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ISOGEOMETRIC GALERKIN METHODS FOR GRADIENT-ELASTIC BARS, BEAMS, MEMBRANES AND PLATES

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Abstract. *Isogeometric Galerkin methods are used to analyse plate and beam bending problems as well as membrane and bar models based on Mindlin's strain gradient elasticity theory for generalized continua. The current strain gradient models include higher-order displacement gradients combined with length scale parameters enriching the strain and kinetic energies of the classical elasticity and hence resulting in higher-order partial differential equations with corresponding non-standard boundary conditions. The problems are first formulated within appropriate higher-order Sobolev space settings and then discretized by utilizing Galerkin methods with isogeometric NURBS basis functions providing appropriate higher-order continuity properties. Example benchmark problems illustrate the convergence properties of the methods.*

1 INTRODUCTION

Isogeometric Analysis (IGA) was introduced by Hughes et al. [9] roughly ten years ago having its primary focus on the integration of industrial design-analysis processes by performing finite element analysis by B-splines and NURBS utilized in computer aided design. Compared to the classical finite element analysis based on polynomial basis functions, isogeometric analysis provides some significant benefits originating from higher-order continuities of basis functions provided by basic isogeometric methods in a natural and straightforward manner, as C^{p-1} continuity for standard NURBS patches of order p. As a related particular implication, isogeometric Galerkin methods have turned out to be applicable for solving problems governed by higher-order partial differential equations which require higher-order reqularities for function spaces of Galerkin methods. In the present work, this capability is utilized for a group of problems following a strain gradient elasticity theory.

Typically, the aim of the generalized continuum theories as gradient elasticity is to take into account the effect of the microstructure of the material on its mechanical behaviour. The current strain gradient models, in particular, include higher-order displacement gradients combined with one length scale parameter [1, 2] enriching the strain energy of the classical elasticity theory [3]. Accordingly, these models result in higher-order partial differential equations with related non-standard boundary conditions [5, 6, 7, 8].

The paper is organized as follows: In Section 2, we introduce our notation by recalling the variational energies for the three-dimensional theory of gradient elasticity. Section 3 is devoted to the governing equations and weak forms of the corresponding dimensionally reduced bar, beam, membrane and plate problems, whereas in Section 4 isogeometric Galerkin discretizations are briefly formulated. Finally, in Section 5, we give some examples of numerical benchmarks illustrating the convergence properties of the methods.

2 GRADIENT ELASTICITY THEORY

Let us denote the classical linear (second order) strain tensor by $\boldsymbol{\varepsilon}$, defined as the symmetric part of the gradient of the displacement vector $\boldsymbol{u}$ in the form

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\big(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\big). \tag{1}$$

As in the classical elasticity theory, the strain tensor is assumed to be related to its work conjugate, the classical (second order) Cauchy stress tensor, through the generalized Hooke's law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}\,\boldsymbol{\varepsilon}\boldsymbol{I}, \tag{2}$$

with the Lame material parameters μ and λ, and $\boldsymbol{I}$ denoting the identity tensor.

In Mindlin's gradient elasticity theory of Form II [3], the (third-order) micro-deformation gradient tensor is defined as $\boldsymbol{\gamma} = \boldsymbol{\nabla}\boldsymbol{\varepsilon}$. Its work conjugate, the (third-order) double stress tensor $\boldsymbol{\tau}$, is defined in the simplest one parameter variant [1, 2] of Mindlin's theory as $\boldsymbol{\tau} = g^2\boldsymbol{\nabla}\boldsymbol{\sigma}$, with g denoting the gradient-elastic modulus describing the length scale of the micro-structure of the material. The virtual internal work expression over a body $\mathcal{B} \subset \mathbb{R}^3$ then takes the form

$$\delta W^g_{\text{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\,\mathrm{d}\mathcal{B} + \int_{\mathcal{B}} g^2\boldsymbol{\nabla}\boldsymbol{\sigma} \,\vdots\, \boldsymbol{\nabla}\boldsymbol{\varepsilon}(\delta\boldsymbol{u})\,\mathrm{d}\mathcal{B}. \tag{3}$$

Applying integration by parts for the second term above gives the so called total stress tensor [1] in the form $\boldsymbol{\sigma}^g = (1 - g^2\Delta)\boldsymbol{\sigma}$, where g is assumed to be constant.

