A three-dimensional phenomenological constitutive model for porous shape memory alloys including plasticity effects

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Abstract

Porous shape memory alloys are a class of very interesting materials exhibiting features typical of porous metals and of shape memory alloys. In contrast to dense shape memory alloys, considerable plastic strain accumulates in porous shape memory alloys even during phase transformation. Moreover, due to the microstructure of porous materials, phase transformation and plasticity phenomena are significantly pressure-dependent.

In this article, we propose a three-dimensional phenomenological constitutive model for the thermomechanical behavior of porous shape memory alloys able to predict shape memory effect, pseudo-elastic behavior and plastic behavior under proportional as well as non-proportional multiaxial loadings. To this end, proper internal variables together with free energy and limit functions are introduced. The material parameters are determined for several available porous shape memory alloys with a wide range of porosity (from 16% to 51%), and a good agreement between numerical predictions of the proposed model and experimental results is observed. Moreover, plastic and transformation strain evolution during increasing and decreasing cyclic loadings are demonstrated through several examples. Finally, to show model capabilities, shape memory effect and pressure-dependent behavior as well as model response under non-proportional loadings are also presented.

Keywords

Porous shape memory alloys, plastic strain, phase transformation, multiaxial loading, pressure dependency

Introduction

Research on the behavior of smart materials has been rapidly increasing due to their innovative applications. Among different types of smart materials, shape memory alloys (SMAs) have two unique features due to recoverable martensitic phase transformation, known as pseudo-elasticity and shape memory effect, observed in both dense and porous SMAs (see e.g. Nemat-Nasser et al., 2005; Shishkovsky, 2012a; Zhao et al., 2005). These features, together with good corrosion resistance and biocompatibility, are utilized in several applications, such as actuators, stents, implants, and devices for orthodontic and endodontic applications in the medical industry (Lagoudas, 2008; Yamauchi et al., 2011; Yoneyama and Miyazaki, 2009).

Moreover, porous SMAs benefit from porous metal characteristics and can be used in applications such as lightweight structures, biomedical implants, filters, heat exchangers, and energy (shock and vibration) absorbers (Bansiddhi et al., 2008; Bram et al., 2011; Gibson and Ashby, 1997; Lefebvre et al., 2008). As an example, in biomedical applications a relatively high porosity level (up to 70%) is required to reduce stress shielding and to increase tissue ingrowth, while in structural applications a low porosity level (below 40%) is normally required (DeGiorgi and Qidwai, 2002; Zhao et al., 2006). Additionally, recent research (Shishkovsky, 1 Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran
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2012a,b) has exploited the shape memory effect of porous SMAs in drug delivery systems.

In the above applications, the porous SMA element is part of a load-bearing structure, thus experiencing thermal and mechanical loadings. In this regard, the understanding of porous SMA thermomechanical behavior and development of a reliable constitutive model is essential to performing an efficient design.

In addition to recoverable transformation strains, irrecoverable permanent strains may also evolve in both dense and porous SMAs. The evolution of such strains depends on several aspects such as porosity ratio, material microstructure, and stress loading level (Bram et al., 2011; Hosseini et al., 2014; Kockar et al., 2013). Therefore, permanent effects are more important in porous SMAs. For clarity, here and in the following we distinguish two kinds of permanent strain, indicated as plastic strain and transformation-induced plastic strain, addressed in the following:

- **Plastic strain**: with this term we refer to a permanent strain effect occurring after a well-defined plastic limit. According to material plastic hardening, when a cyclic loading (with constant stress amplitude) is applied, plastic strain evolves in the first cycle and remains constant during the next cycles.

- **Transformation-induced plastic strain**: with this term we refer to a permanent strain induced by phase transformation. In particular, a portion of the martensite phase generated during forward transformation may remain in the material even after reverse transformation. This strain may accumulate during cyclic loading (even with constant stress amplitude) and usually saturates after a certain number of cycles.

