Non-prismatic beams: A simple and effective Timoshenko-like model

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\textbf{A B S T R A C T}

The present paper discusses simple compatibility, equilibrium, and constitutive equations for a non-prismatic planar beam. Specifically, the proposed model is based on standard Timoshenko kinematics (i.e., planar cross-section remain planar in consequence of a deformation, but can rotate with respect to the beam center-line). An initial discussion of a 2D elastic problem highlights that the boundary equilibrium deeply influences the cross-section stress distribution and all unknown fields are represented with respect to global Cartesian coordinates. A simple beam model (i.e. a set of Ordinary Differential Equations (ODEs)) is derived, describing accurately the effects of non-prismatic geometry on the beam behavior and motivating equation’s terms with both physical and mathematical arguments. Finally, several analytical and numerical solutions are compared with results existing in literature. The main conclusions can be summarized as follows. (i) The stress distribution within the cross-section is not trivial as in prismatic beams, in particular the shear stress distribution depends on all generalized stresses and on the beam geometry. (ii) The derivation of simplified constitutive relations highlights a strong dependence of each generalized deformation on all the generalized stresses. (iii) Axial and shear-bending problem are strictly coupled. (iv) The beam model is naturally expressed as an explicit system of six first order ODEs. (v) The ODEs solution can be obtained through the iterative integration of the right hand side term of each equation. (vi) The proposed simple model predicts the real behavior of non-prismatic beams with a good accuracy, reasonable for the most of practical applications.

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1. Introduction

Non-prismatic beams – sometime mentioned also as beams with non-constant cross-section or beams of variable cross-section – are a particular class of slender bodies, object of the practitioners interest due to the possibility of optimizing their geometry with respect to specific needs (Auricchio et al., 2015; Timoshenko and Young, 1965). Despite the advantages that engineers can obtain from their use, non-trivial difficulties occurring in the non-prismatic beam modeling often lead to inaccurate predictions that vanish the gain of the optimization process (Hodges et al., 2010). As a consequence, an effective non-prismatic beam modeling still represents a branch of the structural mechanics where significant improvements are required.

Within the large class of non-prismatic beams it is possible to recognize several families of beams characterized by peculiar properties and intrinsic modeling problems. Unfortunately, the literature lexicon is not thorough and the language ambiguities could lead to some annoying misunderstanding. Therefore, in order to discuss the existing approaches, we introduce some terminology that we are going to use in this document.

- A prismatic beam is a prismatic slender body with straight center-line and constant cross-section.
- A curved beam is a body with a curvilinear center-line and constant cross-section (orthogonal to the center-line).
- A beam of variable cross-section is a beam with straight center-line and non-constant cross-section; sometimes authors refer to this kind of bodies with the expression “non-prismatic beams” (Attarnejad et al., 2010; Beltempo et al., 2015; Shooshtari and Khajavi, 2010).
- A tapered beam is a beam of variable cross-section with the additional property that the cross-section size varies linearly with respect to the axis coordinate; sometime authors refer

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to this kind of bodies with the expression “beam of variable cross-section” (Cicla, 1939; Franciosi and Mecca, 1998; Romano, 1996; Sapountzakis and Panagos, 2008).

- A twisted beam is a beam of variable cross-section with the additional property that the cross-section rotation around the axis coordinate varies.
- A non-prismatic beam is the most general case we can consider, i.e. a beam with curvilinear center-line and non-constant cross-section.

Classical reference books treat separately curved beams and beams of variable cross-section, but do not provide any specific indications for the non-prismatic beams (Timoshenko, 1955; Timoshenko and Young, 1965). This distinction is substantially confirmed also in more recent books (Bruhns, 2003; Cross et al., 2012) and papers. As an example, Hodges et al. (2008; 2010) propose a model for a tapered beam whereas Rajagopal et al. (2012; Yu et al. (2002) focus on curvilinear beams.

Focusing on curvilinear beams, the classical approach describes the beam geometry through a curvilinear coordinate that runs along the beam center-line (Bruhns, 2003; Capurso, 1971). As a consequence, the cross-sections are defined as the intersection between the beam body and the plane orthogonal to the center-line at a fixed curvilinear coordinate. Furthermore, the resulting forces – ensuring the beam equilibrium – are expressed as function of a local coordinate system with the axes tangential, normal, and binormal to the center-line, respectively. For planar beams, the so far introduced choices lead to express the equilibrium through a system of 3 ODEs that are coupled and use non-linear coefficients (Arunkirinathar and Reddy, 1992; 1993; Rajagopal et al., 2012; Rajasekaran and Padmanabhan, 1989; Yu et al., 2002). The immediate consequence of this approach is that analytical solutions are available only for simple geometries, typically beams with constant curvature. An alternative approach was proposed by Gimena et al. (2008a) that express both displacements and resulting forces in a global coordinate system. The main advantage of the proposed approach is the simplicity of the resulting ODEs that could be solved with successive integrations.

Focusing on beams of variable cross-section, the literature presents a number of possible models that can be classified as follows:

- simple models, based on suitable modifications of Euler–Bernoulli or Timoshenko beam model coefficients (Banerjee and Williams, 1985; 1986; Friedman and Kosmatka, 1993; Sapountzakis and Panagos, 2008; Shooshntari and Khatjavi, 2010);
- enhanced models, that consider an accurate description of stress (Aminbaghai and Binder, 2006; Rubin, 1999) and are often derived from variational principles (Auricchio et al., 2015; Beltempo et al., 2015; Hodges et al., 2008; 2010);
- models based on 2D or 3D Finite Elements (FE), that often appear as the only possible path, especially when an accurate description of unknown fields is required (Balkaya, 2001; El-Mezaini et al., 1991; Kechter and M.Gurtkowski, 1984).

It is well known since the half of the past century that the simple models are no-longer effective in predicting the real behavior of non-prismatic beams (Boley, 1963; Hodges et al., 2010). On the other hand, 2D FE models lead to a high computational effort. As a consequence, the enhanced models seem the best compromise.

It is worth noting that non-prismatic beams are often treated as beams of variable cross-section. In fact, both researchers and practitioners neglect the effects of the non-straight center-line for modeling simplicity (Portland Cement Associations, 1958; Balkaya and Citipitioglu, 1997; El-Mezaini et al., 1991; Ozay and Topcu, 2000; Tena-Colunga, 1996; Timoshenko and Young, 1965). To the author’s knowledge, the few attempts of a complete modeling of non-prismatic beams are Auricchio et al. (2015); Balduzzi (2013); Beltempo (2013); Beltempo et al. (2015) that use mixed variational principles and the dimension reduction method to derive planar beam models and Gimena et al. (2008b) that proposes a model for 3D non-prismatic beams and an effective numerical procedure for the resolution of the Ordinary Differential Equations (ODEs) governing the beam model. Unfortunately, the dimension reduction leads to equations with an unclear physical meaning and a complexity which seems scarcely manageable. Conversely, the model proposed by Gimena et al. (2008b) presents some limitations that will be discussed in the following.