For analysing vibrations or time-dependent problems within the current gradient elasticity theory, an additional gradient parameter introducing a micro-inertia term has been proposed [3] in order to achieve a physically satisfactory dispersion relation for a large range of wave numbers [4]. The variation of the kinetic energy is then written in the form

$$\delta \int_T W_{\text{kin}}^{\gamma} \, \mathrm{d}\tau = - \int_T \Big(\int_{\mathcal{B}} \rho \ddot{\boldsymbol{u}} \cdot \delta \boldsymbol{u} \, \mathrm{d}\mathcal{B} + \int_{\mathcal{B}} \gamma^2 \rho \boldsymbol{\nabla} \ddot{\boldsymbol{u}} : \boldsymbol{\nabla} \delta \boldsymbol{u} \, \mathrm{d}\mathcal{B} \Big) \, \mathrm{d}\tau, \tag{4}$$

with ρ and T denoting the mass density and a time interval of the time variable τ, respectively, and finally γ standing for the gradient parameter related to the micro-inertia.

3 GOVERNING EQUATIONS AND WEAK FORMS

In this section, we briefly recall the governing equations and variational formulations for the bar and beam problems and then for the membrane and plate problems. For simplicity, we focus on statics, whereas for extensions to dynamics we refer to [5, 6, 8].

3.1 Bars and beams

Let us consider a thin bar or beam structure which occupies a three-dimensional body $\mathcal{B} = \Omega \times A$, where $\Omega = (0, L) \subset \mathbb{R}$ denotes the central axis of the structure with L standing for the length of the structure, and $A \subset \mathbb{R}^2$, with $\operatorname{diam}(A) \ll L$, denoting a cross-section of the structure. For simplicity, the cross-section is assumed to be constant.

With a constant Young's modulus E, the governing equation of the bar problem reads as [5]

$$EAu''(x) - g^2 EAu''''(x) + Af(x) = 0 \quad \forall x \in \Omega, \tag{5}$$

with *doubly* clamped, *singly* clamped and free boundaries, respectively:

$$u(x) = \overline{u}, \quad u'(x) = \overline{w} \quad \forall x \in \Gamma_{\mathrm{C_d}}, \tag{6}$$

$$u(x) = \overline{u}, \quad g^2 EAu''(x) = \overline{G} \quad \forall x \in \Gamma_{\mathrm{C_s}}, \tag{7}$$

$$EAu'(x) - g^2 EAu'''(x) = \overline{P}, \quad g^2 EAu''(x) = \overline{G} \quad \forall x \in \Gamma_{\mathrm{F}}. \tag{8}$$

The weak formulation of the problem reads as follows [5]: For $f \in L^2(\Omega)$, find $u \in U \subset H^2(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in V \subset H^2(\Omega), \tag{9}$$

where the bilinear form $a : U \times V \to \mathbb{R}$, $a(u, v) = a^{\mathrm{c}}(u, v) + a^{\nabla}(u, v)$, and the load functional $l : V \to \mathbb{R}$, respectively, are defined by

$$a^{\mathrm{c}}(u, v) = \int_\Omega EAu'v' \, \mathrm{d}x, \quad a^{\nabla}(w, v) = \int_\Omega g^2 EAu''v'' \, \mathrm{d}x, \quad l(v) = \int_\Omega Af \, v \, \mathrm{d}x. \tag{10}$$

The trial and test function sets are denoted by U and V, respectively. The problem is continuous and coercive with respect to the H^2 norm [5].