Another consequence of porous microstructure is pressure dependency which is usually observed in porous materials but not in dense SMAs (Altenbach and Ochsner, 2010; Gibson and Ashby, 1997). Therefore, the modeling of transformation and plastic strains should contain both volumetric and deviatoric contributions.

Although several models are available in the literature to describe the mechanical response of porous SMAs utilizing a phenomenological approach or an averaging approach in conjunction with a dense SMA model, such models do not seem to be adequate in describing the response due to pressure dependency and plastic behavior observed in porous SMAs. Accordingly, “Experimental works on permanent strain effects in porous SMAs” reviews experimental works on irrecoverable effects of dense and porous SMAs; then, “Dense SMA models with permanent strain effects” addresses some of the dense SMA models available in the literature, considering permanent strain effects. We also discuss the applicability of such models to porous SMAs. Moreover, “Porous SMA models with permanent strain effects” presents research works on modeling permanent strain effects in porous SMAs. Finally, the last part of this section expresses the goals of the present work.

**Experimental works on permanent strain effects in porous SMAs**

In the literature, several experimental works have explored the mechanical behavior of porous SMAs and shown the appearance of irrecoverable permanent strain mainly in the pseudo-elastic range, and a limited number of studies in the shape memory effect range. In the following, we discuss in detail some of the most significant available experimental works.

To clarify the permanent strain evolution in porous SMAs compared to dense SMAs, Figure 1 reports the experimental results of Greiner et al. (2005), Lagoudas and Vandygriff (2002), Bram et al. (2011), and Köhl et al. (2011), performed under increasing cyclic loading. In particular, Figure 1(a), (b), (c), and (d) reports three successive cycles of stress–strain curves for dense NiTi, 16%, 42% and 51% porous NiTi samples, respectively.

As can be observed, in the dense SMA sample a small permanent strain is accumulated during the cyclic loading. However, in porous SMA samples, a considerable permanent strain evolves during the cyclic loading which becomes more evident with increasing porosity. In fact, in the 16% porous sample, 3.5% permanent strain accumulates after cyclic loading up to 9% strain (Figure 1(b)), while for the 51% porous sample it reaches 10% when 16% strain is applied (Figure 1(d)). Moreover, according to Figure 1(a), the permanent strain in the dense SMA sample seems to saturate after several cycles, while according to Figure 1(b) to (d) there is no evidence of saturation in porous SMA samples.

To get a better insight, Figure 2 compares recoverable strain (sum of elastic and transformation strains) with maximum strain loading for dense and porous NiTi samples under increasing cyclic loading. It is interesting to observe that for the dense sample the recoverable strain increases almost linearly with maximum strain in each cycle. On the other hand, as a common feature, all the porous samples show asymptotic behavior, observable when the maximum strain increases with the cyclic loading amplitude. An explanation for such a different observation in porous SMAs under increasing cyclic loading is discussed in the following.

In dense SMAs, the idealized behavior is elastic deformation, followed by forward phase transformation, elastic deformation of martensite phase, and finally plastic yielding. During unloading, reverse transformation occurs if the material is in the pseudo-elastic
range. In other words, transformation and plastic strains evolve separately, while a transformation-induced plastic strain may accumulate under cyclic loading.

In porous SMAs, from a macroscopic view point, the regions of elastic, transformation and plastic behavior are to some extent combined, due to stress concentrations around the pores of the microstructure (Liu et al., 2012; Shariat et al., 2013; Zhu et al., 2013). Therefore, transformation and plastic strains may evolve simultaneously. In other words, both kinds of permanent strain (plastic and transformation-induced plastic) may evolve during phase transformation when an increasing cyclic load is applied. However, such a transformation-induced plastic strain is small compared to the plastic strain evolving during or after completion of phase transformation.

