin{bmatrix}
    A & B^T \\
    B & 0
\end{bmatrix}
\begin{bmatrix}
    \hat{u} \\
    \hat{p}
\end{bmatrix}
=
\begin{bmatrix}
    f_1 \\
    f_2
\end{bmatrix}
\tag{12}
\]

where the matrices \( A \), \( B \), and vectors \( f_1 \), \( f_2 \) are given by

\[
    A_{ij} = a^h(N_u^j, N_u^i) \quad f_{1i} = l_i^h(N_u^i)
\]

\[
    B_{ij} = b(N_p^j, N_u^i) \quad f_{2i} = l_2(N_p^i).
\]

In our numerical computations, we extract the approximate solution from (12). The discrete form of the mixed FCM formulation can also serve to compute the discrete inf–sup constant \( \gamma^h \) in (9) for a particular pair of velocity and pressure approximation spaces. The discrete inf–sup constant coincides with the square root of the smallest non-zero eigenvalue of the following generalized eigenvalue problem (see, e.g., [53]):

\[
    B M_{uu}^{-1} B^T q = (\gamma^h)^2 M_{pp} q.
\]
where $M_{u u}$ and $M_{p p}$ are the Gramian matrices associated with the inner products in $V^h$ and $Q^h$, respectively:

$$(M_{u u})_{i j} = (N_i^u, N_j^u)_{V^h}, \quad (M_{p p})_{i j} = (N_i^p, N_j^p)_{Q^h}.$$  

The norms associated to $(\cdot, \cdot)_{V^h}$ and $(\cdot, \cdot)_{Q^h}$ are specified by (5).

If the Neumann boundary is not empty, all eigenvalues of (13) are strictly positive, and the discrete inf–sup constant corresponds to the smallest eigenvalue. In the case of pure Dirichlet boundary conditions, the smallest eigenvalue is zero and it has algebraic and geometric multiplicity one. The associated eigenvector corresponds to the constant pressure mode. In this case, the discrete inf–sup constant is the second smallest eigenvalue.

The essential advantage of FCM is that it admits regular and structured meshes even for complex geometries. Regular, structured meshes facilitate the definition and evaluation of higher-order bases and bases with increased smoothness compared to standard FEA, viz. $C^k$-continuous bases with $k \geq 1$. Recently, several inf–sup stable approximation-space pairs have been introduced, based on the increased smoothness provided by B-spline bases. In this paper, we introduce the B-spline pairs of velocity and pressure spaces herein studied. In Section 4, we review common pairs of approximation spaces and their construction, while, in Section 5, we present a numerical investigation of these spaces in the context of the FCM.

4. Isogeometric analysis and mixed elements

In this section we present a brief overview of the isogeometric analysis concepts relevant to this work; then, we introduce the B-spline pairs of velocity and pressure spaces herein studied.

4.1. Fundamentals of B-splines

Given two integers $p \geq 0$ and $n > 0$, $n$ B-spline basis functions of degree $p$ can be defined over a knot vector $\Xi = [\xi_1, \xi_2, \ldots, \xi_{n+p+1}]$, which is a non-decreasing sequence of parametric coordinates, $\xi_i \leq \xi_{i+1}, i = 1, \ldots, n+p$. If all interior knots are equally spaced the knot vector is called uniform; otherwise, it is called non-uniform. If the first and the last knots are repeated $p+1$ times, the knot vector is called open. In what follows, we always employ open knot vectors, and without loss of generality, we also assume the parameter domain to be $[0, 1]$. Basis functions formed from open knot vectors are interpolatory at the boundaries of the parameter domain.

A B-spline basis function is $C^\infty$-continuous inside knot spans and, at most, $C^{p-1}$-continuous at a knot. If an interior knot value is repeated more than one time, it is called a multiple knot. We introduce the vector $[\xi_1, \ldots, \xi_m]$ of knots without repetitions, and the vector $[r_1, \ldots, r_m]$ of their associated multiplicities such that

$$\Xi = [\xi_1, \ldots, \xi_{r_1}, \ldots, \xi_{r_2}, \ldots, \xi_{r_m}, \ldots, \xi_m]$$

where $\sum_{i=1}^m r_i = n + p + 1$.

At a knot of multiplicity $r_i$, the continuity is $C^{\alpha_i}$ where $\alpha_i = p - r_i$ is the regularity.

The associated knot mesh on the parameter domain $[0, 1]$ is defined as

$$\mathcal{T}_h = \{ I = (\xi_i, \xi_{i+1}), 1 \leq i \leq m \}.$$

For an element $I \in \mathcal{T}_h$, we set $h_I = \text{diam}(I)$, and the global mesh parameter is indicated as $h = \max\{ h_I \}_{I \in \mathcal{T}_h}$.