The present paper aims at discussing a simple and effective non-prismatic planar beam model. The derivation procedure is based on a rigorous treatment of the 2D elastic problem and exploits the simple Timoshenko kinematics (i.e., planar cross-section remain planar in consequence of a deformation, but can rotate with respect to the beam center-line). The simplicity of resulting ODEs allows to provide the analytical solution for tapered beams and helps the understanding of several aspects that influence the effectiveness of the non-prismatic beam modeling.

The outline of the paper is as follows. Section 2 introduces the 2D elastic problem we are going to tackle. Section 3 illustrates how to derive the non-prismatic beam model equations. Section 4 focuses on tapered beams for which some analytical results are provided and compared with other approaches existing in literature. Section 5 provides few numerical examples that highlight critical aspects and advantages of the proposed approach, and Section 6 discusses the final remarks and delineates future research’s developments.

2. 2D problem formulation

The object of our study is the 2D non-prismatic beam that behaves under the hypothesis of small displacements and plane stress state. Moreover, the material that constitutes the beam body is homogeneous, isotropic, and linear-elastic.

We introduce the beam longitudinal axis \( L \), defined as follows

\[
L := \{ x \in [0, L] \}
\]

where \( L \) is the beam length. Moreover, we define the beam center-line \( c : L \rightarrow \mathbb{R} \) and the cross-section height \( h : L \rightarrow \mathbb{R}^+ \) where \( \mathbb{R}^+ \) indicates strictly positive real values. As usual in prismatic beam modeling, we assume that \( h(x) \neq 0 \) for each \( x \) indicating that this ratio plays a central role in determining the model accuracy. Furthermore, we assume that the beam longitudinal axis and the beam center-line are reasonably next to each other, this recommendation will become more clear in Section 4.3.2. Finally, we assume that \( c(x) \) and \( h(x) \) are sufficiently smooth functions which properties will be detailed in the following.

The cross-section lower and upper limits, \( h_l, h_u : L \rightarrow \mathbb{R} \) are defined as follows

\[
h_l(x) := c(x) - \frac{1}{2} h(x), \quad h_u(x) := c(x) + \frac{1}{2} h(x)
\]

Therefore, the cross-section \( A(x) \) is defined as follows

\[
A(x) := \{ y \mid \text{given } x \in L \Rightarrow y \in [h_l(x), h_u(x)] \}
\]

It is worth noting that Definition (3) introduces a small notation abuse: \( A(x) \) is a set and not a function. Nevertheless, it highlights the dependence of set definition on the axis coordinate. Furthermore, Definition (3) leads to cross-sections \( A(x) \) that are orthogonal to the longitudinal axis and not to the beam center-line, according to the approach proposed by Gimena et al. (2008a).

Using all the so far introduced notations, the 2D problem domain \( \Omega \) results defined as follows

\[
\Omega := \{ (x, y) \mid \forall x \in L \rightarrow y \in A(x) \}
\]
where $\nabla^3$ ($\cdot$) is the operator that provides the symmetric part of the gradient, $\nabla : (\cdot)$ represents the divergence operator, $\cdot$ ($\cdot$) represents the double dot product, and $\mathbf{D}$ is the fourth order tensor that defines the mechanical properties of the material. Eq. (5a) is the 2D compatibility relation, Eq. (5b) is the 2D material constitutive relation, and Eq. (5c) represents the 2D equilibrium relation. Eq. (5d) and (5e) are respectively the boundary equilibrium and the boundary compatibility conditions, where $\mathbf{n}$ is the outward unit vector, defined on the boundary.

As illustrated in Fig. 2, the outward unit vectors on the lower and upper limits result as follows

$$
\mathbf{n}|_{h_l}(x) = \frac{1}{\sqrt{1 + (h'_l(x))^2}} \begin{bmatrix} h'_l(x) \\ -1 \end{bmatrix}
$$

$$
\mathbf{n}|_{h_u}(x) = \frac{1}{\sqrt{1 + (h'_u(x))^2}} \begin{bmatrix} -h'_u(x) \\ 1 \end{bmatrix}
$$

(6)

where $(\cdot)'$ denotes the derivative with respect to the independent variable $x$. The boundary equilibrium (5d) on lower and upper limits (i.e., $(\boldsymbol{\sigma} \cdot \mathbf{n})|_{h_l/h_u} = 0$) could be expressed as follows

$$
\begin{bmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = 0 \quad \Rightarrow \quad \sigma_x n_x + \tau n_y = 0 \quad \tau n_x + \sigma_y n_y = 0
$$

(7)

where we omit the indication of the restriction $(\cdot)|_{h_l/h_u}$ for notation simplicity. Manipulating Eq. (7), we express $\tau$ and $\sigma_y$ as functions of $\sigma_x$; using the outward unit vector $\mathbf{n}$ definition (6), we obtain the following expressions for boundary equilibrium

$$
\tau = -\frac{h_x}{n_y} \sigma_x = h'h_x
$$

(8a)

$$
\sigma_y = \frac{n_x^2}{n_y^2} \sigma_x = (h')^2 \sigma_x
$$

(8b)

where $h$ indicates either $h_l(x)$ or $h_u(x)$, accordingly to the point where we are evaluating the function. Eq. (8) is defined only for non-vanishing $n_y$, therefore we need to require that first derivatives of lower and upper limits –and consequently of center-line and cross-section height– are bounded.

As already highlighted in (Auricchio et al., 2015), $\sigma_x$ could be seen as the independent variable that completely defines the stress state on the lower and upper surfaces. Moreover, as stated by Boley (1963) and Hodges et al. (2008, 2010) the lower and upper limit slopes $h'_l$ and $h'_u$ play a central role in boundary equilibrium, determining the stress distribution within the cross-section $A(x)$.

3. Simplified 1D model

This section aims at deriving the ODEs representing the beam model. In fact, the solution of the 2D problem introduced in Section 2 is in general not available. As a consequence, in order to obtain an approximated solution, practitioners consider simplified models properly called beam models. The non-prismatic beam model derivation consists of 4 main steps.
1. derivation of compatibility equations,
2. derivation of equilibrium equations,
3. stress representation, and
4. derivation of simplified constitutive relations

that correspond to the subdivision of the present section.

Fig. 1 represents the domain of the beam model we are going to
develop. It is worth noting that Eq. (5) considers a 2D region as
problem domain, the classical curved beam models considers
the center-line c(x), and the beam model we are going to develop
considers the beam axis L. Furthermore, we remark that all the
generalized variables we are going to use are referred to the global
Cartesian coordinate system Ox y.