With the moment of inertia I, the governing equation of the Euler–Bernoulli beam bending problem reads as [6]

$$EIw''''(x) - g^2 EIw''''''(x) = f(x) \quad \forall x \in \Omega, \tag{11}$$

while the boundary conditions can be found in [6]. The weak formulation of the problem reads as follows [6]: For $f \in L^2(\Omega)$, find $w \in W \subset H^3(\Omega)$ such that

$$a(w,v) = l(v) \quad \forall v \in V \subset H^3(\Omega), \tag{12}$$

where the bilinear form $a : W \times V \to \mathbb{R}$, $a(w,v) = a^{\mathrm{c}}(w,v) + a^{\nabla}(w,v)$, and the load functional $l : V \to \mathbb{R}$, respectively, are defined as

$$a^{\mathrm{c}}(w,v) = \int_\Omega EIw''v''\,\mathrm{d}x, \quad a^{\nabla}(w,v) = \int_\Omega g^2 EIw'''v'''\,\mathrm{d}x, \quad l(v) = \int_\Omega f\,v\,\mathrm{d}x. \tag{13}$$

The trial and test function sets are denoted by W and V, respectively. The problem is continuous and coercive with respect to the H^3 norm [6].

3.2 Membranes and plates

Let us consider a thin planar membrane or plate structure which occupies a three-dimensional body $\mathcal{B} = \Omega \times (-t/2, t/2)$, where the domain $\Omega \subset \mathbb{R}^2$ denotes the midsurface of the structure and $t \ll \mathrm{diam}(\Omega)$ stands for the thickness of the structure. For simplicity, the thickness is assumed to be constant.

The governing equation of the (plane stress) membrane problem is written as [5]

$$(1 - g^2\Delta)\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \quad \text{in}\,\Omega, \tag{14}$$

while the boundary conditions can be found in [5]. The weak formulation of the problem reads as follows [5]: For $\boldsymbol{f} \in [L^2(\Omega)]^2$, find $\boldsymbol{u} \in \boldsymbol{U} \subset [H^2(\Omega)]^2$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} \subset [H^2(\Omega)]^2, \tag{15}$$

where the bilinear form $a : \boldsymbol{U} \times \boldsymbol{V} \to \mathbb{R}$, $a(\boldsymbol{u},\boldsymbol{v}) = a^{\mathrm{c}}(\boldsymbol{u},\boldsymbol{v}) + a^{\nabla}(\boldsymbol{u},\boldsymbol{v})$, and the load functional $l : \boldsymbol{V} \to \mathbb{R}$, respectively, are defined as

$$a^{\mathrm{c}}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{I}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\Omega, \tag{16}$$

$$a^{\nabla}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{\nabla} g^2(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{I}) \vdots \boldsymbol{\nabla}\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\Omega, \quad l(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega. \tag{17}$$

The trial and test function sets are denoted by $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. The problem is continuous and coercive with respect to the H^2 norm [5].

The governing equation of the Kirchhoff plate bending problem is written as

$$D\Delta^2 w - g^2 D\Delta^3 w = f \quad \text{in}\,\Omega, \tag{18}$$

with the bending rigidity defined as

$$D = \frac{Et^3}{12(1-\nu^2)}. \tag{19}$$

The boundary conditions of the problem can be found in [7].

The weak formulation of the problem reads as follows [7]: For $f \in L^2(\Omega)$, find $w \in W \subset H^3(\Omega)$ such that

$$a(w,v) = l(v) \quad \forall v \in V \subset H^3(\Omega), \tag{20}$$

where the bilinear form $a : W \times V \to \mathbb{R}$, $a(w,v) = a^{\mathrm{c}}(w,v) + a^{\nabla}(w,v)$, and the load functional $l : V \to \mathbb{R}$, respectively, are defined as

$$a^{\mathrm{c}}(w,v) = \int_\Omega \boldsymbol{E}\boldsymbol{\varepsilon}(\nabla w) : \boldsymbol{\varepsilon}(\nabla v)\,\mathrm{d}\Omega, \tag{21}$$

$$a^{\nabla}(w,v) = \int_\Omega g^2 \boldsymbol{\nabla}(\boldsymbol{E}\boldsymbol{\varepsilon}(\nabla w)) \,\vdots\, \boldsymbol{\nabla}\boldsymbol{\varepsilon}(\nabla v)\,\mathrm{d}\Omega, \quad l(v) = \int_\Omega f\, v\,\mathrm{d}\Omega. \tag{22}$$

The symmetric positive definite bending moduli tensor is defined, in the case of constant bending rigidity, by the relation

$$\boldsymbol{E}\boldsymbol{\varepsilon} = D\big((1-\nu)\boldsymbol{\varepsilon} + \nu \operatorname{tr}\boldsymbol{\varepsilon}\boldsymbol{I}\big). \tag{23}$$

The trial and test function sets are denoted by W and V, respectively. The problem is continuous and coercive with respect to the H^3 norm [7].