Figure 1. Three successive cycles of experimental stress–strain curves for (a) dense NiTi (Greiner et al., 2005), (b) 16% porous NiTi (Greiner et al., 2005), (c) 42% porous NiTi (Lagoudas and Vandygriff, 2002), (d) 51% porous NiTi (Bram et al., 2011).

Figure 2. Recoverable strain vs maximum strain diagram for cyclic experiments on different dense and porous NiTi samples.
With this in mind, we can interpret the experimental results reported in Figures 1 and 2. In the dense sample, the loading is below the plastic limit; thus, only a small transformation-induced plastic strain evolves during the cyclic loading which tends to saturate after several cycles. In the porous samples, the plastic limit is lower than that of the dense sample, and therefore a considerable plastic strain as well as a small transformation-induced plastic strain evolve during increasing cyclic loading.

In other words, the mechanism for plastic deformation of porous SMAs is similar to what is observed in conventional metal foams (Bram et al., 2011). Since the pore microstructure is not regular, the thickness of struts near the pores of SMA samples is different. Even at very low stress levels, pseudo-elastic deformation starts at regions with smaller wall thickness. By increasing the load, thin-walled regions start to deform plastically, while martensitic phase transformation occurs simultaneously in other regions with thick-walled regions. This is a level-by-level deformation mechanism taking place in porous materials. In dense SMAs, incomplete reverse phase transformation is usually observed after a considerable plastic strain. However, in porous SMAs, plastic yielding which occurs during phase transformation is local and does not considerably influence the reverse phase transformation. Therefore, as reported by Bram et al. (2011), Köhl et al. (2011) and Scalzo et al. (2009), even after a compression loading with a considerable plastic strain, pseudo-elastic and shape memory strain recovery is observed.

Although experiments which specifically demonstrate the pressure dependency of porous SMAs are not available, it is well-known in the literature that the plastic behavior of porous materials is pressure-dependent (Altenbach and Ochsner, 2010; Deshpande and Fleck, 2000). Therefore, the mechanical behavior of porous SMAs depends on hydrostatic pressure as well. In other words, from the macroscale viewpoint, the transformation and plastic strains in porous SMAs contain both deviatoric and volumetric parts. Recently, Ashrafi et al. (2015) studied the phase transformation of porous SMAs and presented a three-dimensional constitutive model by considering deviatoric and volumetric parts of transformation strain as internal variables. They presented pseudo-elasticity and shape memory effect of porous SMAs under proportional as well as non-proportional loadings.

Finally, as a summary of the above discussion, we list the main features of permanent strain effects in porous SMAs:

- Plastic strain effects become more considerable with increasing porosity.
- The evolution of transformation and plastic strains is pressure-dependent.
- Transformation strain recovers even after considerable accumulation of plastic strain.

**Dense SMA models with permanent strain effects**

Much research has tried to describe permanent strain evolution in dense SMAs. As stated before, two kinds of permanent strain (plastic and transformation-induced plastic) are present in dense SMAs.

For the first kind, there are several models attempting to describe the plastic yield of dense SMAs (see e.g. Hartl and Lagoudas, 2009; Paiva et al., 2005; Yan et al., 2003, among others). Such models assume independent variables to describe transformation and plastic strains and propose corresponding limit functions and evolution equations. The plasticity effect is observed after completion of phase transformation and under high levels of stress. It is noted that during unloading of dense SMAs which have experienced considerable plastic yielding, incomplete reverse phase transformation is usually observed. While most of the models do not consider simultaneous evolution of transformation and plastic strains, Hartl and Lagoudas (2009) modeled the simultaneous transformation and plastic yield for high-temperature dense SMAs.