In Fig. 1, q(x), m(x), and p(x) are the horizontal, bending, and
vertical resulting loads defined as

\[ q(x) = \int_{A(x)} f_s(x,y) \, dy \quad \text{m}(x) = -\int_{A(x)} y f_s(x,y) \, dy \]
\[ p(x) = \int_{A(x)} f_y(x,y) \, dy \]  

(9)

where \( f_s(x,y) \) and \( f_y(x,y) \) are the horizontal and vertical com-
ponents of the load vector \( f \).

Finally, for convenience during the model derivation, we intro-
duce the linear function \( \tilde{b}(y) \) defined as

\[ \tilde{b}(y) = \frac{2(c(x)-y)}{h(x)} \]  

(10)

We note that \( \tilde{b}(y) \) represents an odd function with respect to
the cross-section \( A(x) \), it vanishes at \( y = c(x) \), and it is equal to \( \pm 1 \) at
\( y = \tilde{h}_{u_{xy}}(x) \).

3.1. Compatibility equations

In order to develop the non-prismatic beam model, we as-
sume that the kinematics usually adopted for prismatic Timo-
shenko beam models is still valid. Therefore, we represent 2D dis-
placement field \( s(x,y) \) in terms of three 1D functions indicated as
generalized displacements: the horizontal displacement \( u(x) \), the rota-
tion \( \varphi(x) \), and the vertical displacement \( v(x) \). Specifically, we assume that
the beam body displacements are approximated as follows:

\[ s(x,y) \approx \left\{ \begin{array}{l}
\frac{u(x) - h(x)\varphi(x)}{2} \\
\frac{v(x) - \tilde{b}(y)\varphi(x)}{2} \\
\end{array} \right\} \]  

(11)

We introduce the generalized deformations i.e., the horizontal de-
formation \( \varepsilon_0(x) \), the curvature \( \chi(x) \), and the shear deformation \( \gamma(x) \)
respectively defined as follows

\[ \varepsilon_0(x) = \frac{1}{h(x)} \int_{A(x)} \varepsilon_x \, dy \quad \chi(x) = \frac{12}{h^2(x)} \int_{A(x)} \varepsilon_y (c(x) - y) \, dy \]
\[ \gamma(x) = \frac{1}{h(x)} \int_{A(x)} \varepsilon_{xy} \, dy \]  

(12)

where \( \varepsilon_x \) and \( \varepsilon_{xy} \) are the components of the strain tensor \( \varepsilon \).

Therefore, the beam compatibility is expressed through the fol-
lowing equations

\[ \varepsilon_0(x) = u'(x) - c'(x)\varphi(x) \]  

(13a)
\[ \chi(x) = -\varphi'(x) \]  

(13b)
\[ \gamma(x) = v'(x) + \varphi(x) \]  

(13c)

We note that, with respect to the prismatic beam compatibility,
a new term \( c'(x)\varphi(x) \) appears in Eq. (13a). Its physical meaning
is illustrated in Fig. 3 that shows that, if the center-line is non-
horizontal, a rotation induces non-negligible horizontal displace-
ment. On the other hand, considering a straight center-line parallel
to the beam axis, this term vanishes recovering the more familiar
compatibility equations usually adopted for prismatic beams.

3.2. Equilibrium equations

We introduce the generalized stresses i.e., the resulting hori-
Zontal stress \( H(x) \), the resulting bending moment \( M(x) \), and the resulting
vertical stress \( V(x) \) respectively defined as follows

\[ H(x) = \int_{A(x)} \sigma_x \, dy \]  

(14a)
\[ M(x) = \int_{A(x)} \sigma_y (c(x) - y) \, dy \]  

(14b)
\[ V(x) = \int_{A(x)} \tau \, dy \]  

(14c)

We use the virtual work principle in order to obtain the beam
equilibrium equations. The internal work \( L_{\text{int}} \) can be calculated
multiplying the virtual generalized deformations \( \delta \varepsilon_0(x), \delta \chi(x), \) and
\( \delta \gamma(x) \) (that satisfy Eq. (13)) for the corresponding equilibrated
generalized stresses \( H(x), M(x), \) and \( V(x) \). Analogously, the external
work \( L_{\text{ext}} \) can be calculated multiplying the virtual generalized dis-
placements \( \delta u(x), \delta \varphi(x), \) and \( \delta v(x) \) for the corresponding resulting
loads. Equalizing internal and external works we obtain

\[ L_{\text{int}} = \int_L [\delta \varepsilon_0(x) \cdot H(x) + \delta \chi(x) \cdot M(x) + \delta \gamma(x) \cdot V(x)] \, dx = \int_L x \]  

(15)

Substituting compatibility Eq. (13) in Eq. (15) we obtain:

\[ \int_L [\delta \varepsilon_0(x) \cdot H(x) + \delta \chi(x) \cdot M(x) + \delta \gamma(x) \cdot V(x)] \, dx = \int_L [\delta u(x) \cdot q(x) - \delta \varphi(x) \cdot m(x) + \delta v(x) \cdot p(x)] \, dx \]  

(16)

Integrating by parts the terms that apply derivatives to virtual gen-
eralized displacements and later collecting generalized displace-
ments, we obtain

\[ -\int_L \delta u(x) \cdot H'(x) \, dx + \int_L \delta \varphi(x) \cdot [M' - c'(x)H(x) + V(x)] \, dx \]
\[ -\int_L \delta v(x) \cdot V'(x) \, dx = \int_L [\delta u(x) \cdot q(x) - \delta \varphi(x) \cdot m(x) + \delta v(x) \cdot p(x)] \, dx \]  

(17)

In order to satisfy Eq. (17) for all possible virtual generalized dis-
placements, generalized stresses must satisfy the following equa-
tions

\[ H'(x) = -q(x) \]  

(18a)
that therefore are the equilibrium relations.

It is worth noting that equilibrium Eq. (18) can be obtained considering the equilibrium of a portion of non-prismatic beam of length \( dx \) (see Fig. 4). Furthermore, Eq. (18b) is a generalization of prismatic beam rotation equilibrium. Specifically, the term \( H(x) \cdot c'(x) \) takes into account the moment induced by horizontal resulting forces that are applied at different \( y \) coordinate in each cross-sections. As expected, the coefficient \( c'(x) \) vanishes for prismatic beams, leading to the well known prismatic beam equilibrium equation.

3.3. Recovery of cross-section stress distributions

The Timoshenko beam recovers the stress distributions within the cross-section through the following assumptions:

- the horizontal stress \( \sigma_x \) has a linear distribution,
- the vertical stress \( \sigma_y \) has a vanishing value,
- the shear stress \( \tau \) has a parabolic distribution, according to Jourawski theory (Timoshenko, 1955, Chapter IV).