Given a knot vector, the B-spline basis functions $N_{i, p}(\xi)$ are defined starting with the piecewise constants ($p = 0$)

$$N_{i, 0}(\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0, & \text{otherwise}; \end{cases} \quad (14)$$

and for $p \geq 1$, they are defined recursively by the Cox–de Boor formula [54]

$$N_{i, p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i, p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1, p-1}(\xi). \quad (15)$$

We denote the space of open B-splines spanned by the basis functions $N_{i, p}$ with regularity $\alpha$ at all internal knots by

$$S_{a, h}^p = S_{a}^p(\mathcal{T}_h) := \text{span}\{N_{i, p}\}_{i=1}^n.$$
For each B-spline basis function \( N_{i,p} \) we also define the associated Greville abscissa, i.e., the knot average
\[
\bar{\xi}_{i,p} = \frac{\xi_{i+1} + \cdots + \xi_{i+p}}{p}, \quad i = 1, \ldots, n.
\]
When the multiplicity of the internal knots is not greater than \( p \), all the Greville abscissae are distinct, i.e., for each Greville abscissa there is only one associated B-spline basis function.

In two dimensions, we consider the parameter domain \( \hat{\Omega} = (0, 1)^2 \subset \mathbb{R}^2 \) (Fig. 4(a)). Given the integers \( p_d, \alpha_d \), and knot vectors \( \Xi_d, \Gamma_d \) with associated knot mesh \( \mathcal{I}_{h,d} \), where \( d = 1, 2 \), the knot mesh \( \mathcal{M}_h \) for the two dimensional parametric domain is defined as
\[
\mathcal{M}_h = \bigotimes_{d=1,2} \mathcal{I}_{h,d}.
\]
For an element \( Q \in \mathcal{M}_h \), we set \( h_Q = \text{diam}(Q) \), and the global mesh is indicated as \( h = \max \{ h_Q \}_{Q \in \mathcal{M}_h} \).

The space of bivariate B-splines is defined as
\[
S_{\alpha_1,\alpha_2,h}^{p_1, p_2} := \bigotimes_{d=1,2} S_{\alpha_d}^{p_d} (\mathcal{I}_{h,d}) = \text{span} \{ N_{i_1,p_1}(\xi) N_{j_2,p_2}(\eta) \}_{i=1,j=1}^{n_1,n_2}
\]
where \( \{ N_{i_1}(\xi) \}_{i=1}^{n_1} \) and \( \{ N_{j_2}(\eta) \}_{j=1}^{n_2} \) are canonical bases of \( S_{\alpha_1}^{p_1} \) and \( S_{\alpha_2}^{p_2} \). The Greville abscissae of a bivariate B-spline are defined as \( \hat{\gamma}_{i,j} = (\bar{\xi}_{i,p_1}, \bar{\eta}_{j,p_2}) \) with \( \bar{\xi}_{i,p_1}, \bar{\eta}_{j,p_2} \) being respectively the Greville abscissae of \( N_{i_1,p_1} \) and \( N_{j_2,p_2} \).

B-spline surfaces are obtained from linear combinations of bivariate B-spline basis functions
\[
S(\xi, \eta) = \sum_{i,j} P_{i,j} N_{i_1,p_1}(\xi) N_{j_2,p_2}(\eta)
\]
where \( P_{i,j} \) are the so-called control points.

For the regular grids considered herein, the control points and the Greville abscissae have coincident locations.

An illustration of a bivariate B-spline and its associated control points for \( p_1 = 3 \), \( p_2 = 2 \), \( \alpha_1 = 2 \), \( \alpha_2 = 1 \) is presented in Fig. 4.

In order to approximate the unknown fields, isogeometric analysis employs the isoparametric concept as in standard FEM, i.e., the same B-spline basis functions are used for the description of both the geometry and the unknown fields. In the case of mixed formulations, the pressure field and each component of the velocity field possess their own control nets, which are all defined on the same geometry. The basis functions associated with these control nets are then used to approximate the fields; see Section 4.2.

### 4.2. Mixed B-spline discretizations

As mentioned in the previous sections, one important advantage of FCM is that the background mesh of the immersed domain is regular. This allows one to use only B-splines for the approximation instead of NURBS as in standard IGA. The immersed domain is a rectangle (in 2D, or rectangular cuboid in 3D) and therefore the geometry map becomes affine. As a consequence, pull-back mappings such as the Piola transform become trivial. We note that low-order B-splines can be used for the analysis, even if the physical domain is parameterized by high-order splines.

