Theoretical and numerical modeling of dense and porous shape memory alloys accounting for coupling effects of plasticity and transformation

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1. Introduction

Shape memory alloys (SMAs) have unique features (shape memory effect and pseudoelasticity), which make them good candidates in aerospace and biomedical applications. In addition to dense SMAs, porous SMAs have drawn increasing attention in the past years. Benefiting from the characteristics of porous metals, porous SMAs can be used as a light-weight or graded structure in biomedical implants, and energy absorbers (Bansiddhi et al., 2008; Lefebvre et al., 2008; Qidwai et al., 2001; Zhao et al., 2006).

An important phenomenon related to the thermomechanical behavior of shape memory alloys is plasticity. Pseudo-elasticity and shape memory effect of SMAs are due to martensitic phase transformation under mechanical and thermal loadings. In addition to transformation strain which is recoverable, irrecoverable strain may also evolve in SMAs. According to experimental observations (see, e.g., Song et al. (2014a); Wang et al. (2008)), there are three sources for irrecoverable strain: (1) transformation ratcheting; (2) plastic strain of martensite phase; (3) incomplete reverse transformation due to plasticity.

Considering experimental evidences, the forward and reverse transformations result in a progressive decrease in start stress of martensite transformation and the dissipation energy, which stabilize after a few cycles (Song et al. (2014b); Yu et al. (2015)). This phenomenon is denoted as transformation ratcheting in the literature (Kang et al., 2009) and is the only source for residual strain before reaching plastic limit. Several constitutive models have been proposed for such a phenomenon (see, e.g., Auricchio et al. (2007); Lagoudas and Entchev (2004); Panico and Brinson (2008); Saint-Sulpice et al. (2012); Yu et al. (2014)). However, when the loading reaches plastic limit, the residual strain is mainly due to plastic strain and incomplete reverse transformation. In dense SMAs, plastic strain evolves after completion of phase transformation, but in porous SMAs, plastic strain evolves during phase transformation due to local plastic yielding. The mechanism for plastic deformation of porous SMAs is similar to what is observed in conventional metal foams (Bram et al., 2011). Since the pore microstructure is not regular, the thickness of struts near the pores of SMA samples is different. Even at very low stress levels,
pseudo-elastic deformation starts at regions with smaller wall-thickness. Increasing the load, thin-walled regions start to deform plastically, while martensitic phase transformation occurs simultaneously in thick-walled regions. This is a level by level deformation mechanism taking place in porous materials.

In this study, we focus on the effects of plastic strain on phase transformation behavior and neglect the ratcheting effects which is dominant in cyclic loading of dense and porous SMAs. Fig. 1a and b shows two stress–strain curves for a typical dense SMA sample under two different loading levels. Due to incomplete reverse transformation, the residual strain ($\varepsilon_{r}$) in the sample is, in general, more than the plastic strain ($\varepsilon_{pl}$). Comparing Fig. 1a and b, we observe that reverse transformation has been reduced by increasing the plastic strain. While in Fig. 1a a partial reverse transformation is observed, in Fig. 1b no reverse transformation is observed. Specifically, to show the coupling effects of plasticity and transformation, we define the parameter $q$ as:

$$q = \frac{\varepsilon_{r} - \varepsilon_{pl}}{\varepsilon_{pl}}$$  \hspace{1cm} (1)

If plastic strain has no effect on reverse phase transformation, the residual strain is equal to plastic strain and therefore $q = 0$. In other words, larger values for $q$ means less reverse phase transformation and more residual strain. In general, parameter $q$ is not constant and may vary with the evolution of plastic strain.

Another phenomenon observed in SMAs according to experimental evidences (Song et al. 2014a; Wang et al. 2008) is that, evolution of plastic strain degrades the critical stress of reverse transformation. Fig. 2 shows two typical stress-strain curves; while the curve with solid line experiences no plastic deformation, the curve with dash line experiences plastic strain which degrades the critical stress of the reverse phase transformation during unloading.

From a microscopic viewpoint, plastic strain occurs locally in porous SMAs during phase transformation. Although the effect of plastic strain on the phase transformation of porous SMAs is not negligible, experimental evidences (Bram et al., 2011; Köhl et al., 2011; Scalzo et al., 2009) show that even after a compression loading of porous NiTi samples with a considerable plastic strain, pseudo-elastic or shape memory strain recovery is observed.

Several studies have tried to model plastic yield of dense SMAs (see, e.g., Hartl and Lagoudas (2009); Paiva et al. (2005); Yan et al. (2003); Zhou (2012) among others). Such models utilize independent variables to describe transformation and plastic strains and propose corresponding limit functions and evolution equations. In these models, the plastic strain, which evolves after completion of the phase transformation, influences the reverse phase transformation. While most of the models do not consider simultaneous evolution of transformation and plastic strains, the model proposed by Hartl and Lagoudas (2009) considers simultaneous transformation and plastic yield which usually occurs at elevated temperatures for dense SMAs.

Another important consequence of porous microstructure is pressure dependency which is usually observed in porous but not dense materials (Altenbach & Ochsner, 2010; Bigoni & Piccolroaz, 2004; Gibson & Ashby, 1997). There are few works attempting to propose a phenomenological constitutive model for porous SMAs including plasticity and pressure-dependency. In such models, transformation and plastic strains should contain both volumetric and deviatoric contributions. Sayed et al. (2012) proposed a constitutive model which combines a dense SMA model with a porous plasticity model. Their model incorporates the phase transformation and plastic behavior simultaneously, but does not consider pressure-dependent phase transformation and shape memory effect. Based on the Gurson–Tvergaard–Needleman
formulation, Olsen and Zhang (2012) proposed a constitutive model for SMAs with micro-voids. The model is able to reproduce pressure-dependent plasticity and phase transformation, but not simultaneously. Moreover, Zhu et al. (2014) used a dense SMA model with plasticity effect and studied plates with pore arrays. They have shown that, due to stress concentration around the pores, transformation and plastic strains occur simultaneously. Recently, Ashrafi et al. (2016) proposed a 3-D phenomenological constitutive model for simultaneous evolution of transformation and plastic strains utilizing pressure-dependent limit functions. Although Sayed et al. (2012) and Ashrafi et al. (2016) studied simultaneous phase transformation and plasticity of porous SMAs, they assumed that plasticity has no effect on phase transformation.