Beltempo et al. (2015) apply the so far introduced hypotheses to several non-prismatic beams demonstrating that (i) the assumption on horizontal stress seems reasonable in all the considered cases; (ii) the Jourawski theory is completely ineffective in predicting the real shear stress distribution since it leads to violate the boundary equilibrium (8a) on \( h_x(x) \) and \( h_y(x) \).

Therefore, we modify the recovery of cross-section shear-stress distribution as illustrated in Fig. 5. Specifically, given the generalized stresses \( H(x) \) and \( M(x) \) it is possible to reconstruct the horizontal stress distribution within the cross-section and, in particular, to evaluate the horizontal stress magnitude on lower and upper limits \( h_x(x) \) and \( h_y(x) \) (Step 1). Boundary equilibrium (8a) allows to evaluate the shear stress at lower and upper limits of the cross-section – \( \tau_{h_x} \) and \( \tau_{h_y} \) respectively– that thereafter we interpolate through linear functions (Step 2). In order to satisfy Eq. (14c) we add a bubble function to the linear shear stress distribution obtaining a parabolic shear stress distribution (Step 3). Finally, we decompose the linear shear stress distributions in odd and even contributions (Step 4).

Recalling the \( h_x \) and \( h_y \) definitions (2) and performing simple calculations, the stress distributions result as follows

\[
\sigma_x(x, y) = \sigma_{x0}(x) + \tilde{b}(y) \cdot \sigma_{x1}(x)
\]

(19a)

\[
\tau(x, y) = c'(x) \sigma_{x0}(x) - \frac{h'(x)}{2} \sigma_{x1}(x)
\]

\[
+ \left( - \frac{h'(x)}{2} \sigma_{x0}(x) + c'(x) \sigma_{x1}(x) \right) \tilde{b}(y) + \frac{3}{2} \tilde{\tau}(x) \left( 1 - \tilde{b}^2(y) \right)
\]

(19b)

where the variables \( \sigma_{x0}(x), \sigma_{x1}(x), \) and \( \tilde{\tau}(x) \) result defined as follows

\[
\sigma_{x0}(x) = \frac{H(x)}{h(x)}, \quad \sigma_{x1}(x) = \frac{M(x)}{h^2(x)};
\]

\[
\tilde{\tau}(x) = \left( \frac{V(x)}{h(x)} - c'(x) \sigma_{x0}(x) + \frac{h'(x)}{2} \sigma_{x1}(x) \right)
\]

(20)

Finally, few algebraic steps allow us to express the shear stress distribution as illustrated in the following

\[
\tau(x, y) = \left( c'(x) \sigma_{x0}(x) - \frac{h'(x)}{2} \sigma_{x1}(x) \right) \tilde{b}(y) + \frac{3}{2} \tilde{\tau}(x) \left( 1 - \tilde{b}^2(y) \right)
\]

(21)

where the variable \( \tau_0(x) \) results defined as follows

\[
\tau_0(x) = - \frac{V(x)}{h(x)}
\]

(22)

It is worth noting that the previously introduced quantities have a clear physical meaning:

- \( \sigma_{x0}(x) \) is the mean value of the horizontal stress within the cross-section,
- \( \sigma_{x1}(x) \) is the maximum horizontal stress value induced by bending moment that occurs at the cross-section lower limit, and
- \( \tau_0(x) \) is the shear stress mean value.

In Eq. (21), the shear distribution \( \tau \) depends not only on \( V(x) \), but also on \( h(x) \) and \( M(x) \) that determine not only the magnitude but also the shape of the shear distribution. Finally, Eq. (21) leads to conclude that the maximum shear stress does not occur in correspondence of the beam center-line, as noted by Paglietti and Carta (2007, 2009).

The assumption of vanishing vertical stresses \( \sigma_y(x, y) = 0 \) agrees with the assumptions of prismatic beam stress recovery, significantly simplifying the beam model equations, but leads to violate Eq. (8b). Fortunately, this choice will not deeply worsen the model capability, as the numerical examples of Section 5 will illustrate. On the other hand, readers may refer to (Auricchio et al., 2015; Beltempo et al., 2015) for more refined models that do not neglect the contribution of vertical stresses.

The recovery relation (21) is well known since the first half of the twentieth century. In particular, referring to the analytical solution of the 2D equilibrium Eq. (5c) for an infinite long wedge, Atkin (1938); Cicala (1939); Timoshenko and Goodier (1951) express the stress distribution as the combination of some trigonometric functions. More in detail, Timoshenko and Goodier (1951) state that a parabolic shear distribution is an approximation reasonably accurate in the case of small boundary slope. Later on, Krahula (1975) extends the so far mentioned results to tapered beams, recovering equations substantially identical to (19a) and (21). Furthermore, more recently, Bruhns (2003, Example 3.9) notes that (i) the shear distribution in a tapered beam has no longer a distribution similar to the prismatic beams, (ii) the shear distribution depends not only on resulting shear, but also in bending moment and resulting horizontal stress, and (iii) the maximum shear could occur everywhere in the cross-section.
However, it is worth noting that all the so far mentioned references consider the stress representation (19a) and (21) valid only for tapered beams whereas we are extending the representation’s validity to more general cases. The numerical examples reported in (Auricchio et al., 2015; Beltempo et al., 2015) partially confirm that the recovery relations (19a) and (21) remain valid also in more general situations. Obviously, the effectiveness of stress description as well as of the beam model will decrease increasing the boundary slope and decreasing the beam slenderness, as it happens for tapered beams.

3.4. Simplified constitutive relations

To complete the Timoshenko-like beam model we need to introduce some simplified constitutive relations that define the generalized deformations as a function of generalized stresses.

The constitutive relation of the Timoshenko prismatic beam model needs a correction factor $k$ that is introduced in order to equalize the work of generalized shear stresses and deformations that considers only their mean values and the work of real shear stresses and strains that takes into account the real stress and strain distributions within the cross-section. In particular, for the simple case of a rectangular cross-section, the shear correction factor could be evaluated through the following formula

$$k = \frac{\int \tau^2 dy}{\int \tau dy} = \frac{1}{h(x)} \int \frac{9}{4} (1 - \tilde{b}^2 y)^2 dy = \frac{5}{6} \quad (23)$$

It is worth noting that Eq. (23) is effective because only the magnitude varies while the shear distribution has the same shape in all the cross-sections. Unfortunately, this does not hold for the non-prismatic beam. This simple consideration, together with the non-trivial dependence of $\tau(x, y)$ on $H(x)$ and $M(x)$—see Eq. (21)—lead to conclude that the prismatic beam constitutive relations are not satisfactory for the non-prismatic beams and a more refined approach must be adopted.