In this work, we investigate the behavior of the following families of mixed elements in the IGA-FCM setting, which are graphically depicted in Fig. 5:

- **Taylor-Hood element** [33–35] (Fig. 5(a)):
  \[
  \begin{align*}
  V_h^{TH} &= (S_{p-1,p-1}^{p+1,p+1})^2, \\
  Q_h^{TH} &= S_{p-1,p-1}^{p,p}.
  \end{align*}
  \]
  The velocity and pressure spaces are defined on the same knot mesh with the same regularity. The two components of velocity are defined with the same space. The velocity space is one order higher than the pressure space. This implies that the multiplicity of the knots of the velocity space is increased by one with respect to that of the pressure space.
Fig. 4. An illustration of a bivariate B-spline basis function on a rectangular domain with $p = (3, 2), \alpha = (2, 1)$. (a) Parameter domain: The triangles denote the Greville abscissae (and the one associated with the considered basis function is marked in red). (b) Physical domain: The dashed line denotes the control net; the solid lines denote the knot lines (i.e., the mesh); the circles denote the control points (and the one associated with the considered basis function is marked in red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- Raviart–Thomas element [34,37] (Fig. 5(b)):

$$
\begin{align*}
\mathbf{v}_{h}^{RT} &= \mathbf{S}_{p, p-1, h}^{p+1, p} \times \mathbf{S}_{p-1, p, h}^{p, p+1} , \\
Q_{h}^{RT} &= \mathbf{S}_{p-1, p-1, h}^{p, p} .
\end{align*}
$$

(17)

This is a $H(\text{div})$ conforming element. The velocity space is anisotropic with respect to the degree. Both velocity and pressure spaces are defined on the same knot mesh with their highest regularities.
• Nédélec element [34] (Fig. 5(c)):

\[
\begin{align*}
V_h^N & = S_{p-1,p-1,h}^{p+1,p+1} \times S_{p-1,p,h}^{p+1,p+1}, \\
Q_h^N & = S_{p-1,p-1,h}^{p-1,p-1,1}. 
\end{align*}
\]

(18)

This element “lies” between the Taylor-Hood and the Raviart–Thomas elements. The velocity space uses equal degrees for both components as in the Taylor-Hood element, while maintaining the regularities of Raviart–Thomas element. Both velocity and pressure spaces are defined on the same knot mesh.
• Sub-grid element [35,36] (Fig. 5(d)):

\[
\begin{align*}
V^\text{SG}_h &= (s^p+1,p+1)^2, \\
Q^\text{SG}_h &= S^{p-1,p-1}. \\
\end{align*}
\tag{19}
\]

The velocity and the pressure spaces are defined on different knot meshes. The velocity knot mesh is obtained by subdividing each element of the pressure knot mesh into four elements (in 2D, and 8 elements in 3D.) Both velocity and pressure spaces have their highest regularities.

Since the velocity mesh is refined with respect to the pressure mesh, the Gauss points are associated with the velocity mesh. Consequently, in order to evaluate the pressure basis functions in these points, a special data structure is needed to allow the determination of the pressure element associated with a particular velocity element.

5. Numerical experiments

In this section we investigate the numerical performance of IGA-FCM for mixed formulations. System (12) is solved using the four mixed element families discussed in Section 4.2. The numerical inf–sup values are computed by solving the generalized eigenvalue problem (13).

The accuracy of the IGA-FCM approximation depends on the applied integration depth. If the integration depth is insufficient, the accuracy of the approximation is determined by the integration error, rather than the error in the IGA approximation. This is particularly manifest for higher-order approximations, due the fact that the integration scheme is typically of low order (1 or 2); see Section 3.2. In our test cases, we have selected integration depth \( k = 13 \). This integration depth ensures that for the considered range of polynomial orders and mesh-sizes, the integration error is subordinate to the IGA-approximation error. Hence, it allows to test and compare the different methods in the considered range of mesh-sizes and polynomial orders. We use Jacobi preconditioning in the solution procedure for the discrete systems, to reduce the effect of ill-conditioning and the corresponding solution error on the accuracy of the results. The condition numbers of the linear systems corresponding to the IGA-FCM discretization generally increase as the mesh is refined, because mesh refinement typically leads to the occurrence of cut cells with smaller volume fractions due to the fact that the number of cut cells increases, and as the polynomial order of the approximation is increased [48].

In all computations, we have used a global, uniform \( p \)-dependent stabilization parameter according to \( \beta = 5(p + 1)^2 \). The \( p \)-dependence of the stabilization parameter has been selected in accordance with the corresponding dependence of the constant in the underlying discrete trace inequality [46]. It is to be mentioned that on fine meshes and at higher orders of approximation, the convergence results presented in Section 5.3 display some minor (non-essential) sensitivity to the stabilization parameter.