Numerical implementation of constitutive models is an essential step to use them in simulation of different structures. Toward this end, time discretization and solution algorithm for the models should be developed to account for simultaneous evolution of transformation and plastic strains. Among the above-mentioned constitutive models, Sayed et al. (2012) used an explicit solution algorithm and developed a user material subroutine VUMAT in ABAQUS. Also, Hartl and Lagoudas (2009) proposed an implicit solution algorithm which contains four possible branches: elastic, transformation (no plastic), plastic (no transformation) and simultaneous transformation and plastic. However, the coupling between transformation and plasticity needs an active set search. Auricchio et al. (2014) modeled multiple phase transformation and reorientation of SMAs, replacing the Kuhn–Tucker inequality conditions by the equivalent Fischer–Burmeister function (Fischer, 1992) which omit an active set search. Moreover, Kiefer et al. (2012) present two alternative algorithms for the integration of the coupled, non-linear and inelastic constitutive equations for magnetic shape memory alloy, i.e., the classical predictor-corrector return-mapping scheme and the Fischer–Burmeister based algorithm. As another numerical approach, Ashrafi et al. (2016) considered a stress-driven problem which decouples the constitutive equations and simplifies the solution algorithm. However, in a general problem, the constitutive equations are coupled and it is necessary to develop a general solution algorithm.

The purpose of the present study is two-fold: first, to propose 3-D phenomenological models for porous and dense SMAs accounting for coupling effects of transformation and plasticity, and second, to present a general solution algorithm for simultaneous evolution and coupling effects of transformation and plastic strains. To this end, the porous model proposed by Ashrafi et al. (2016) is extended to include the effects of plasticity on phase transformation. Two versions of the model for dense and porous SMAs are presented, and a general solution algorithm is addressed to numerically implement these models. Such an algorithm is achieved by eliminating the need for a predictor-corrector-type scheme through an effective and efficient procedure. It consists of replacing the Kuhn–Tucker inequality conditions by the equivalent Fischer–Burmeister function (Auricchio et al., 2014; Fischer, 1992). Moreover, we implement the model using corotational formulation and perform finite element analysis for several examples to show the efficiency of the proposed model for large rotations and general multiaxial loadings.

The present paper is organized as follows. The proposed three-dimensional phenomenological model is presented in Section 2. Section 3 describes the numerical implementation of model equations and the full solution algorithm. Section 4 presents validation, numerical results, and a comprehensive study on the material parameters of the models. Summary and conclusions are finally drawn in Section 5.

2. 3-D constitutive models for porous SMAs

This section addresses a three-dimensional phenomenological constitutive model for SMAs along the lines of the recent work by Ashrafi et al. (2016). In this regard, the model is extended to include plasticity effects on phase transformation. It is noted that we neglect ratcheting since the main goal of the present study is to study the behavior of dense and porous SMAs which undergoes plastic strain. In the following, two versions of such a model for dense and porous SMAs are presented. The difference between dense and porous SMA models originates from the pressure-dependent behavior observed in porous materials.

2.1. Dense SMA model

Total strain \( \mathbf{e} \) is decomposed into deviatoric strain \( \mathbf{e} \) and volumetric strain \( \theta \) as:

\[
\mathbf{e} = \mathbf{e} + \frac{\theta}{3} \mathbf{I}
\]

(2)

where \( \mathbf{I} \) is the second-order identity tensor. Assuming small strains and pressure independency, we consider the additive decomposition of deviatoric strain:

\[
\mathbf{e} = \mathbf{e}^\text{el} + \mathbf{e}^\text{tr} + \mathbf{e}^\text{pl}
\]

(3)

where \( \mathbf{e}^\text{el} \), \( \mathbf{e}^\text{tr} \) and \( \mathbf{e}^\text{pl} \) are elastic, transformation and plastic deviatoric strains, respectively.

In the model, we assume the deviatoric and volumetric strains \( \mathbf{e}, \theta \) and the absolute temperature \( T \) as control variables, and the transformation and plastic strains \( \mathbf{e}^\text{tr} \) and \( \mathbf{e}^\text{pl} \) as internal variables. The Helmholtz free energy function \( \Psi \) is considered as a function of such control and internal variables, i.e., \( \Psi = \Psi(\mathbf{e}, \theta, T, \mathbf{e}^\text{tr}, \mathbf{e}^\text{pl}) \). To include the coupling effects discussed in Section 1, we extend the Helmholtz free energy introduced by Ashrafi et al. (2016) as:

\[
\Psi = \frac{1}{2} K(\theta)^2 + G\left(\mathbf{e} - \mathbf{e}^\text{tr} - \mathbf{e}^\text{pl}\right)^2 - 3\alpha K\theta (T - T_0)
\]

\[
\quad + \beta (T - T_m)\left\| \mathbf{e}^\text{tr} - q\mathbf{e}^\text{pl} \right\| - A e^\text{tr} : \mathbf{e}^\text{tr}
\]

\[
+ \frac{1}{2} h^\text{tr} \left\| \mathbf{e}^\text{tr} \right\|^2 + \frac{1}{2} h^\text{pl} \left\| \mathbf{e}^\text{pl} \right\|^2 + \varphi_\text{pl} \left( \left\| \mathbf{e}^\text{pl} \right\| \right)
\]

\[
+ (\mu_0 - T\eta_0) + e(T - T_0) - T \ln(T/T_0)
\]

(4)

In the following we explain the extensions and the idea behind them:

- Utilizing the parameter \( q \) defined in Section 1, the term \( \beta (T - T_m)\left\| \mathbf{e}^\text{tr} - q\mathbf{e}^\text{pl} \right\| \) in the model by Ashrafi et al. (2016) is replaced by \( \beta (T - T_m)\left\| \mathbf{e}^\text{tr} - q\mathbf{e}^\text{pl} \right\| \). In this way, instead of complete recovery (\( \mathbf{e}^\text{pl} = 0 \)) partial recovery of transformation (\( \mathbf{e}^\text{tr} = q\mathbf{e}^\text{pl} \)) can be predicted by the present model. Therefore, when the plastic strain increases the recoverable strain decreases. Here, we assumed \( q \) as a material parameter to be constant which is an acceptable approximation according to experimental data.

- By introducing the term \( -A e^\text{tr} : \mathbf{e}^\text{tr} \) into Helmholtz energy function, the degradation effect of transformation and plasticity on each other is considered. \( A \) is a parameter controlling the degradation effects. In this way, evolution of plastic strain decreases critical stress of reverse phase transformation.