In particular, we consider the stress potential, defined as follows

$$\Psi^* = \frac{1}{2} \left( \frac{\sigma_2(x,y)^2}{E} + \frac{\tau^2(x,y)}{G} \right) \quad (24)$$

where $E$ is the Young’s modulus and $G$ is the shear modulus. Substituting the stress recovery relations (19a) and (21) in Eq. (24), the generalized deformations result as the derivatives of the stress...
potential with respect to the corresponding generalized stress. Therefore we have
\[ \varepsilon_0(x) = \int_{A(x)} \frac{\partial \Psi}{\partial H(x)} \, dy = \varepsilon_H(x) + \varepsilon_M(x) + \varepsilon_V(x) \quad (25a) \]
\[ \chi(x) = \int_{A(x)} \frac{\partial \Psi}{\partial M(x)} \, dy = \chi_H(x) + \chi_M(x) + \chi_V(x) \quad (25b) \]
\[ \gamma(x) = \int_{A(x)} \frac{\partial \Psi}{\partial V(x)} \, dy = \gamma_H(x) + \gamma_M(x) + \gamma_V(x) \quad (25c) \]

where
\[ \varepsilon_H = \left( \frac{c^2(x)}{5Gh(x)} + \frac{b^2}{12Gh(x)} + \frac{1}{Eh(x)} \right) \]
\[ \varepsilon_M = \chi_H = -\frac{8c^2(x)h(x)}{5Gh^2(x)}; \quad \varepsilon_V = \gamma_H = \frac{c^2(x)}{5Gh(x)} \]
\[ \chi_M = \left( \frac{9h^2}{5Gh^2(x)} + \frac{12c^2(x)}{Gh^2(x)} + \frac{12}{Eh^2(x)} \right) \]
\[ \chi_V = \gamma_M = -\frac{9h^2}{5Gh^2(x)}; \quad \gamma_V = \frac{6}{5Gh(x)} \]

It is worth noting the following statements.

• Unlike the prismatic beam models, the generalized deformations depend on all generalized stresses.
• It is possible to recognize in $\varepsilon_H$, $\chi_M$, and $\gamma_V$ the terms of prismatic beam constitutive laws
\[ \varepsilon_0(x) = \frac{H(x)}{Eh(x)}; \quad \chi(x) = \frac{12M(x)}{Eh^3(x)}; \quad \gamma(x) = \frac{V(x)}{kGh(x)} \quad (26) \]
• The shear correction factor, as well as the other coefficients, results naturally from the derivation procedure, bypassing the problem of their evaluation.

To the author’s knowledge, Vu-Quoc and Léger (1992) are the former researchers mentioning the non-trivial dependence of all generalized deformations on all generalized stresses. Considering the shear bending of a tapered beam, the authors derive the beam’s flexibility matrix using the principle of complementary virtual work. Later on, deriving a model for a tapered beam, Rubin (1999) and Aminbaghi and Binder (2006) consider the fact that the bending moment produces also shear deformation and the shear force produces a curvature. Unfortunately, they do not consider the effect of a non-horizontal center-line and Rubin (1999) use different coefficients with respect to the ones reported in Eqs. (25), resulting energetically inconsistent. Conversely, for the model derivation, Auricchio et al. (2015); Beltempo et al. (2015); Hodges et al. (2008) use variational principles that have the advantage to naturally derive the constitutive relations, but could lead to more complicated equations, as already highlighted in Section 1.

3.5. Remarks on beam model’s ODEs

In the following we resume the beam model's ODEs according to the notation adopted by Gimena et al. (2008a).
\[ \begin{bmatrix} H'(x) \\ V'(x) \\ M'(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c(x) & 1 & 0 \end{bmatrix} \begin{bmatrix} H(x) \\ V(x) \\ M(x) \end{bmatrix} - \begin{bmatrix} q(x) \\ p(x) \\ m(x) \end{bmatrix} \]
\[ \begin{bmatrix} \varphi'(x) \\ \psi'(x) \\ \upsilon'(x) \end{bmatrix} = \begin{bmatrix} \chi_H & \chi_V & \chi_M \\ \gamma_H & \gamma_V & \gamma_M \\ \varepsilon_H & \varepsilon_V & \varepsilon_M - c(x) \end{bmatrix} \begin{bmatrix} \varphi(x) \\ \psi(x) \\ \upsilon(x) \end{bmatrix} \]

With respect to Eq. (27), it is worth noting what follows.

• The equations are naturally expressed as an explicit system of first order ODEs.
• Similar to Gimena et al. (2008a), the matrix that collects equations’ coefficients has a lower triangular form with vanishing diagonal terms. As a consequence, the analytical solution can easily be obtained through an iterative process of integration done row by row, starting from $H(x)$ and arriving at $u(x)$.
• Reducing the beam model proposed by Gimena et al. (2008a) to a 2D case, it is easy to recognize the same structure in equilibrium and kinematics relations. Specifically Gimena et al. (2008a) use trigonometric coefficients whereas we use the corresponding linearized ones (i.e., we assume that $\sin(\theta) \approx \tan(\theta) \approx \theta$).
• The sub-matrix that collects the constitutive relation coefficients is symmetric with respect to the anti-diagonal.
• Finally, the sub-matrix that collects the constitutive relation coefficients is completely full whereas Gimena et al. (2008a) neglect coupling between curvature, horizontal, and vertical resulting stresses (i.e., $\chi_H = \chi_V = \chi_M = \gamma_H = \gamma_M = 0$). Examples reported in Section 5 will illustrate that all the terms of the simplified constitutive relations plays a crucial role and are in general not negligible.

Now we move our attention back to the equilibrium (18) and the compatibility (13) equations. It is worth noting that the use of a global Cartesian coordinate system presents the following main advantages.

• Both the beam equilibrium and compatibility are not influenced by the cross-section size, which on the contrary plays a central role in constitutive relations.
• Horizontal and vertical expressions are obtained through independent equations, whereas in classical formulation (Bruhns, 2003, Chapter 5) tangential and normal equilibrium equations are coupled.

Furthermore, we note that few researchers follow the path of using a global Cartesian coordinate system with cross-sections orthogonal to x-axis. Fung and Kaplan (1952) investigate the buckling of beams with small curvatures and Borri et al. (1992); Popescu et al. (2000); Rajagopal and Hodges (2014) propose beam models considering oblique cross-sections. On the other hand, Vogel (1993) develops a second order Euler–Bernoulli beam model for curved beams in large displacement regime and Gimena et al. (2008b, 2008a) develop 3D non-prismatic beam models expressing the resulting forces in a global coordinate system.

Finally, if we neglect shear deformation in the proposed model and the second order terms in (Vogel, 1993) –i.e., if we lead both models to use the same assumptions– we obtain the same equations, showing once more that the proposed model has the capability to recover simpler models available in literature.