5.1. Dirichlet quarter-annulus problem

We investigate the properties of the IGA-FCM mixed formulation in combination with the element families in (16)–(19) on the basis of (3) on an open quarter-annulus domain

\[
\Omega = \left\{ (x, y) \in \mathbb{R}^2_{>0} : 1 < x^2 + y^2 < 16 \right\},
\]

with inner radius \( R_1 = 1 \) and outer radius \( R_2 = 4 \); see Fig. 6. Dirichlet boundary conditions are prescribed on the entire boundary \( \partial \Omega = \Gamma_D \) and, accordingly, it holds that \( \Gamma_N = \emptyset \). The data \( f \) and \( g \) are selected such that

\[
\begin{align*}
\mathbf{u}_1 &= 10^{-6}x^2y^4(x^2 + y^2 - 1)(x^2 + y^2 - 16)(5x^4 + 18x^2y^2 - 85x^2 + 13y^4 - 153y^2 + 80), \\
\mathbf{u}_2 &= 10^{-6}xy(x^2 + y^2 - 1)(x^2 + y^2 - 16)(102x^2 + 34y^2 - 10x^4 - 12x^2y^2 - 2y^4 - 32), \\
p &= 10^{-7}xy(y^2 - x^2)(x^2 + y^2 - 16)(x^2 + y^2 - 1)^2 \exp(14(x^2 + y^2)^{-1/2}) \quad \tag{20}
\end{align*}
\]

satisfy (3). The sample solution (20) has been taken from [55]. Note that \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) vanish on \( \partial \Omega \) and, hence, \( g = 0 \). Moreover, the pressure complies with \( \int_\Omega p = 0 \). The exact velocity and pressure contours are plotted in Fig. 7.
Table 1
Discrete inf–sup constants of IGA-FCM for the quarter annulus problem using the four considered mixed elements with polynomial degrees $p \in \{1, 2, 3\}$ for the pressure field.

<table>
<thead>
<tr>
<th></th>
<th>$C_{\text{inf–sup}}$</th>
<th>Mesh</th>
<th>8 x 8</th>
<th>16 x 16</th>
<th>32 x 32</th>
<th>64 x 64</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p = 1$</td>
<td>TH</td>
<td>.5551</td>
<td>.5174</td>
<td>.4900</td>
<td>.4703</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RT</td>
<td>.2931</td>
<td>.2860</td>
<td>.2324</td>
<td>.2482</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ND</td>
<td>.5195</td>
<td>.5173</td>
<td>.4899</td>
<td>.4703</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SG</td>
<td>.6393</td>
<td>.5619</td>
<td>.5157</td>
<td>.4861</td>
<td></td>
</tr>
<tr>
<td>$p = 2$</td>
<td>TH</td>
<td>.5469</td>
<td>.5137</td>
<td>.4879</td>
<td>.4690</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RT</td>
<td>.2266</td>
<td>.1979</td>
<td>.1995</td>
<td>.1621</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ND</td>
<td>.4303</td>
<td>.4538</td>
<td>.3893</td>
<td>.3964</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SG</td>
<td>.6221</td>
<td>.5562</td>
<td>.5132</td>
<td>.4848</td>
<td></td>
</tr>
<tr>
<td>$p = 3$</td>
<td>TH</td>
<td>.5438</td>
<td>.5122</td>
<td>.4870</td>
<td>.4685</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RT</td>
<td>.2110</td>
<td>.1897</td>
<td>.1753</td>
<td>.1696</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ND</td>
<td>.3588</td>
<td>.3701</td>
<td>.3058</td>
<td>.3095</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SG</td>
<td>.6155</td>
<td>.5538</td>
<td>.5121</td>
<td>.4842</td>
<td></td>
</tr>
</tbody>
</table>

5.2. Numerical inf–sup test

To test the stability of the IGA-FCM formulation for the four different families of mixed elements, we compute the associated discrete inf–sup values by means of (13) for pressure polynomial degrees $p \in \{1, 2, 3\}$ and on a sequence of uniform background meshes with 8 x 8, 16 x 16, 32 x 32, and 64 x 64 elements. The results are reported in Table 1 and visualized in Fig. 8. Table 1 conveys that for all four families of mixed finite-elements the corresponding discrete inf–sup constants are bounded from below away from 0. In particular, the Taylor–Hood and Sub-grid elements display very similar inf–sup values for all considered polynomial orders. On the finest mesh (64 x 64), these elements exhibit inf–sup values of approximately 0.47–0.49, essentially independent of $p$. For the Nédélec element, the inf–sup value tends to decrease as the polynomial order increases. While for $p = 1$ the inf–sup constant of the Nédélec element is comparable to that of the Taylor–Hood and Sub-grid elements, for $p \in \{2, 3\}$ it exhibits smaller values.
The Raviart–Thomas element yields the smallest inf–sup values. This result is consistent with the fact that the divergence of the velocity space is smallest for the Raviart–Thomas elements, in the sense that $\text{div} V_{RT}^h$ is a proper subset of $\text{div} V_{TH}^h$, $\text{div} V_{ND}^h$ and $\text{div} V_{SG}^h$. One may also note that the inf–sup value of the Nédélec element generally lies between the inf–sup values of the Raviart–Thomas element and of the Taylor–Hood element, which is in agreement with the inclusion relations $V_{RT}^h \subset V_{ND}^h \subset V_{TH}^h$. Fig. 8 corroborates that for the considered range of polynomial orders and meshes, all four elements pass the inf–sup test. The numerical results thus suggest that all four elements are stable on uniform meshes, irrespective of the mesh-size and polynomial order. The results in Fig. 8 also convey that the inf–sup constant of the Raviart–Thomas element exhibits some fluctuations as the mesh-size varies. This behavior can be attributed to the fact that the Raviart–Thomas element belongs to the $H(\text{div})$-conforming family, i.e., the divergence operator maps $V_{RT}^h$ into and onto $Q_{RT}^h$. On account of the compatibility between $\text{div} V_{RT}^h$ and $Q_{RT}^h$, the Raviart–Thomas element is sensitive to the manner in which constraints such as Dirichlet conditions are imposed, because such constraints generally interfere with the congruence of $\text{div} V_{RT}^h$ and $Q_{RT}^h$. 
5.3. Convergence study