In Eq. (4). \( K \) and \( G \) are the bulk and shear elastic moduli; \( h^\text{tr} \) and \( h^\text{pl} \) are material parameters defining the transformation and plastic kinematic hardening, respectively; \( T_m \) is a reference temperature related to starting temperature of transformation; \( \beta \) is a material parameter related to the dependency of the critical stress on the temperature; \( \zeta^\text{el} \) is the positive part function. Moreover, \( \alpha \) and \( \varphi_\text{pl} \) are the thermal expansion and the heat capacity coefficients, while \( \mu_0 \) and \( \eta_0 \) are internal energy and entropy at reference temperature \( T_0 \). Finally, the material parameter \( \delta_{1} \) corresponds to the
maximum effective transformation strain reached at the end of the transformation during a uniaxial test. To satisfy such a constraint, the saturation function $\psi_{\text{t}}$ is introduced as:

$$\psi_{\text{t}}(\|\varepsilon^t\|) = \begin{cases} 0 & \text{if } \|\varepsilon^t\| \leq \varepsilon_L \\ +\infty & \text{otherwise} \end{cases}$$  \hspace{1cm} (5)

To derive the constitutive equations and thermodynamic forces, the second law of thermodynamics should be satisfied. Considering the Helmholtz free energy function in Eq. (22), the mechanical dissipation energy $D^m$ is expressed using the Clausius–Duhamel inequality as follows:

$$D^m = \sigma : \dot{\varepsilon} - (\Psi + \eta \dot{T}) \geq 0$$  \hspace{1cm} (6)

where $\sigma$ is the stress tensor, $\eta$ the entropy and a dot super-script indicates the derivative with respect to time. Substituting Eqs. (2) and (3) into Eq. (6), we obtain:

$$D^m = \left( s - \frac{\partial \Psi}{\partial e} \right) : \dot{\varepsilon} + \left( p - \frac{\partial \Psi}{\partial \theta} \right) \dot{T} - \frac{\partial \Psi}{\partial e^\theta} : \dot{e}^\theta - \frac{\partial \Psi}{\partial \varepsilon^p} : \dot{\varepsilon}^p \geq 0$$  \hspace{1cm} (7)

where we also decomposed the stress $\sigma$ into its deviatoric and volumetric parts $s$ and $p$ as:

$$\sigma = s + p1$$  \hspace{1cm} (8)

To satisfy inequality (7), following standard arguments we can derive the constitutive equations and thermodynamic forces as below:

$$s = \frac{\partial \Psi}{\partial e} : \dot{\varepsilon} + \frac{\partial \Psi}{\partial \theta} \dot{T}$$

$$\eta = -\frac{\partial \Psi}{\partial e^\theta} : \dot{e}^\theta$$

$$X = -\frac{\partial \Psi}{\partial e} : \dot{\varepsilon}$$

$$Q = -\frac{\partial \Psi}{\partial \varepsilon^p} : \dot{\varepsilon}^p$$  \hspace{1cm} (9)

where $X$ and $Q$ are the thermodynamic forces associated to the transformation and plastic strains, respectively. Based on relations (9), the mechanical dissipation inequality (7) reduces to:

$$D^m = X : \dot{\varepsilon}^t + Q : \dot{\varepsilon}^p \geq 0$$  \hspace{1cm} (10)

Recalling the transformation and plasticity behavior described in Section 1, we introduce two distinct convex limit functions for phase transformation and plastic behavior:

$$F^t = ||X|| - R^t$$

$$F^p = ||Q|| - R^p$$  \hspace{1cm} (11)

where $R^t$ and $R^p$ are the radii of the transformation and plastic domain, respectively, and the norm operator $||\cdot||$ is defined as $||A|| = \sqrt{A^T A}$.

To satisfy the second law of thermodynamics or the dissipation inequality (10), we introduce the following associative flow rules for the internal variables:

$$\dot{\varepsilon}^t = \tilde{\varepsilon}^t \frac{\partial F^t}{\partial X} = \tilde{\varepsilon}^t \frac{X}{||X||}$$  \hspace{1cm} (12)

$$\dot{\varepsilon}^p = \tilde{\varepsilon}^p \frac{\partial F^p}{\partial Q} = \tilde{\varepsilon}^p \frac{Q}{||Q||}$$  \hspace{1cm} (13)

where the Lagrange multipliers $\tilde{\varepsilon}^t$ and $\tilde{\varepsilon}^p$ should satisfy the Kuhn-Tucker conditions:

$$\tilde{\varepsilon}^t \geq 0, F^t \leq 0, \tilde{\varepsilon}^t F^t = 0$$

$$\tilde{\varepsilon}^p \geq 0, F^p \leq 0, \tilde{\varepsilon}^p F^p = 0$$  \hspace{1cm} (14)

Therefore, the constitutive equations and thermodynamic forces introduced in Eq. (9) may be expressed as:

$$\begin{bmatrix}
\frac{\partial \Psi}{\partial e}
\frac{\partial \Psi}{\partial \theta}
\frac{\partial \Psi}{\partial e^\theta}
\frac{\partial \Psi}{\partial \varepsilon^p}
\end{bmatrix} =
\begin{bmatrix}
s
p
\eta
\dot{\varepsilon}^t
\dot{\varepsilon}^p
\end{bmatrix}$$

$$\begin{bmatrix}
s
p
\eta
\dot{\varepsilon}^t
\dot{\varepsilon}^p
\end{bmatrix} =
\begin{bmatrix}
s
p
\eta
\dot{\varepsilon}^t
\dot{\varepsilon}^p
\end{bmatrix}$$

where the variable $\gamma$ results from the saturation function subdifferential and is defined as:

$$\frac{\partial \psi_{\text{h}}}{\partial ||\varepsilon^p||} = \begin{cases} 0 & \text{if } ||\varepsilon^p|| < \varepsilon_L \\ \gamma \text{ if } ||\varepsilon^p|| = \varepsilon_L \end{cases}$$  \hspace{1cm} (16)

$$\gamma \geq 0, F^s \leq 0, \gamma F^s = 0$$  \hspace{1cm} (17)

$$F^s = ||\varepsilon^T|| - \varepsilon_L$$  \hspace{1cm} (18)

The dense SMA model in the time-continuous frame is summarized in Table 1.