4. Tapered beam analytical solution

This section discusses the analytical solution of the ODEs governing the problem (27) for the simple case of a tapered beam. Specifically, we consider a beam with an inclined center-line, as illustrated in Fig. 6.

The center-line and the thickness have the following analytical expressions
\[ c(x) = c_1 x; \quad h(x) = h_1 x + h_0 \quad (28) \]

4.1. Homogeneous solution

Considering the geometry so far introduced, the homogeneous solution of the beam model ODEs (27) is
The analytical solution of non-prismatic beam (29) uses rational and logarithmic terms whereas the analytical solution of prismatic beam uses polynomial terms. As a consequence, the polynomials usually adopted in structural analysis in order to reconstruct the prismatic beam displacement given the nodal displacements or as shape functions for FE software, are no longer effective for non-prismatic beams. Furthermore, the non-prismatic beam stiffness matrix has a completely different structure that is influenced by the non-trivial distribution of displacements along the beam longitudinal axis and the strong coupling of all equations. The homogeneous solution (29) can be used to overcome these problems, nonetheless these aspects lie outside this paper’s aims and will be object of a further scientific paper.

4.2. Particular solution

Considering a constant vertical load \( p(x) = p \) we obtain the following particular solution for the beam model ODEs (27).

\[
\begin{align*}
    H(x) &= 0 \\
    M(x) &= \frac{1}{2} px^2 \\
    V(x) &= -px
\end{align*}
\]

Fig. 6. Tapered beam considered for the evaluation of the analytical solution: geometry and parameter’s definitions.

\[
\begin{align*}
    \varphi(x) &= \frac{1}{10E_Gh_1^2h^4(x)} \left( h_0(c_1 C_6 - C_3) - h_1 C_4 \right) \\
    &+ \frac{c_1(60EC_1^2 + Eh_1^2 + 60G) C_6}{5EGh_1^2h(x)} \\
    &- \frac{6(10EC_1^2 + Eh_1^2 + 10G) C_5 + C_3}{5EGh_1^2h(x)} \\
    u(x) &= \frac{\log(h(x))}{60EGh_1^2} \left( 72c_1(10EC_1^2 - Eh_1^2 + 10G) (c_1 C_6 - C_3) \right) \\
    &+ 5h_1(Eh_1 + 12G) C_6 + 12c_1 Eh_1 C_3) \\
    &- \frac{c_1(60EC_1^2 - 7Eh_1^2 + 60G) h_0(C_5 - C_1 C_6) + h_1 C_4)}{10EGh_1^2h(x)} \\
    &+ c_1 C_2 + C_2 \\
    v(x) &= -\frac{\log(h(x))}{EGh_1^2} \left( c_1(60EC_1^2 - Eh_1^2 + 60G) C_6 \right) \\
    &- 3(20EC_1^2 + 3Eh_1^2 + 20G) C_5 \\
    &- \frac{3(20EC_1^2 + Eh_1^2 + 20G)}{10EGh_1^2h(x)} \left( c_1 h_0 C_6 - h_0 C_5 - h_1 C_4 \right) - xc_3 + C_1 \\
    &+ \left( \frac{20EC_1^2 + Eh_1^2 + 20G}{2h(x)} \right) \left( \frac{3xh_1^2}{10} \right)
\end{align*}
\]

where the parameters \( C_1, C_2, C_3, C_4, C_5, \) and \( C_6 \) depend on boundary conditions.

4.3. Asymptotic analysis

In this section, we perform some tests in order to evaluate robustness and correctness of the beam model. Specifically we are going to investigate two main aspects:

1. the behavior of the beam solution when the geometry tends to become prismatic,
2. the capability of the beam model to recover the solution of a tapered beam that has the straight axis rotated with respect to the principal Cartesian coordinate system.

We consider a tapered cantilever, clamped in the initial cross-section and loaded with a vertical concentrated force in the final cross-section i.e., \( u(0) = \varphi(0) = v(0) = H(l) = M(l) = 0 \). and \( V(l) = 1 \). Furthermore, we assume the following values:

\[
\begin{align*}
    l &= 10 \text{ mm} \quad h(0) = \alpha h(l) \quad h(l) = 1 \text{ mm} \quad E &= 10^5 \text{ MPa} \\
    G &= 4 \cdot 10^4 \text{ MPa}
\end{align*}
\]

where \( \alpha \) is the ratio between the maximum and the minimum cross-section sizes.

4.3.1. Beam behavior for vanishing taper slope

To evaluate the non-prismatic beam behavior for vanishing taper slope, we consider two cases:

1. a cantilever with an horizontal center-line i.e., \( c(x) = 0 \) where all the cross-sections are symmetric with respect to the beam longitudinal axis, denoted in the following as \( \text{symm} \)
2. a cantilever with an horizontal lower limit i.e., \( h_1(l) = 0 \) and \( c(x) = (1 - \alpha) h(l) \) \( \frac{2}{5} \), denoted in the following as \( \text{unsymm} \).

We investigate the model behavior when the parameter \( \alpha \rightarrow 1 \) i.e., when the beam becomes prismatic. Since it is not possible to evaluate analytically the displacement limit, we evaluate the maximum vertical displacement \( v(l) \) varying \( \alpha \) between 2 and 1 + 1 \( \cdot 10^{-9} \) mm.

The maximum vertical displacement for the Timoshenko beam solution, indicated in the following as \( v_{\text{ms}} \) assume the following value:

\[
\begin{align*}
    v_{\text{ms}} &= \frac{1}{3} \frac{V(l)l^3}{E I_x} + \frac{6}{5} \frac{V(l)}{G h} = 0.0403 \text{ mm}
\end{align*}
\]
Fig. 7(a) plots the variation of the maximum displacement as a function of the parameter $\alpha$ whereas Fig. 7(b) plots the asymptotic error defined as

$$e_{as} = \frac{|V(l) - V_{as}|}{|V_{as}|}$$

(33)

We note that the effect of the non-horizontal center-line is negligible. Furthermore, the parameter $\alpha$ has a meaningful influence on the beam behavior for values greater than $1 + 1 \cdot 10^{-3}$. Performing calculations with standard double precision numbers, the solution starts to worsen for $\alpha < 1 + 1 \cdot 10^{-4}$ and calculations do not run for $\alpha < 1 + 1 \cdot 10^{-7}$. Therefore, we use 50 digits precision numbers for the evaluation of the results reported in Fig. 7.

Concluding, the model converges to the Timoshenko solution as expected. Furthermore, the analysis highlights some numerical problems in the model solution that fortunately occur for geometries without practical interest.

4.3.2. Effects of beam rotation

This section discusses the effect of a rotation of the global Cartesian coordinate system with respect to which we express the beam geometry. Due to the rotation $\theta$ of the Cartesian coordinate system, the cross-sections will not remain perpendicular to the beam center-line and the beam geometry will change as illustrated in Fig. 8. Nevertheless, we expect that the magnitude and the direction of the displacements evaluated in the final cross-section will not change.