4 CONFORMING ISOGEOMETRIC GALERKIN APPROACH

Within the gradient elasticity theory, the bar and mebrane problems deal with H^2 spaces, whereas for the beam and plate problems H^3 is the appropriate Sobolev space. Therefore, C^1 and C^2 continuity, respectively, are required for the corresponding conforming Galerkin approximations. Within the classical elasticity theory, instead, these problems deal with less regular function spaces, H^1 and H^2, respectively, requiring C^0 and C^1 continuity, respectively, for conformity. Isogeometric methods provide higher-order regularities for the approximation spaces in a straightforward manner. Furthermore, with continuity and coercivity, conformity implies Cea's lemma type convergence results for the methods [5, 6, 8].

Let us next briefly recall the isogeometric tensor product discretizations [9] which can be applied for solving the problems formulated in the previous section. As an example, we construct a discrete space for the plane stress problem.

First, we introduce an isoparametric discrete space S_h for the approximation of the displacement field such that $\boldsymbol{u}_h \in [S_h]^2$ with

$$S_h = \{R_{i,j}^{p,q} \circ \boldsymbol{F}^{-1}\}. \tag{24}$$

The geometrical mapping between the two-dimensional parametric space $[0,1] \times [0,1]$ and the midsurface $\bar{\Omega}$ is defined by $\boldsymbol{F} : [0,1] \times [0,1] \to \bar{\Omega}$ as

$$\boldsymbol{F}(\xi,\eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi,\eta) \boldsymbol{B}_{i,j} \tag{25}$$

providing an isogeometric NURBS discretization. Above, $\boldsymbol{B}_{i,j}$, $i = 1 \dots n, j = 1 \dots m$, denote the control points, while the NURBS basis functions are defined as

$$R_{i,j}^{p,q}(\xi,\eta) = \frac{N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}{\sum_{\hat{i}=1}^{n} \sum_{\hat{j}=1}^{m} N_{\hat{i},p}(\xi) M_{\hat{j},q}(\eta) w_{\hat{i},\hat{j}}}. \tag{26}$$

The B-spline basis functions $N_{i,p}$ and $M_{j,q}$ of order p and q, respectively, associated to the open knot vectors $\{0 = \xi_1, \dots, \xi_{n+p+1} = 1\}$, $\{0 = \eta_1, \dots, \eta_{m+q+1} = 1\}$, respectively, are defined as follows:

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \le \xi < \xi_{i+1}, \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \tag{28}$$

The tensor product mesh of the midsurface, with the mesh size h, is denoted by

$$\mathcal{T}_h = \{\boldsymbol{F}([\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]) | i \in [n+p], j \in [m+q]\}. \tag{29}$$

By assuming $p = q$ and global regularity C^{p-1} over $\mathcal{T}_h$, with $p \geq 2$, it holds that $S_h \subset H^2(\Omega)$, which provides a conforming and consistent Galerkin method following the formulation (15) with $\boldsymbol{U}_h = [S_h]^2 \cap \boldsymbol{U}$, $\boldsymbol{V}_h = [S_h]^2 \cap \boldsymbol{V}$.

5 NUMERICAL RESULTS

As simple examples of convergence results for the proposed methods, we study the beam and plate bending problems with constant material values and smooth distributed loadings. For numerical results concerning free vibrations, we refer again to [5, 6, 8].

In Figures 1 and 2, for the beam and plate problems, respectively, the convergence of the relative error in the H^3 norm is plotted against the number of degrees of freedom (in logarithmic scales). It can be seen that for the NURBS order $p = 3, 4, 5$ the convergence rates follow the theoretical rates [6, 8] depicted by the dashed slopes.