In this section, we test the convergence of the IGA-FCM formulation for the four considered mixed element families. Table 2 presents the relative $L^2$ and $H^1$ errors of the velocity field, and the $L^2$ error of the pressure field, when the degree of the pressure is $p = 1$. For Taylor–Hood, Nédélec and Sub-grid elements, the velocity spaces are isotropic and one order higher than that of the pressure space. The optimal convergence rate for the velocity approximations that can be obtained with these elements is therefore 2 in the $H^1$-norm and 3 in the $L^2$-norm. For the Raviart–Thomas element, the velocity approximation space is anisotropic with respect to the degree, and the space is complete only up to degree $p$. The optimal convergence rates of the velocity approximation in the $H^1$-norm and the $L^2$-norm are then restricted to 1 and 2, respectively. Table 2 corroborates that the aforementioned optimal convergence rates are indeed obtained.

Considering the convergence of the pressure approximation, the results in Table 2 indicate that the Taylor–Hood, Nédélec and Sub-grid element yield optimal convergence rates, in particular, $\|p - p_h\|_{L^2} = O(h^{p+1})$ as $h \to 0$. The Raviart–Thomas element appears to display a suboptimal convergence rate, although the asymptotic convergence
velocity-approximation error decays as $\|v - v_h\|_{L^2}$ and $\|p - p_h\|_{H^1}$ for the Taylor–Hood, Raviart–Thomas, Nédélec and Sub-grid elements, respectively.

The corresponding results are presented in Table 3 and Fig. 10, and in Table 4 and Fig. 11, respectively. For $p = 1$, the observed convergence rates for the Taylor–Hood and Nédélec elements are close to the optimal convergence rates. For the Sub-grid element, the results indicate an optimal rate of convergence for the velocity approximation. The results for the pressure approximation are inconclusive. For the Raviart–Thomas element, the velocity approximation displays optimal convergence rates in both the $H^1$-norm and the $L^2$-norm. The convergence rate for the pressure

Table 2
Relative error of IGA-FCM for the quarter annulus problem for the Taylor–Hood (TH), Raviart–Thomas (RT), Nédélec (ND) and Sub-grid (SG) elements with $p = 1$ for the pressure field.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>8 x 8</th>
<th>16 x 16</th>
<th>32 x 32</th>
<th>64 x 64</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u - u_h^{TH}|_{L^2}$</td>
<td>7.76e−2</td>
<td>1.49e−2</td>
<td>2.24e−3</td>
<td>3.11e−4</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>2.38</td>
<td>2.74</td>
<td>2.85</td>
</tr>
<tr>
<td>$|u - u_h^{TH}|_{H^1}$</td>
<td>3.68e−1</td>
<td>1.28e−1</td>
<td>2.98e−2</td>
<td>6.23e−3</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>1.53</td>
<td>2.10</td>
<td>2.26</td>
</tr>
<tr>
<td>$|p - p_h^{TH}|_{L^2}$</td>
<td>7.57e−1</td>
<td>2.27e−1</td>
<td>3.97e−2</td>
<td>8.23e−3</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>1.74</td>
<td>2.51</td>
<td>2.27</td>
</tr>
<tr>
<td>$|u - u_h^{RT}|_{L^2}$</td>
<td>3.37e−1</td>
<td>1.29e−1</td>
<td>3.85e−2</td>
<td>1.02e−2</td>
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<tr>
<td>Order</td>
<td>–</td>
<td>1.38</td>
<td>1.74</td>
<td>1.92</td>
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<td>$|u - u_h^{RT}|_{H^1}$</td>
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<td>6.01e−1</td>
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<td>0.53</td>
<td>0.71</td>
<td>1.50</td>
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<td>$|u - u_h^{ND}|_{L^2}$</td>
<td>9.46e−2</td>
<td>1.67e−2</td>
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<td>3.23e−4</td>
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<td>–</td>
<td>2.51</td>
<td>2.78</td>
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<td>1.97</td>
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<td>2.31</td>
<td>2.28</td>
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<td>$|u - u_h^{SG}|_{L^2}$</td>
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<td>2.13e−2</td>
<td>2.82e−3</td>
<td>3.51e−4</td>
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<tr>
<td>Order</td>
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<td>2.68</td>
<td>2.92</td>
<td>3.01</td>
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<td>$|u - u_h^{SG}|_{H^1}$</td>
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<td>1.89</td>
<td>1.95</td>
</tr>
<tr>
<td>$|p - p_h^{SG}|_{L^2}$</td>
<td>8.30e−2</td>
<td>1.77e−1</td>
<td>4.79e−2</td>
<td>7.83e−3</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>2.23</td>
<td>1.89</td>
<td>2.61</td>
</tr>
</tbody>
</table>