<table>
<thead>
<tr>
<th>Internal variables: $\varepsilon^t, \varepsilon^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material parameters: $K, G, h^p, h^t, \beta, T_m, \alpha, \varepsilon_L, R^t, R^p, Q$</td>
</tr>
<tr>
<td>Constitutive equations and thermodynamic forces:</td>
</tr>
<tr>
<td>$s = 2G(e^t - e^p)$</td>
</tr>
<tr>
<td>$p = K[\theta - 3\alpha(T - T_0)]$</td>
</tr>
<tr>
<td>$\eta = -\frac{\partial \Psi}{\partial e^\theta} : \dot{e}^\theta = \eta_0 + 3\alpha K\theta - \frac{\beta(T - T_m)}{</td>
</tr>
<tr>
<td>$X = -\frac{\partial \Psi}{\partial e} : \dot{\varepsilon} = s - \beta(T - T_m) \frac{e^t - qe^p}{</td>
</tr>
<tr>
<td>$Q = -\frac{\partial \Psi}{\partial \varepsilon^p} : \dot{\varepsilon}^p = s + q\beta(T - T_m) \frac{e^p - qe^t}{</td>
</tr>
<tr>
<td>Limit function:</td>
</tr>
<tr>
<td>$F^t =</td>
</tr>
<tr>
<td>$F^p =</td>
</tr>
<tr>
<td>Evolution equations:</td>
</tr>
<tr>
<td>$\dot{\varepsilon}^t = \tilde{\varepsilon}^t \frac{1}{R^t}$</td>
</tr>
<tr>
<td>$\dot{\varepsilon}^p = \tilde{\varepsilon}^p \frac{1}{R^p}$</td>
</tr>
<tr>
<td>Kuhn-Tucker conditions:</td>
</tr>
<tr>
<td>$\tilde{\varepsilon}^t \geq 0, F^t \leq 0, \tilde{\varepsilon}^t F^t = 0$</td>
</tr>
<tr>
<td>$\tilde{\varepsilon}^p \geq 0, F^p \leq 0, \tilde{\varepsilon}^p F^p = 0$</td>
</tr>
</tbody>
</table>

### Table 1

Dense SMA constitutive model in the time-continuous frame.

2.2. Porous SMA model

Here we extend the dense SMA model presented in Section 2.1 to include pressure dependent behavior observed in porous SMAs. The total strain is decomposed as $\varepsilon = e^t + e^p + e^p$  \hspace{1cm} (19)

where $e^t, e^p$ and $e^p$ are total, elastic, transformation and plastic strains, respectively. Recalling that the inelastic behavior of porous SMAs is pressure-dependent, we also introduce the following decompositions:

$$e^t = e^t + \frac{\partial^t}{3}$$  \hspace{1cm} (20)

$$e^p = e^p + \frac{\partial^p}{3}$$  \hspace{1cm} (21)
where $\mathbf{e}^t$ and $\mathbf{e}^p$ are deviatoric components, $\mathbf{e}^{tr}$ and $\mathbf{e}^{tp}$ are volumetric components.

To introduce Helmholtz free energy function, in addition to the control and internal variables introduced in Section 2.1, we assume volumetric parts of the transformation and plastic strains $\mathbf{e}^{tr}$ and $\mathbf{e}^{tp}$ as internal variable, i.e., $\Psi = \Psi(\mathbf{e}, \mathbf{e}^{tr}, \mathbf{T}, \mathbf{e}^{tp}, \mathbf{e}^p, \theta^p, \theta^{tr})$. Moreover, we introduce the following equivalent transformation and plastic strain tensors $\mathbf{g}^{tr}$, $\mathbf{g}^p$ as functions of internal variables (Ashrafi et al., 2015; Deshpande & Fleck, 2000):

$$
\mathbf{g}^{tr} = \sqrt{(1 + \kappa/6)} \left( \mathbf{e}^{tr} + \sqrt{\frac{1}{3\kappa}} \theta^{tr} \mathbf{1} \right)
$$

$$
\mathbf{g}^p = \sqrt{(1 + \kappa/6)} \left( \mathbf{e}^p + \sqrt{\frac{1}{3\kappa}} \theta^{tp} \mathbf{1} \right)
$$

where $\kappa$ is a porosity parameter representing the pressure-dependent behavior of the porous material.

Similar to the dense SMA model, considering the coupling effects, linear kinematic hardening for transformation and plastic behavior as well as linear dependency of transformation behavior on temperature, we introduce the following form for the Helmholtz free energy function:

$$
\Psi = \frac{1}{2} K(\theta - \theta^{tr} - \theta^{tp})^2 + G \left( ||\mathbf{e} - \mathbf{e}^{tr} - \mathbf{e}^p||^2 - 3\alpha K \theta (T - T_0) \right)
$$

$$
+ \beta (T - T_m) ||\mathbf{g}^{tr} - \mathbf{g}^p||^2 - \mathbf{g}^{tr} : \mathbf{g}^p
$$

$$
+ \frac{1}{2} h_t ||\mathbf{e}^{tr}||^2 + \frac{1}{2} h_t ||\mathbf{g}^{tr}||^2 + \varphi_{\mathbf{g}^p}(||\mathbf{g}^{tr}||) + (u_0 - T_1 \theta)
$$

$$
+ \zeta (T - T_0 - T \ln(T/T_0))
$$

(24)

where the saturation function $\varphi_{s_1}$ is introduced as:

$$
\varphi_{s_1}(||\mathbf{g}^{tr}||) = \begin{cases} 
0 & \text{if } ||\mathbf{g}^{tr}|| \leq s_1 \\
+\infty & \text{otherwise}
\end{cases}
$$

To derive the constitutive equations and thermodynamic forces, the second law of thermodynamics should be satisfied. Following standard arguments, we can derive constitutive equations and thermodynamic forces:

$$
s = \frac{\partial \Psi}{\partial e^{tr}} = 2G(e^{tr} - e^{tp} - e^p)
$$

$$
p = \frac{\partial \Psi}{\partial e^{tp}} = K(\theta - \theta^{tr} - \theta^{tp}) - 3\alpha K \theta (T - T_0)
$$

$$
\eta = \frac{\partial \Psi}{\partial \theta^{tr}} = n_0 + 3\alpha K \theta - \beta (T - T_m) ||\mathbf{e}^{tr}||^2 \left[ \frac{1}{|\mathbf{T}|} \right] + c \ln (T/T_0)
$$

$$
\mathbf{X}_d = \frac{\partial \Psi}{\partial \mathbf{e}^{tr}} = s - (1 + \kappa/6)
$$

$$
\times \left( \beta (T - T_m) \frac{\mathbf{e}^{tr} - \mathbf{g}^{tr}}{||\mathbf{g}^{tr}||} + \sqrt{\frac{1}{3\kappa}} \theta^{tr} \mathbf{1} \right)
$$

$$
\mathbf{X}_p = \frac{\partial \Psi}{\partial \mathbf{e}^{tp}} = p - (1 + \kappa/6)
$$

$$
\times \left( \beta (T - T_m) \frac{\mathbf{e}^{tr} - \mathbf{g}^{tr}}{||\mathbf{g}^{tr}||} + h_t ||\mathbf{e}^{tr}|| + \eta \theta^{tr} - \mathbf{g}^{tr} \right)
$$

$$
\mathbf{Q}_d = \frac{\partial \Psi}{\partial \mathbf{g}^{tr}} = q - (1 + \kappa/6)
$$

$$
\times \left( h_t ||\mathbf{e}^{tr}|| - \beta (T - T_m) \mathbf{e}^{tr} - \mathbf{g}^{tr} \right)
$$

$$
\mathbf{Q}_p = \frac{\partial \Psi}{\partial \mathbf{g}^{tp}} = q - (1 + \kappa/6)
$$

$$
\times \left( \eta \theta^{tr} \theta^{tr} - \beta (T - T_m) \mathbf{g}^{tr} - \mathbf{g}^{tr} \right)
$$