We define the displacement magnitude relative error $e_s$ and the displacement direction error $\Delta \theta$ as follows:

$$e_s = \frac{|\sqrt{u^2(l) + v^2(l)} - V(l)|_{\theta = 0}|}{V(l)}$$

$$\Delta \theta = |\arctan\left(\frac{u}{v}\right) - \theta|$$

(34)

Fig. 9(a) displays the relative error $e_s$ evaluated for different values of $\theta$ and $\alpha$. The error becomes more significant for higher values of the rotation angle $\theta$. Nevertheless, the error magnitude remains in a reasonable range for the most of practical applications.

Fig. 8. Symmetric tapered cantilever, geometry definition through different Cartesian coordinate systems $l = 10$ mm, $h(0) = ah(l)$, $h(l) = 1$ mm, $P = 1$ N, $E = 10^3$ MPa, and $G = 4 \cdot 10^4$ MPa.

Fig. 7. Asymptotic behavior of a tapered cantilever loaded with a shear force $V(l) = 1$ N applied in the final cross-section.

Fig. 9. Effects of the Cartesian coordinate system rotation on the behaviour of a tapered cantilever loaded with a shear force $P = 1$ N applied in the final cross-section.
Fig. 9(b) displays the displacement direction error $\Delta \theta$ for different values of $\theta$ and $\alpha$. The results are reasonably accurate in predicting the right direction of displacement, at least in the considered cases.

4.4. Maximum displacement for symmetric, double-tapered beams

This section considers the double tapered beam loaded with a constant vertical load depicted in Fig. 10. In particular, this geometry is of interest since it is often used to shape beams that support double pitched roofs.

According to several results reported in literature, we express the beam maximum displacement $v_{\text{max}}$ as the sum of the bending $v_E$ and the shear $v_G$ contributions as follows:

$$v_{\text{max}} = v_E + v_G = \frac{5}{384} k_E \frac{p l^4}{h_0^2 b} - \frac{1}{8} k_C \frac{p l^2}{G h_0 b}$$

(35)

Exploiting the beam symmetry, using both the homogeneous (29) and the particular (30) solutions, and imposing suitable boundary values (i.e., $u(0) = v(0) = M(0) = H(l/2) = V(l/2) = \varphi(l/2) = 0$) the coefficients $k_E$ and $k_C$ results defined as follows

$$k_E = -\frac{6}{5} \frac{8\alpha^3 - 11\alpha^2 + 4\alpha - 1 - 2\alpha^2 \log (\alpha)(2\alpha + 1)}{\alpha^2(\alpha - 1)^4}$$

$$k_C = -\frac{12}{2} \frac{29\alpha^3 - 40\alpha^2 + 15\alpha - 4 - 2\alpha^2 \log (\alpha)(8\alpha + 3)}{\alpha^2(\alpha - 1)^2}$$

(36)

On the other hand, considering the model proposed by Rubin (1999), Schneider and Albert (2014) proposes the following expressions for the coefficients $k_E$ and $k_C$:

$$k_E = \frac{1}{\alpha^3(0.15 + \frac{0.65}{\alpha})}; \quad k_C = \frac{2}{1 + \alpha^{2/3}}$$

(37)

Finally, considering the Timoshenko prismatic beam equations, assuming only that cross-section area and inertia vary along the beam axis, Ozelson and Baird (2002) proposed the following expressions for the coefficients $k_E$ and $k_C$:

$$k_E = \frac{19.2}{(\alpha - 1)^3} \left( 2\frac{\alpha + 2}{\alpha - 1} \log \left( \frac{\alpha + 1}{2} \right) + \frac{3}{\alpha + 1} - \frac{2}{(\alpha + 1)^2} - 4 \right)$$

$$k_C = \frac{4}{\alpha - 1} \left( \frac{\alpha + 1}{\alpha - 1} \log \left( \frac{\alpha + 1}{2} \right) - 1 \right)$$

(38)

Fig. 11 reports the coefficients evaluated according to Eqs. (36)–(38).

All the functions reported in Fig. 11 converge to 1 for $\alpha \to 1$. This is an expected behavior since if $\alpha = 1$ we are considering a prismatic beam and therefore no correction factors must be applied to Eq. (35) in order to evaluate the beam maximum displacement. Fig. 11(a) highlights that Eqs. (36) and (37) provide substantially identical evaluation of the coefficient $k_E$. On the other hand, with respect to the proposed model, Eq. (38) overestimates the bending contribution of values that could exceed 100%. Fig. 11(b) highlights that each model provides completely different evaluations of the coefficient $k_C$. Furthermore, with respect to the model derived in this paper, both the formulas proposed in literature underestimate the shear contribution of values that could exceed the 40%.

The adoption of Eqs. (36) in engineering practice needs additional considerations on material constitutive law and rigorous validations that lie outside this paper’s aims. This specific aspect, as well as of other aspects of interest for practitioners, will be object of a future scientific paper.

5. Numerical examples

This section aims at providing further details about the obtained model capabilities. We consider two examples: (i) a tapered cantilever and (ii) an arch shaped beam. Both cases were already analyzed in (Auricchio et al., 2015) considering a more refined model.

5.1. Tapered beam

We consider the symmetric tapered beam illustrated in Fig. 12 ($l = 10\text{ mm}$, $h(0) = 1\text{ mm}$, and $h(l) = 0.5\text{ mm}$) and we assume $E = 10^5\text{ MPa}$ and $G = 4 \cdot 10^3\text{ MPa}$ as material parameters. Moreover, the beam is clamped in the initial cross-section $A(0)$ and a concentrated load $P = [-1, 0]\text{N}$ acts on the final cross-section $A(l)$.

Solving Eqs. (18), we obtain the following expressions for the generalized stresses

$$H(x) = 0; \quad M(x) = x - 10; \quad V(x) = 1$$

(39)
Fig. 12. Symmetric tapered beam: \( l = 10 \text{ mm} \), \( h(0) = 1 \text{ mm} \), \( h(l) = 0.5 \text{ mm} \), \( P = 1 \text{ N} \), \( E = 10^5 \text{ MPa} \), and \( G = 4 \cdot 10^4 \text{ MPa} \).

Fig. 13 depicts the distributions of the stresses \( \sigma_x \) and \( \tau \) in the cross-section \( A(0.5l) \). The label mod indicates the stress distribution obtained using Eqs. (19a) and (21), whereas the label ref indicates the 2D FE solution, computed using the commercial software ABAQUS (Simulia, 2011), considering the full 2D problem, and using a structured mesh of 7680 × 512 bilinear elements. Fig. 13(b) allows to appreciate a difference between the model and the reference solution, nevertheless the relative error magnitude is smaller than \( 1 \cdot 10^{-3} \).