rate is not yet fully apparent from the considered sequence of meshes. The following a-priori estimate (see [37, Theorem 6.2])

$$\|p - p_h\| \leq \left(1 + \frac{1}{\gamma_h}\right) \inf_{q^h \in Q_h} \|p - q_h\|_{Q_h} + \frac{C_a}{\gamma_h} \|u - u_h\|_{V_h}$$

however conveys that the convergence rate of the pressure is potentially restricted by the convergence rate of the velocity in the $H^1$-norm, which is only of order $p = 1$ for the Raviart–Thomas element, and therefore one order lower than the interpolation error in the pressure. For the Taylor–Hood, Nédélec and Sub-grid elements, the velocity-approximation error decays as $\|u - u_h\|_{V_h} = O(h^{p+1})$ as $h \to 0$, and this rate coincides with the rate of convergence of the pressure-interpolation error.

To facilitate a comparison of the results obtained by means of the four mixed elements in the context of IGA-FCM, Fig. 9 plots the relative errors with respect to the total degrees of freedom (of the velocity and pressure spaces). One can observe that the Sub-grid element provides the most efficient approximation, in that it achieves the lowest error per degree of freedom for both pressure and velocity. This result is consistent with the theory of $k$-refinement, since both the velocity and pressure of the Sub-grid element possess the highest order of continuity that can be attained without degenerating to a single element.

In a similar manner, we test the four considered mixed elements with pressure spaces of degree $p = 2$ and $p = 3$. The corresponding results are presented in Table 3 and Fig. 10, and in Table 4 and Fig. 11, respectively. For $p = 2$, the observed convergence rates for the Taylor–Hood and Nédélec elements are close to the optimal convergence rates. For the Sub-grid element, the results indicate an optimal rate of convergence for the velocity approximation. The results for the pressure approximation are inconclusive. For the Raviart–Thomas element, the velocity approximation displays optimal convergence rates in both the $H^1$-norm and the $L^2$-norm. The convergence rate for the pressure
approximation again appears to be suboptimal, similar to the case of $p = 1$. Also for $p = 3$, the Raviart–Thomas element exhibits optimal convergence rates for the velocity and a suboptimal convergence rate for the pressure. The results for the Taylor–Hood and Nédélec elements for $p = 3$ suggest optimal convergence rates for these elements. According to Table 4, the Sub-grid element exhibits suboptimal convergence rates for $p = 3$. Especially the observed rate of decay of the $L^2$-norm of the error in the pressure approximation, which is approximately 2.6, falls short of the optimal convergence rate of 4. It appears that this significantly suboptimal convergence rate of the error in the pressure approximation of the Sub-grid element is due to the fact that the FCM acts differently on the micro-elements of the velocity approximation than on the macro-elements of the pressure approximation. However, the precise mechanism and the relation to the polynomial degree $p$ of the approximation remain topics for further study.

Inspection of the pressure approximations reveals that all four elements exhibit pressure oscillations near the cut boundary. These pressure oscillations are particularly manifest on coarse meshes and at low orders of approximation. Fig. 12 presents the pressure approximation provided by the Taylor–Hood element for $p = 1$ on a mesh with $32 \times 32$ elements. This result is representative of the pressure approximations provided by the other three element families.
Table 3
Relative error of IGA-FCM for the quarter annulus problem for the Taylor–Hood (TH), Raviart–Thomas (RT), Nédélec (ND) and Sub-grid (SG) elements with \( p = 2 \) for the pressure field.