(26)

where $\mathbf{X}_d$ and $\mathbf{Q}_d$ are the thermodynamic forces associated to the deviatoric part of transformation and plastic strains, respectively; $\chi_d$ and $\bar{\chi}_d$ are the thermodynamic forces associated to the volumetric part of transformation and plastic strains, respectively. Moreover, the variable $\gamma$ which results from the saturation function sub-differential is defined as:

$$
\frac{\partial \varphi_{s_1}}{\partial ||\mathbf{g}^{tr}||} = \begin{cases} 
0 & \text{if } ||\mathbf{g}^{tr}|| \leq s_1 \\
\frac{\gamma}{||\mathbf{g}^{tr}||} & \text{if } ||\mathbf{g}^{tr}|| = s_1
\end{cases}
$$

(27)

$$
\gamma \geq 0, \quad F^t \geq 0, \quad \gamma F^t = 0
$$

(28)

$$
F^t = ||\mathbf{g}^{tr}|| - s_1
$$

(29)

Recalling the transformation and plasticity behavior described in Section 1 and the pressure-dependent behavior of porous SMAs, we introduce two different convex limit functions for phase transformation and plastic behavior:

$$
\bar{\mathbf{F}}^t = \bar{X} - \bar{R}^t
$$

$$
\bar{F}^p = \bar{Q} - \bar{R}^p
$$

(30)

where $\bar{R}^t$ and $\bar{R}^p$ are the radii of the transformation and plastic domain, respectively. Moreover, $\bar{X}$ and $\bar{Q}$ are effective transformation and plastic thermodynamic forces, defined as follows:

$$
\bar{X} = \sqrt{\left( ||\mathbf{X}_d|| + \kappa (n_0)^2 \right)^2 / (1 + \kappa/6)}
$$

$$
\bar{Q} = \sqrt{\left( ||\mathbf{Q}_d|| + \kappa (n_0)^2 \right)^2 / (1 + \kappa/6)}
$$

(31)

(32)

To satisfy the second law of thermodynamics, we introduce the following associative flow rules for the internal variables:

$$
\mathbf{e}^{tr} = \hat{\zeta}^{tr} \frac{\partial F^t}{\partial X_d} = \hat{\zeta}^{tr} \frac{\mathbf{X}_d}{(1 + \kappa/6)}
$$

$$
\mathbf{e}^{tp} = \hat{\zeta}^{tp} \frac{\partial F^p}{\partial Q_d} = \hat{\zeta}^{tp} \frac{\mathbf{Q}_d}{(1 + \kappa/6)}
$$

(33)

(34)

where the Lagrange multipliers $\hat{\zeta}^{tr}$ and $\hat{\zeta}^{tp}$ should satisfy the Kuhn–Tucker conditions:

$$
\hat{\zeta}^{tr} \geq 0, \quad F^t \leq 0, \quad \hat{\zeta}^{tr} F^t = 0
$$

$$
\hat{\zeta}^{tp} \geq 0, \quad F^p \leq 0, \quad \hat{\zeta}^{tp} F^p = 0
$$

(35)

The pressure-dependent SMA constitutive model in the time-continuous frame is summarized in Table 2. We observe that, setting $R_p = \infty$ and $A = \bar{Q} = \kappa = 0$, the model degenerates to an SMA model without plastic strain effects similar to the one originally proposed by Souza et al. (1998), and improved by Auricchio and Petrini (2004) and Arghavani et al. (2011a; 2011b).

3. Time discretization and solution algorithm

In order to numerically implement the dense and porous SMA models, in the following we sketch a time discrete version of each model and propose a general simple solution algorithm accounting for simultaneous evolution and coupling effects of transformation and plastic strains.
3.1. Time discretization

3.1.1. Dense SMA model

Assuming to be given the state \((s_n, p_n, e_{tr}^n, e_p^n)\) at time \(t_n\), the actual deviatoric and volumetric strain \((e, \theta)\) and temperature \(T\) at time \(t_n\), the time-discrete values \((s, p, e_{tr}, e_p)\) can be computed using an implicit backward Euler method. It should be noted that for notation simplicity here, and in the following, we drop the subindex \(n + 1\) for variables computed at time \(t_{n+1}\). The discretized version of dense SMA constitutive equations takes the following form:

\[
\begin{align*}
\dot{p} &= K(\theta - \theta_T - \theta_0 - 3\alpha(T - T_0)) \\
\dot{s} &= 2G(e - e_{tr} - e_p) \\
X_d &= -\frac{\partial \Psi}{\partial e_{tr}} = s - (1 + \kappa/6) \beta(T - T_m) e_{tr} - q e_p + \gamma \frac{e_p}{\|e_p\|} + h^r e_{tr} + A e_p \\
Q_d &= s + q \beta(T - T_m) e_{tr} - q e_p + h^r e_{tr} + A e_p \\
e_{tr} &= e_{tr} + \Delta e_{tr} \\
e_p &= e_p + \Delta e_p, \\
\Delta e_{tr} &= \dot{\zeta}_T \frac{e_p}{\|e_p\|} \\
\Delta e_p &= \frac{\partial e_e}{\partial p} \frac{Q}{\|Q\|} \\
\end{align*}
\]

(36)

where \(\Delta e_{tr}\) and \(\Delta e_p\) are the consistency parameters, time integrated over the interval \([t_n, t]\).