As for the second kind, transformation-induced plastic strain has also been studied under cyclic loading by many authors. Specifically, several phenomenological constitutive models accounting for such permanent effects have been proposed in the literature (see e.g. Auricchio et al., 2007; Bo and Lagoudas, 1999; Lagoudas and Entchev, 2004; Malécot et al., 2006; Saint-Sulpice et al., 2009, 2012). Most of these models assume transformation-induced plastic strain is related to a portion of detwinned martensite remains in the material after unloading to zero stress. Moreover, according to experimental evidence, this strain saturates after several cycles. Another assumption of such models is the evolution law during forward and reverse transformation. Lagoudas and Entchev (2004) assumed an evolution equation in which such a permanent strain increases during forward transformation and decreases during reverse transformation; however, it increases on average in each cycle. Malécot et al. (2006) assumed that this kind of permanent strain only accumulates during forward phase transformation, while in the model developed by Auricchio et al. (2007) transformation-induced plastic strain accumulates with a same law during forward and reverse transformation.

According to the reviewed literature, it is concluded that the proposed dense SMA models are not quite appropriate for modeling the plasticity of porous SMAs. The models of the first kind assume separate
transformation and plastic strain while those of the second kind assume that permanent strain accumulates only during phase transformation and saturates after several cycles of loading. Although the model proposed by Hartl and Lagoudas (2009) contains the feature of simultaneous transformation and plastic strain evolution, as a dense model, it does not consider pressure-dependent behavior. Therefore, the main features of permanent strain evolution in porous SMAs mentioned in “Experimental works on permanent strain effects in porous SMAs” are not properly captured utilizing the cited dense models.

**Porous SMA models with permanent strain effects**

There are few works attempting to describe permanent strain effects in porous SMAs by employing a homogenization approach or by developing appropriate phenomenological models.

Entchev and Lagoudas (2004) utilized a dense SMA model including transformation-induced plastic strain and studied the pseudo-elastic behavior of porous SMAs using a micromechanical averaging approach. Panico and Brinson (2008) developed a dense SMA model and used it in a porous finite element model. As discussed in “Dense SMA models with permanent strain effects”, the employed dense models are not pressure-dependent and only capture the effects of transformation-induced plastic strain. However, as discussed previously, the inelastic strain evolution in porous SMAs is pressure-dependent and mainly due to locally plastic yielding near pores.

Using a phenomenological approach, Sayed et al. (2012) proposed a constitutive model which combines a dense SMA model with a porous plasticity model. The model incorporates the phase transformation and plastic behavior simultaneously, but does not consider pressure-dependent phase transformation and shape memory effect. Based on the Gurson–Tvergaard–Needleman formulation, Olsen and Zhang (2012) proposed a constitutive model for SMAs with micro-voids. The model is able to reproduce pressure-dependent plasticity and phase transformation, but not simultaneously.

**Goals of the present work**

From the experimental evidence and the described mechanisms for porous SMAs, two strains (plastic and transformation-induced plastic) with different mechanisms may evolve during phase transformation. However, experiments show that plastic yielding near the pores of the material is the dominant permanent behavior observed during or after phase transformation. Therefore, in this work, we focus on the constitutive modeling of porous SMAs, capturing plastic strain effects.

As discussed previously, during loading of porous SMAs, transformation and plastic strains may evolve simultaneously; upon unloading, in the pseudo-elastic range, only transformation strain is recovered, while in the shape memory effect range, the transformation strain is recovered after heating. During an increasing cyclic loading, transformation strain reaches its maximum; however, plastic strain increases with loading. Moreover, transformation and plastic strain evolution in porous SMAs are pressure-dependent in contrast to dense SMAs.

As discussed in detail in “Dense SMA models with permanent strain effects” and “Porous SMA models with permanent strain effects”, the existing dense and porous SMA models do not properly capture the features of porous SMA behavior discussed in “Experimental works on permanent strain effects in porous SMAs”. In the present work, considering the mechanism for plastic strain evolution, we propose a three-dimensional phenomenological constitutive model for the thermomechanical behavior of porous SMAs. The model can predict shape memory effect, pseudo-elastic behavior, and plastic strain evolution under proportional as well as non-proportional multi-axial loadings. Since the main goal of the present study is to model plasticity effects observed in porous SMAs during phase transformation, employing the proposed model for dense SMAs experiencing considerable plastic strain needs some modifications. Moreover, compared to micromechanical models, the proposed model significantly reduces the computational cost.