| Mesh  | \(|\mathbf{u} - \mathbf{u}_h^{TH}\|_2^2\) | \(\|\mathbf{p} - p_h^{TH}\|_2\) | \(\|\mathbf{u} - \mathbf{u}_h^{RT}\|_2^2\) | \(\|\mathbf{p} - p_h^{RT}\|_2\) |
|-------|-------------------------------------|-------------------|-----------------|-------------------|
|       | \(8 \times 8\)                      | \(16 \times 16\)  | \(32 \times 32\) | \(64 \times 64\) |
| Order | \(1.64 \times 10^{-2}\)             | \(1.79 \times 10^{-3}\) | \(1.54 \times 10^{-4}\) | \(1.15 \times 10^{-5}\) |
| Order | \(5.29 \times 10^{-2}\)             | \(9.93 \times 10^{-3}\) | \(1.57 \times 10^{-3}\) | \(2.32 \times 10^{-4}\) |
| Order | \(9.49 \times 10^{-2}\)             | \(1.26 \times 10^{-2}\) | \(1.65 \times 10^{-3}\) | \(2.24 \times 10^{-4}\) |
| Order | \(1.40 \times 10^{-1}\)             | \(2.00 \times 10^{-2}\) | \(2.53 \times 10^{-3}\) | \(3.07 \times 10^{-4}\) |
| Order | \(2.68 \times 10^{-1}\)             | \(6.94 \times 10^{-2}\) | \(1.73 \times 10^{-2}\) | \(4.21 \times 10^{-3}\) |
| Order | \(5.08 \times 10^{-1}\)             | \(1.93 \times 10^{-1}\) | \(2.38 \times 10^{-1}\) | \(6.31 \times 10^{-2}\) |
| Order | \(2.52 \times 10^{-2}\)             | \(2.25 \times 10^{-3}\) | \(1.72 \times 10^{-4}\) | \(2.14 \times 10^{-5}\) |
| Order | \(7.33 \times 10^{-2}\)             | \(1.22 \times 10^{-2}\) | \(7.48 \times 10^{-3}\) | \(3.54 \times 10^{-4}\) |
| Order | \(2.68 \times 10^{-1}\)             | \(3.47 \times 10^{-2}\) | \(4.16 \times 10^{-3}\) | \(4.14 \times 10^{-4}\) |
| Order | \(4.07 \times 10^{-2}\)             | \(3.09 \times 10^{-3}\) | \(2.04 \times 10^{-4}\) | \(1.35 \times 10^{-5}\) |
| Order | \(8.99 \times 10^{-2}\)             | \(1.36 \times 10^{-2}\) | \(1.84 \times 10^{-3}\) | \(4.53 \times 10^{-4}\) |
| Order | \(1.75 \times 10^{-1}\)             | \(3.37 \times 10^{-2}\) | \(3.68 \times 10^{-3}\) | \(9.10 \times 10^{-4}\) |
| Order | \(8.99 \times 10^{-2}\)             | \(1.36 \times 10^{-2}\) | \(1.84 \times 10^{-3}\) | \(4.53 \times 10^{-4}\) |

Table 4
Relative error of IGA-FCM for the quarter annulus problem for the Taylor–Hood (TH), Raviart–Thomas (RT), Nédélec (ND) and Sub-grid (SG) elements with \( p = 3 \) for the pressure field.

| Mesh  | \(|\mathbf{u} - \mathbf{u}_h^{TH}\|_2^2\) | \(\|\mathbf{p} - p_h^{TH}\|_2\) | \(\|\mathbf{u} - \mathbf{u}_h^{RT}\|_2^2\) | \(\|\mathbf{p} - p_h^{RT}\|_2\) |
|-------|-------------------------------------|-------------------|-----------------|-------------------|
|       | \(8 \times 8\)                      | \(16 \times 16\)  | \(32 \times 32\) | \(64 \times 64\) |
| Order | \(3.28 \times 10^{-3}\)             | \(1.99 \times 10^{-4}\) | \(9.28 \times 10^{-6}\) | \(3.60 \times 10^{-7}\) |
| Order | \(9.96 \times 10^{-3}\)             | \(1.06 \times 10^{-3}\) | \(9.26 \times 10^{-5}\) | \(7.50 \times 10^{-6}\) |
| Order | \(1.46 \times 10^{-2}\)             | \(3.37 \times 10^{-3}\) | \(2.56 \times 10^{-4}\) | \(2.35 \times 10^{-5}\) |
| Order | \(3.98 \times 10^{-2}\)             | \(2.85 \times 10^{-3}\) | \(1.79 \times 10^{-4}\) | \(1.14 \times 10^{-5}\) |
| Order | \(7.30 \times 10^{-2}\)             | \(9.88 \times 10^{-3}\) | \(1.25 \times 10^{-3}\) | \(1.57 \times 10^{-4}\) |
| Order | \(2.59 \times 10^{-1}\)             | \(2.75 \times 10^{-1}\) | \(4.00 \times 10^{-2}\) | \(3.25 \times 10^{-3}\) |
| Order | \(5.78 \times 10^{-3}\)             | \(2.62 \times 10^{-4}\) | \(1.05 \times 10^{-5}\) | \(3.87 \times 10^{-7}\) |
| Order | \(1.69 \times 10^{-2}\)             | \(1.45 \times 10^{-3}\) | \(1.07 \times 10^{-4}\) | \(7.59 \times 10^{-6}\) |
| Order | \(1.31 \times 10^{-1}\)             | \(7.15 \times 10^{-3}\) | \(3.75 \times 10^{-4}\) | \(2.91 \times 10^{-5}\) |
Table 4 (continued)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>8 × 8</th>
<th>16 × 16</th>
<th>32 × 32</th>
<th>64 × 64</th>
</tr>
</thead>
<tbody>
<tr>
<td>|u − u_p^SG|_{L^2}</td>
<td>9.58e−3</td>
<td>3.91e−4</td>
<td>1.64e−5</td>
<td>9.14e−7</td>
</tr>
<tr>
<td>Order</td>
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<td>4.62</td>
<td>4.57</td>
<td>4.17</td>
</tr>
<tr>
<td>|u − u_p^SG|_{H^1}</td>
<td>2.13e−2</td>
<td>1.73e−3</td>
<td>1.31e−4</td>
<td>1.28e−5</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>3.62</td>
<td>3.73</td>
<td>3.35</td>
</tr>
<tr>
<td>|p − p_h^SG|_{L^2}</td>
<td>6.56e−2</td>
<td>8.77e−3</td>
<td>1.44e−3</td>
<td>2.32e−4</td>
</tr>
<tr>
<td>Order</td>
<td>–</td>
<td>3.14</td>
<td>2.63</td>
<td>2.63</td>
</tr>
</tbody>
</table>