3.1.2. Porous SMA model

Assuming to be given the state \((s_n, p_n, e_{tr}^n, \theta_{tr}^n, e_p^n, \theta_p^n)\) at time \(t_n\), the actual deviatoric and volumetric strain \((e, \theta)\) and temperature \(T\) at time \(t_{n+1}\), the time-discrete values \((s, p, e_{tr}, \theta_{tr}, e_p, \theta_p)\) can be computed using an implicit backward Euler method. The discretized version of porous SMA constitutive equations takes the following form:

\[
\begin{align*}
\dot{p} &= K(\theta - \theta_T - \theta_0 - 3\alpha(T - T_0)) \\
\dot{s} &= 2G(e - e_{tr} - e_p) \\
X_d &= -\frac{\partial \Psi}{\partial e_{tr}} = s - (1 + \kappa/6) \beta(T - T_m) e_{tr} - q e_p + \gamma \frac{e_p}{\|e_p\|} + h^r e_{tr} + A e_p \\
Q_d &= s + q \beta(T - T_m) e_{tr} - q e_p + h^r e_{tr} + A e_p \\
e_{tr} &= e_{tr} + \Delta e_{tr} \\
e_p &= e_p + \Delta e_p, \\
\Delta e_{tr} &= \dot{\zeta}_T \frac{e_p}{\|e_p\|} \\
\Delta e_p &= \frac{\partial e_e}{\partial p} \frac{Q}{\|Q\|} \\
\Delta \zeta_T &= \alpha_T (1 + \kappa/6) X_d \\
\Delta \zeta_T &= \alpha_T (1 + \kappa/6) X_d \\
\Delta \zeta_T &= \alpha_T (1 + \kappa/6) X_d \\
\Delta \zeta_T &= \alpha_T (1 + \kappa/6) X_d \\
\end{align*}
\]

(37)
Fig. 3. Comparison of stress–strain curves with the experiments of Wang et al. (2008) for dense NiTi samples used for calibration of dense SMA model.

Fig. 4. Validation of stress–strain curves predicted by the dense SMA model with other experiments of Wang et al. (2008) for dense NiTi samples.

### Table 3

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
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<td>$E$ (GPa)</td>
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<td>$R^f$ (MPa)</td>
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<tr>
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<td>$A$ (MPa)</td>
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<tr>
<td>$q$ (-)</td>
<td>3500</td>
</tr>
<tr>
<td>$0.1$</td>
<td></td>
</tr>
</tbody>
</table>

3.2. Solution algorithm

A solution algorithm should contain a search method for detecting the active sets of transformation and plastic regions. An effective approach to the detection of an active set search has been introduced by Schmidt-Baldassari (2003) in the context of crystal plasticity. The approach replaces the Kuhn–Tucker complementarity inequality conditions, $a \leq 0, b \geq 0, ab = 0$, by the equivalent Fischer–Burmeister complementarity function $\varphi$, defined as follows (Fischer, 1992):

$$\varphi(a, b) = \sqrt{a^2 + b^2} + a - b$$

(38)

It can be easily shown that $\varphi(a, b) = 0 \iff a \leq 0, b \geq 0, ab = 0$. Therefore, we can rewrite the complementarity inequality constraints as a non-linear equality constraint by their replacement with a Fischer–Burmeister complementarity function. The application of such functions makes possible to omit an active set search, a fundamental advantage when dealing with coupled evolution
The Kuhn–Tucker and saturation conditions can be thus substituted by the following set of functions:

\[
\begin{align*}
\sqrt{(F^r)^2 + (\Delta \xi^r)^2 + F^r - \Delta \zeta^r} &= 0 \\
\sqrt{(F^p)^2 + (\Delta \xi^p)^2 + F^p - \Delta \zeta^p} &= 0 \\
\sqrt{(F^s)^2 + (\gamma)^2 + F^s - \gamma} &= 0
\end{align*}
\]  

\( (39) \)

In this paper, utilizing equality relations (40) instead of Kuhn–Tucker and saturation inequality relations, we solve the nonlinear equations of each SMA models (dense or porous) in every increment using Newton–Raphson method. In this regard, we determine consistent tangent matrix by exact linearization of the non-linear finite element (FE) equations. The FE implementation for dense and porous SMAs was performed in a user defined subroutine UMAT in ABAQUS/Standard.

Remark. In many structural applications, porous and dense SMA components such as beams, springs and torque tubes usually exhibit large global displacements with small strains. We should remark that using corotational formulation, small-strain constitutive
models can be successfully applied to solve such large displacement problems, where strains are small though rotations can be arbitrarily large.

However, for the numerical implementation of such large displacement problems (where the updated Lagrangian formulation is employed) the time-discretization of the rate form equations should satisfy the objectivity requirements. For a typical Jaumann objective rate form equation $\dot{A} = F(t, T, A, ...)$, the incrementally-objective time-discrete form is obtained via Hughes–Winget algorithm as (Hughes & Winget, 1980):

$$A_{n+1} = QAQ^T + \Delta t F(t, T, A, ...)$$  \hspace{1cm} (41)

where $A$ and $F$ are second order tensors in the current configuration and $Q$ is the incremental rotation tensor.

It is noted that, to use a small-strain constitutive model for solution of large displacement problems in software ABAQUS/Standard, the user should activate the option NLGEOM.

However, in this case, only stress and strains are automatically rotated incrementally by the software. The rotation of user-defined tensorial internal variables ($\epsilon^e$ and $\sigma^e$) are left to the user, while the tensor $Q$ is passed through the UMAT as a 3x3 matrix DROT\(^1\).

4. Numerical examples

In this section, we first validate the proposed models through testing its ability to reproduce several dense and porous SMA experiments. Then, we perform a study on material parameters related to coupling effects and pressure dependency. Finally, two structural examples are presented to show effectiveness of porous SMA model and its numerical implementation for large rotation condition, as well as non-proportional loading.

\(^1\) DROT is just the name of the variable used by user subroutine UMAT for rotation of coordinates.

Fig. 7. (a) Finite element mesh of the porous RVE using dense SMA material model, (b) Finite element mesh of the dense RVE using porous SMA material model.

Fig. 8. Comparison of porous NiTi response under uniaxial loading using two methods: porous RVE (with dense model) and dense RVE (with porous model), (a) 4\% strain loading, (b) 8\% strain loading.
It is noted that in the porous SMA model, the coefficient $\sqrt{1/(1+\kappa/6)}$, introduced in the definitions (31) and (32) for the effective thermodynamic forces, is adopted so that the stress limits for transformation and plasticity under uniaxial loading are independent of the porosity parameter \( \kappa \). Therefore, using only uniaxial experiments the porosity parameter \( \kappa \) in addition to Poisson’s ratio \( \nu \) cannot be determined. In the following, we identify the material parameters of both models and validate them with uniaxial experiments. The remaining parameters are essential for predictions under general multiaxial loadings which are determined using computational simulation of porous SMAs under hydrostatic loading.