Since \( H(x) = 0 \) therefore \( \varepsilon_H H(x) = \chi_H H(x) = \gamma_H H(x) = 0 \); moreover, since \( c(x) = 0 \) also \( \varepsilon_M = \varepsilon_V = 0 \). Fig. 14 depicts the plots of the generalized deformations \( \chi(x) \) and \( \gamma(x) \). The curvature

Fig. 14. Curvature (a), and shear deformation (b) axial distributions, evaluated for a tapered cantilever with a shear load \( P = 1 \text{ N} \) applied in the final cross-section.

Fig. 15. Rotation (a) and vertical displacements (b) axial distributions, evaluated for a tapered cantilever with a shear load \( P = 1 \text{ N} \) applied in the final cross-section.
induced by vertical forces $\chi_V(x)$ has a negligible magnitude with respect to the curvature induced by resulting bending moment $\chi_M(x)$. On the other hand both shear deformations $\gamma_M(x)$ and $\gamma_V(x)$ have the same order of magnitude and therefore both play a crucial role in determining the tapered beam’s behavior.

Fig. 15 reports the beam displacements $\phi(x)$ and $v(x)$. Table 1 reports the maximum vertical displacement evaluated with different models (i.e., the model proposed in this paper –indicated as Analytical model–, the model developed by Auricchio et al. (2015) –indicated as ABL–, and the FE analysis software ABAQUS –indicated as $v_{ref}$–). We note that all the models are accurate.
Table 1
Mean value of the vertical displacement evaluated on the final cross-section and obtained considering different models for a symmetric tapered cantilever with a vertical load P = 1 N applied in the final cross-section.

<table>
<thead>
<tr>
<th>Beam model</th>
<th>u(x) [mm]</th>
<th>[u(x)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical model</td>
<td>−0.0657826</td>
<td>1.064×10⁻³</td>
</tr>
<tr>
<td>ABL</td>
<td>−0.0657294</td>
<td>2.541×10⁻⁴</td>
</tr>
<tr>
<td>2D solution (u(x))</td>
<td>−0.0657127</td>
<td>0</td>
</tr>
</tbody>
</table>

in predicting vertical displacement. Because it is more refined, ABL provides the most accurate prediction of vertical displacement whereas, being less refined, the analytical model proposed in this paper is the less accurate. However, the analytical model shows a good accuracy, acceptable in most engineering applications.

5.2. Arch shaped beam

We now consider the arch shaped beam depicted in Fig. 16. The beam center-line and cross-section height are defined as

\[ c(x) := -\frac{1}{100}x^2 + \frac{1}{10}x; \quad h(x) := \frac{1}{50}x^2 - \frac{1}{5}x + \frac{3}{5} \quad (40) \]

Moreover, the beam is clamped in the initial cross-section \( A(0) \) and loaded on the final cross-section \( A(l) \) with a constant horizontal load distribution \( f_A(fl) = [1, 0, 0] \) N/ mm.

Solving Eqs. (18), we obtain the following expressions for the generalized stresses

\[ H(x) = \frac{3}{5}; \quad M(x) = \frac{3}{5} \left( -\frac{1}{100}x^2 + \frac{1}{10}x \right); \quad V(x) = 0 \quad (41) \]

Fig. 17 depicts the cross-section distributions of the stresses \( \sigma_h \) and \( r \). The label \( \text{mod} \) indicates the stress distribution obtained using Eqs. (19a) and (21), whereas the label \( \text{ref} \) indicates the 2D ABAQUS solution, computed considering the full 2D problem and using a structured mesh of \( 10240 \times 256 \) bilinear elements. In order to exclude boundary effects, we consider the cross-section \( A(0.75l) \). We note the good agreement between the Analytical model and the 2D FE solution.

Since \( V(x) = 0 \) therefore we have \( \varepsilon_u V(x) = \gamma u V(x) = \gamma r V(x) = 0 \). Fig. 18(a), (c), and (d) depict the generalized deformation \( \varepsilon_0(x), \chi(x), \) and \( \gamma(x) \), respectively. We note that the horizontal deformation induced by the bending moment \( \varepsilon_0 M(x) \) has negligible magnitude compared with the horizontal deformation induced by the resulting horizontal stress \( \varepsilon_u H(x) \). Analogously, the curvature induced by resulting horizontal stress \( \gamma u H(x) \) has a negligible magnitude with respect to the curvature induced by bending moment \( \gamma r M(x) \). On the other hand, both the shear deformations induced by resulting horizontal stress \( \gamma u H(x) \) and bending moment \( \gamma r M(x) \) have non-vanishing magnitudes.

Fig. 18(b) shows the horizontal elongation induced by center-line rotation \( c'(x) \). This quantity is two order of magnitude bigger than the horizontal deformation \( \varepsilon_0(x) \), and plays a central role in determining the horizontal displacements \( u \).

Fig. 19 reports the beam displacements \( u(x), \phi(x), \) and \( v(x) \).
Table 2 reports the maximum displacements $v(l)$ and $u(l)$ evaluated with different models. The reference solutions $v_{\text{ref}}$ and $u_{\text{ref}}$ are calculated using the ABAQUS software. The numerical results confirm that the analytical model is effective in predicting displacements despite it results less accurate than the model ABL.

### 6. Conclusions

The modeling of a generic non-prismatic planar beam proposed in this paper was done through 4 main steps:

1. derivation of compatibility equations
2. derivation of equilibrium equations
3. stress representation
4. derivation of simplified constitutive relations

In particular, compatibility and equilibrium equations are derived considering a global Cartesian coordinate system allowing the beam model to be expressed through simple ODEs. The stress representation takes accurately into account the boundary equilibrium of the body, and is crucial in determining the model effectiveness. The simplified constitutive relations need careful derivation and result in non-trivial equations.

The main conclusions highlighted by the derivation procedure and the discussion of practical examples can be resumed as follows:

- The shear distribution depends not only on vertical resulting stress but also on horizontal resulting stress and bending moment.
- The complex geometry leads each generalized deformation to depend on all generalized stresses, in contrast with prismatic beams.
- The proposed model allows the evaluation of homogeneous and particular solutions in simple cases of practical interest.
- Example discussed in Section 5.2 highlights that non-prismatic beams could behave very differently than prismatic ones, even if they are slender and with very smooth cross-section variations.
- Numerical examples demonstrate that the proposed model gives effective and accurate results for complex geometries, so that the model is a promising tool for practitioners and researchers.

Further developments of the present work will include the application of the proposed model to more realistic cases in particular to the simplified modeling of wood structures, as well as the generalization of the proposed modeling procedure to 3D beams.

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