The article is organized as follows. “Three-dimensional phenomenological model for porous SMAs” presents the proposed constitutive model. Then, the following section discusses time discretization and a solution algorithm. “Numerical results” presents the material parameter identification and model validation. Moreover, numerical predictions of the proposed model for different thermomechanical loadings are the subject of this section. Finally, a summary and conclusions are given in the last section.

**Three-dimensional phenomenological model for porous SMAs**

We now propose a general phenomenological constitutive model for porous SMAs, able to describe plasticity phenomena as well as pseudo-elasticity and shape memory effect under general thermomechanical loadings.

Assuming small strains, we consider the additive strain decomposition

\[ \varepsilon = \varepsilon^e + \varepsilon^T + \varepsilon^p \]

(1)

where \( \varepsilon \), \( \varepsilon^e \), \( \varepsilon^T \) and \( \varepsilon^p \) are total, elastic, transformation and plastic strains, respectively. Recalling that the
inelastic behavior of porous SMAs is pressure-dependent, we also introduce the following decompositions

\[ e = e + \frac{\theta}{3} \mathbf{1} \]  \hspace{1cm} (2)

\[ e_v^\sigma = e_v + \frac{\theta^\sigma}{3} \mathbf{1} \]  \hspace{1cm} (3)

\[ e^\sigma = e + \frac{\theta^\sigma}{3} \mathbf{1} \]  \hspace{1cm} (4)

where \( e, e^\sigma, e_v^\sigma \) are deviatoric components, \( \theta, \theta^\sigma \) and \( \theta^v \) are volumetric components, and \( \mathbf{1} \) is the second-order identity tensor.

In the model, we take the deviatoric and volumetric strains \( e, \theta \) and the absolute temperature \( T \) as control variables, and the deviatoric and volumetric parts of the transformation and plastic strains \( e^\sigma, \theta^\sigma, e_v^\sigma \) and \( \theta^v, \theta^\sigma \) as internal variables. The Helmholtz free energy function \( \Psi \) is considered as a function of such control and internal variables, that is, \( \Psi = \Psi(e, \theta, T, e^\sigma, \theta^\sigma, e_v^\sigma, \theta^v) \).

To derive the constitutive equations and thermodynamic forces, the second law of thermodynamics should be satisfied. Considering the Helmholtz free energy function \( \Psi \), the mechanical dissipation inequality \( D^m \) is expressed using the Clausius–Duhem inequality as follows

\[ D^m = \mathbf{\sigma} : \dot{\mathbf{e}} - \left( \Psi + \eta \dot{T} \right) \geq 0 \]  \hspace{1cm} (5)

where \( \mathbf{\sigma} \) is the stress tensor, \( \eta \) is the entropy and a dot superscript indicates the derivative with respect to time. Substituting strain decomposition equations (1) to (4) into equation (5) then yields

\[ D^m = \left( s - \frac{\partial \Psi}{\partial e} \right) : \dot{\mathbf{e}} + \left( p - \frac{\partial \Psi}{\partial \theta} \right) \dot{\theta} + \left( - \frac{\partial \Psi}{\partial T} - \eta \right) \dot{T} \]

\[ - \frac{\partial \Psi}{\partial e_v^\sigma} : \dot{e_v^\sigma} - \frac{\partial \Psi}{\partial \theta^\sigma} : \dot{e_v^\sigma} - \frac{\partial \Psi}{\partial \theta^v} \theta^\sigma - \frac{\partial \Psi}{\partial \theta^\sigma} \theta^\sigma \geq 0 \]  \hspace{1cm} (6)

where we also decomposed the stress \( \mathbf{\sigma} \) into its deviatoric and volumetric parts \( s \) and \( p \) as

\[ \mathbf{\sigma} = s + p \mathbf{1} \]  \hspace{1cm} (7)