4.1. Validation

We validate the dense and porous SMA models with experimental uniaxial results presented by Wang et al. (2008) and Köhl et al. (2011) for dense and porous NiTi samples, respectively. It is noted that we assume the thermal expansion coefficient equal to zero since it has a secondary effect compared to transformation and plastic strains.

For validation of dense SMA model, we identify material parameters (reported in Table 3) using the two stress–strain curves reported by Wang et al. (2008) for dense NiTi samples (Fig. 3a and b). According to the experiments, evolution of plastic strain has a considerable degradation effect on critical stress of reverse transformation. However, plastic strain does not significantly affect incomplete reverse transformation. The corresponding comparison in addition to model predictions for other loadings are illustrated in Figs. 3 and 4. The comparison of experimental data and model predictions reveal the capability of model in describing transformation and plasticity behavior. The effects of plastic strain on reverse phase transformation is captured in the model and the predictions for residual strain are in good agreement with experiments.

For validation of porous SMA model, we identify material parameters (reported in Table 4) using the two stress–strain curves reported by Köhl et al. (2011) for 51% porous NiTi samples (Fig. 5). The corresponding comparison in addition to model predictions for
other loadings are illustrated in Figs. 5 and 7. The model results show good agreement with experiments. Moreover, Fig. 5 depicts the capability of the model in simultaneous evolution of transformation and plastic strains.

Up to now, we determined two sets of material parameters for dense and 51% porous NiTi. It is obvious that the material properties of the NiTi that used in fabrication of porous samples of Köhl et al. (2011) are not the same as the material properties of dense NiTi samples of Wang et al. (2008). However, we try to recapture porous NiTi response by analysis of a porous RVE using the SMA dense model and compare it with the results of a dense RVE using the porous SMA model. It is noted that quantitative comparison of the results of these two simulation methods are not reasonable; therefore we only perform a qualitative comparison.

In the analysis of the porous RVE, the properties for dense SMA are considered according to Table 3 and a 51% porous mesh was generated. For more detail on the simulation method see Ashrafi et al. (2015); DeGiorgi and Qidwai (2002); Panico and Brinson (2008). Fig. 7a and b shows the finite element mesh for the porous and dense RVEs, respectively. Stress-strain curves for two uniax-

**Fig. 11.** Porous SMA model predictions for different values of $\kappa$ ($A = 3500$ and $q = 0.1$); (a) uniaxial loading, (b) shear loading.

**Fig. 12.** Porous SMA model predictions for different values of $\kappa$ ($A = 3500$ and $q = 0.1$); (a) biaxial loading, (b) hydrostatic loading.

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Material parameters adopted for Köhl et al. (2011) experiments on 51% porous NiTi samples.</th>
</tr>
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<td>$E$ (GPa)</td>
<td>$h^u$ (MPa)</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------</td>
</tr>
<tr>
<td>6.5</td>
<td>2000</td>
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</table>


The results of porous SMA model predictions for different values of $\kappa$ ($A = 3500$ and $q = 0.1$) are depicted in Fig. 13. (a) shows the butterfly-shaped stress loading profile, while (b) displays the strain response.

Fig. 13. Porous SMA model predictions for different values of $\kappa$ ($A = 3500$ and $q = 0.1$): (a) butterfly-shaped stress loading profile, (b) strain response.

Fig. 14 illustrates the finite element analysis of a porous NiTi spring under extensional loading. The figure shows contour plots of equivalent transformation strain (a) and equivalent plastic strain (b).

Fig. 14. Finite element analysis of a porous NiTi spring under extensional loading: (a) contour plot of equivalent transformation strain, (b) contour plot of equivalent plastic strain.

The finite element analysis of a porous NiTi spring after unloading is presented in Fig. 15. Similar to the previous figure, it contains contour plots of equivalent transformation strain (a) and equivalent plastic strain (b).

Fig. 15. Finite element analysis of a porous NiTi spring after unloading: (a) contour plot of equivalent transformation strain, (b) contour plot of equivalent plastic strain.

Material loadings are compared in Fig. 8. According to Fig. 8a, for 4% strain loading, the results of porous and dense meshes are in agreement quantitatively, the global response (phase transformation and maximum stress) is recaptured in both simulations. Similarly, for 8% strain loading (Fig. 8b), the global response (phase transformation, plastic strain evolution and maximum stress) is recaptured. Moreover, comparing the results of Fig. 8a and b reveals that by increasing loading, both simulation methods can recapture typical behaviors of porous SMAs, i.e., simultaneous evolution of transformation and plastic strains and reverse phase transformation after plastic strain evolution.

4.2. Parametric study

To show the main features of the proposed models, we perform a study on the introduced material parameters, i.e., $q$, $A$, and $\kappa$. The parameters $q$ and $A$ control the coupling effects of transformation and plasticity, and the parameter $\kappa$ controls pressure dependent behavior of porous SMAs. Here we use the material parameters reported in Tables 5 for dense and porous NiTi samples. As stated before, the value for $\kappa$ and $\nu$ are determined using computational simulation method (see Ashrafi et al. (2015) for more details).

First we study the effects of coupling parameters ($q$ and $A$). Fig. 9a and b depicts the dense and porous SMA model results, respectively for different values of $q$ under uniaxial loading. Accordingly, increasing $q$ leads to more residual strain and lowers the plastic stress limit. However, the stress limit for reverse transformation remain unchanged for different values of $q$. In other words, the influence of plasticity on recoverable strain increase with increasing $q$. Fig. 10a and b shows the dense and porous SMA model results, respectively for different values of $A$ under uniaxial loading. Larger values for $A$ degrades the material behavior and the stress limits for plasticity and reverse transformation decrease with increasing $A$. Similar to the influence of parameter $q$ on residual
strain, larger values for $A$ leads to larger residual strain. These coupling effects can be explained according to the relations for transformation and plastic thermodynamic forces (Eqs. 36{a} and 36{a}).

After studying the effects of coupling parameters, here we study the effects of porosity parameter $\kappa$. In this regard, we compare the porous SMA model behavior under several multiaxial loadings for three different values of porosity parameter ($\kappa = 0.01, 0.30, 0.65$). It is noted that pressure dependency increases with increasing $\kappa$; therefore the model with $\kappa = 0.01$ behaves similar to a dense model with no pressure dependency. Fig. 11a and b compares porous SMA model results with different porosity parameters under uniaxial and shear proportional loadings, respectively. Moreover, Fig. 12a and b shows similar results for biaxial and hydrostatic loadings, respectively.