(a) Relative $L^2$ error of velocity, $p = 2$.
(b) Relative $H^1$ error of velocity, $p = 2$.
(c) Relative $L^2$ error of pressure, $p = 2$.

Fig. 10. Relative error of IGA-FCM for the quarter annulus problem for the Taylor–Hood (TH), Raviart–Thomas (RT), Nédélec (ND) and Sub-grid (SG) elements with $p = 2$ for the pressure field.

Moreover, it is noted that the pressure oscillations are insensitive to the type of Nitsche stabilization, in the sense that results obtained with a local stabilization parameter (see Section 2.2) or with a (skew-symmetric) parameter-free Nitsche formulation [56] only show non-essential differences with the results obtained using global stabilization as presented here. The pressure oscillations near the cut boundary are clearly discernible. It is to be mentioned that the
Fig. 11. Relative error of IGA-FCM for the quarter annulus problem for the Taylor–Hood (TH), Raviart–Thomas (RT), Nédélec (ND) and Sub-grid (SG) elements with $p = 3$ for the pressure field.

pressure oscillations decay fast under mesh refinement and order elevation, in accordance with the aforementioned convergence rates for the pressure approximations.

6. Conclusions

We investigated the properties of the Isogeometric Finite-Cell Method (IGA-FCM) for mixed formulations, in the context of the Stokes problem. We considered four different families of isogeometric mixed elements, namely, Taylor–Hood, Raviart–Thomas, Nédélec, and Sub-grid elements. For a generic test case corresponding to a quarter-annulus domain, we computed the numerical inf–sup constants for the aforementioned element families for linear, quadratic and cubic pressure approximations. The results convey that all four elements pass the inf–sup stability test in the IGA-FCM setting. We also assessed the convergence behavior of the four element families under mesh refinement and order elevation.

1 It is important to note that, because the velocity space of the RT element in (17) is anisotropic with respect to the degree and is only complete up to order $p$, the corresponding optimal convergence rates for the velocity approximation are one order lower than those for the other elements.
refinement for linear, quadratic and cubic approximations. For the Taylor–Hood and Nédélec elements, optimal convergence rates were observed for the velocity approximation in both the $H^1$-norm and the $L^2$-norm and for the pressure approximation in the $L^2$-norm. The Raviart–Thomas element yields an optimal convergence rate for the velocity approximation, but the pressure approximation is generally suboptimal. The convergence behavior of the Sub-grid element depends on the order of approximation. For linear pressure approximations, we observed optimal convergence rates for both velocity and pressure. For quadratic approximations, the convergence rate for the velocity approximation appears optimal, but the observed convergence for the pressure is irregular and inconclusive. For cubic approximations, the observed convergence rates for the Sub-grid element are suboptimal, both for velocity and for pressure.

For the Taylor–Hood and Nédélec element families, the observed optimal convergence rates of IGA-FCM are in agreement with corresponding results in the literature for boundary-fitted approximations. For the Raviart–Thomas elements, the observed optimal convergence rate for the velocity approximation is in accordance with corresponding results for fitted approximations. However, the suboptimal convergence rate for the pressure approximation of the Raviart–Thomas elements in the IGA-FCM context is at variance with the optimal rates that have been observed in the literature for fitted approximations. The suboptimal convergence rates for higher-order Sub-grid elements are also incongruent with corresponding results in the literature for fitted approximations.

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References