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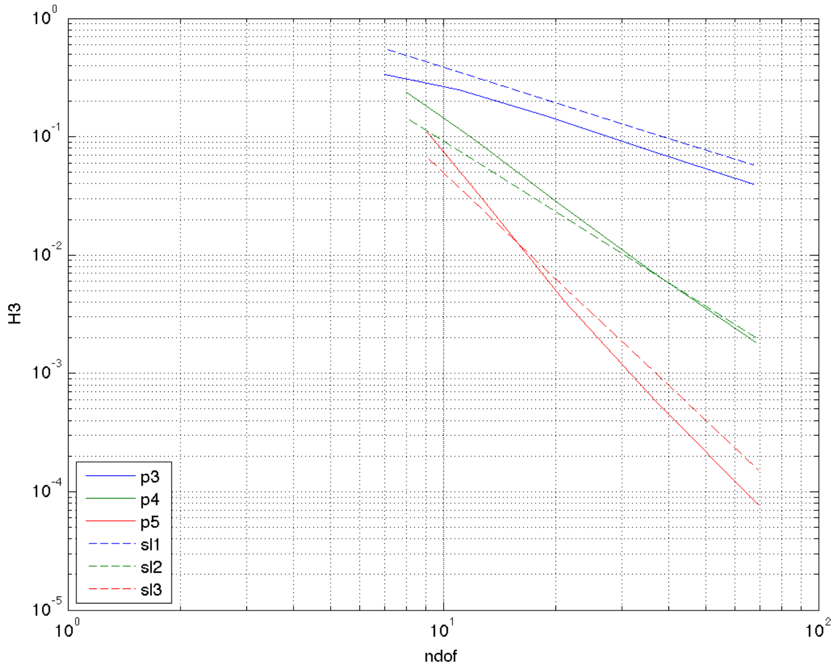

Figure 1: Doubly simply supported beam: Convergence of the relative error in the H^3 norm for $p = 3, 4, 5$ with $g = t = L/20$.

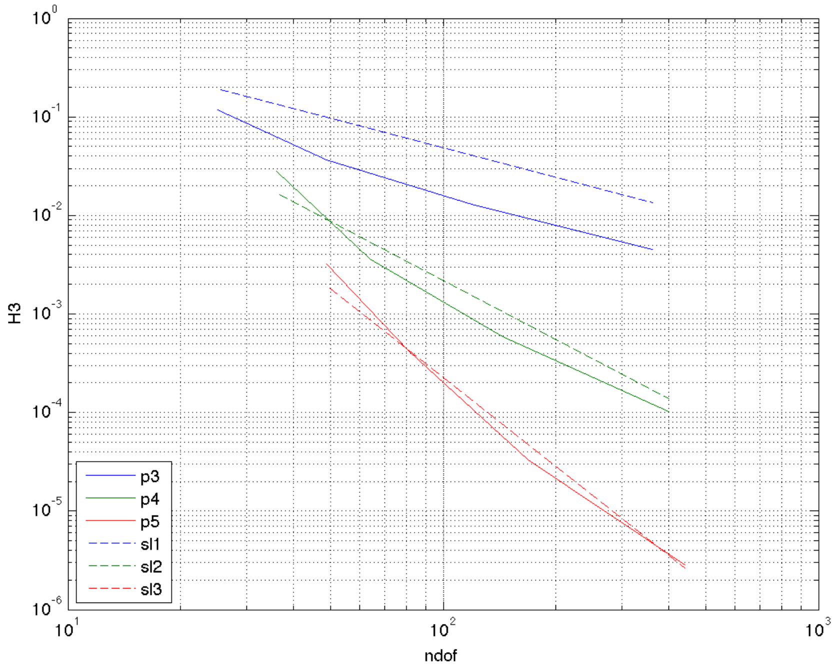

Figure 2: Doubly simply supported square plate: Convergence of the relative error in the H^3 norm for $p = 3, 4, 5$ with $g = t = 0.1$.