As stated before, due to the coefficient $\sqrt{1/(1 + \kappa/6)}$, the results of porous SMA model are independent of $\kappa$ for uniaxial loading; however the model results for other loadings depend on the porosity parameter $\kappa$. It is deduced from Figs. 11 and 12 that smaller values for $\kappa$ predicts larger strain under pure shear loading, but smaller strain under biaxial and hydrostatic loading. In Fig. 12 b, when $\kappa = 0$ elastic response is observed which is similar to that of a dense material under hydrostatic loading in which no transformation or plastic strain occurs. This phenomenon can be explained by comparing the ratio of hydrostatic to deviatoric stress $P/\|S\|$ in these loadings. In shear, uniaxial and biaxial loadings the ratio $P/\|S\|$ is equal to 0, $1/3$ and $\sqrt{2}/3$, respectively. Therefore, if the ratio is smaller than $1/3$ for a loading condition, larger values of $\kappa$ predicts stiffer behavior; if the ratio is larger than $1/3$, smaller values of $\kappa$ predicts stiffer behavior. As an example for non-proportional loading, strain response under a butterfly-shaped stress loading is illustrated in Fig. 13. When both axial stresses are tensile or compressive ($P/\|S\| > 1/3$), porous model with $\kappa = 0.01$

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
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<tbody>
<tr>
<td>Material parameter</td>
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<tr>
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</tr>
<tr>
<td>$E$</td>
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<tr>
<td>$\tilde{A}$</td>
</tr>
<tr>
<td>$\tilde{q}$</td>
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</table>

Fig. 16. Force-displacement curves for a porous NiTi spring in extension (three different loading range).

Fig. 17. Porous NiTi cylinder under non-proportional axial-pressure loading; (a) geometry and loading, (b) three different loading profiles, (c) strain responses.
is stiffer; when one of the axial stresses is tensile and the other is compressive (\(|p|/|s| < 1/3\)), porous model with \(k = 0.65\) is stiffer.

4.3. Finite element analysis of BVPs

In this section, we demonstrate the capabilities of the porous SMA model, i.e., simultaneous transformation and plasticity as well as pressure dependency. Moreover, we assess FE implementation of the model for a small strain large rotation Boundary Value Problem (BVP). For the simulations, we use the commercial nonlinear finite element software ABAQUS/Standard, implementing the described algorithm within a user-defined subroutine UMAT. To this end, we need the material parameters reported in Table 4 as well as the values for Poisson’s ratio (\(\nu\)) and porosity parameter (\(\kappa\)). We utilize a computational simulation method (Ashrafi et al., 2015; Panico & Brinson, 2008) and determine the values of \(\nu\) and \(\kappa\) for a 51% porous SMA sample. In combination with the parameters for the 51% porous NiTi sample in Table 4, we use a set of material parameters reported in Table 5.

4.3.1. Spring actuator

As the first example, we consider a porous SMA spring (with a wire diameter of 4 mm, a spring external diameter of 24 mm, a pitch size of 12 mm and with two coils and an initial length of 28 mm). An axial force is applied to one end while the other end is completely fixed. The force is increased from zero to its maximum value and unloaded back to zero. Specifically, by SDV16 and SDV17, we refer to equivalent transformation and plastic strains (\(|\varepsilon^p|\) and \(|\varepsilon^p|\)), respectively. Fig. 14a and b shows the transformation and plastic strain contours, respectively in the spring after applying a 150 N. Fig. 15a and b shows the same contours after removing the load. For comparison, the initial configuration is also presented in all figures showing that the spring is under extension and undergoes large displacement; therefore this example shows the model capability for large rotation and small strain conditions. Comparing Figs. 17 and 16 reveals that transformation strain is recovered while plastic strain leads to a residual elongation of spring after unloading. Force-displacement curve is also depicted in Fig. 16 for three different loading ranges (90, 150 and 180 N). The results show pseudo-elastic behavior, while a portion of spring displacement is unrecoverable due to simultaneous evolution of plastic strain during loading of spring. Moreover, the residual length of the spring increases with increasing loading.

4.3.2. Tube with internal and axial pressure

In the second example, to show the pressure dependent behavior of porous SMAs under non-proportional loadings, we show model results for a tube with internal and axial pressure (Fig. 17a). One end of the tube is completely fixed while a pressure load is applied to the other end, and a uniform pressure is applied to the internal surface of the tube. The loading is non-proportional as depicted in Fig. 17b. The axial strain versus radial strain are plotted for such loadings (with three different ranges) in Fig. 17c. The results show the capability of the proposed model and its numerical implementation in predicting behavior of porous SMAs under pressure-axial non-proportional loading. Fig. 17c shows the coupling between evolution of axial and volumetric strain. A residual strain remains in the sample after complete unloading due to evolution of plastic strain which is larger for the higher loading range (Loading 3).

5. Summary and conclusions

In this study, we proposed two versions of a 3-D phenomenological constitutive model for dense and porous SMAs. Coupling effects of transformation and plasticity are taken into account in these models. Also, the model for porous SMAs includes pressure dependent behavior observed in porous materials. A general effective solution algorithm for the two constitutive models was proposed. Using Fischer–Burmeister complementary function to satisfy Kuhn–Tucker and saturation conditions, we were able to effectively handle phase transformation and plasticity simultaneously. In addition to phase transformation, the model includes the main features of SMAs (degradation and incomplete reverse transformation) due to plastic deformation.

We numerically implemented the models in user-defined subroutines (UMAT) in ABAQUS and validated the models with experimental data for dense and porous NiTi samples. Also, through a parametric study, we carefully investigated the role of the transformation-plasticity coupling parameters (\(q\) and \(A\)) as well as the porosity parameter (\(k\)). The parameter \(q\) is the ratio of the irrecoverable part of transformation strain to plastic strain. Also, parameter \(A\) is a measure of decrease in the stress limit of the reverse transformation when plastic strain evolves. According to Figs. 9 and 10, Increasing plastic strain, the recoverable portion and the stress limit of the reverse transformation decrease; therefore no reverse transformation occurs when plastic strain reaches a specific threshold. In porous SMAs, the porosity parameter (\(k\)) controls pressure dependent behavior of porous SMAs and its effects depend on the loading. The model formulation is adopted so that the results are identical under uniaxial loading for different values of \(k\), which simplify the calibration process (Fig. 11a). However, the model results are substantially different for other loadings; while the response becomes stiffer by increasing \(k\) under shear loading, the response becomes more compliant by increasing \(k\) under biaxial loading.

Moreover, using corotational formulation in numerical implementation of the models enabled us to study structures undergoing large rotations. Through finite element analysis of two structural examples, we evaluated the capability of the model and solution algorithm in describing large rotation and pressure-dependent behavior of porous SMAs under proportional as well as non-proportional loadings.

References