To satisfy inequality (6), following standard arguments we can derive the constitutive equations and thermodynamic forces as below

\[ s = \frac{\partial \Psi}{\partial e}, \quad p = \frac{\partial \Psi}{\partial \theta}, \quad \eta = - \frac{\partial \Psi}{\partial T}, \]

\[ X_d^\sigma = - \frac{\partial \Psi}{\partial e_v^\sigma}, \quad X_v^\sigma = - \frac{\partial \Psi}{\partial \theta^\sigma}, \]

\[ X_p^\sigma = - \frac{\partial \Psi}{\partial \theta^v}, \quad X_v^\sigma = - \frac{\partial \Psi}{\partial \theta^\sigma} \]  \hspace{1cm} (8)

Here \( X_d^\sigma \) and \( X_v^\sigma \) are the thermodynamic forces associated with the deviatoric part of the transformation and plastic strains, respectively; \( X_d^\sigma \) and \( X_v^\sigma \) are the thermodynamic forces associated with the volumetric part of the transformation and plastic strains, respectively.

Based on relations (8), the mechanical dissipation inequality (6) reduces to

\[ D^m = X_d^\sigma : \dot{e}^\sigma + X_v^\sigma : \dot{e}^\sigma + X_p^\sigma \dot{\theta^\sigma} + X_v^\sigma \dot{\theta^\sigma} \geq 0 \]  \hspace{1cm} (9)

Recalling the transformation and plasticity behavior described in the introduction and the pressure-dependent behavior of porous SMAs, we introduce two distinct convex limit functions for phase transformation and plastic behavior

\[ F^\sigma = X_d^\sigma - R^\sigma \]

\[ F^p = X_v^\sigma - Y^p \]  \hspace{1cm} (10)

where \( R^\sigma \) and \( Y^p \) are the radii of the transformation and plastic domain, respectively. Moreover, \( X_d^\sigma \) and \( X_v^\sigma \) are effective transformation and plastic thermodynamic forces, defined as follows

\[ X_d^\sigma = \sqrt{\frac{\|X_d^\sigma\|^2 + \kappa (X_d^\sigma)^2}{1 + \kappa/6}} \]  \hspace{1cm} (11)

\[ X_v^\sigma = \sqrt{\frac{\|X_v^\sigma\|^2 + \kappa (X_v^\sigma)^2}{1 + \kappa/6}} \]  \hspace{1cm} (12)

Here, \( \kappa \) is a porosity parameter representing the pressure-dependent behavior of the porous material, while the norm operator \( \| \cdot \| \) is defined as \( \| A \| = \sqrt{A : A^T} \). Moreover, we introduce effective transformation and plastic strain rates \( \dot{e}^\sigma \) and \( \dot{\theta^\sigma} \) which are work conjugates of the effective transformation and plastic thermodynamic forces, respectively

\[ X_d^\sigma \dot{e}^\sigma = X_d^\sigma : \dot{e}^\sigma + X_v^\sigma \dot{\theta^\sigma} \]

\[ X_v^\sigma \dot{\theta^\sigma} = X_v^\sigma : \dot{e}^\sigma + X_v^\sigma \dot{\theta^\sigma} \]  \hspace{1cm} (13)

Moreover, we assume an isotropic linear hardening behavior for the plastic domain radius \( Y^p \) expressed as

\[ Y^p(\dot{\theta^\sigma}) = R^p + H_p \dot{\theta^\sigma} \]  \hspace{1cm} (15)

where \( R_p \) is the initial radius for the plastic domain, \( H_p \) is the isotropic hardening coefficient, and the equivalent plastic strain is defined as

\[ \dot{\theta^\sigma} = \int \dot{\theta^\sigma} \, dt \]  \hspace{1cm} (16)

We remark that the limit functions described by equations (10), (11) and (12) are similar to the yield function already proposed by Deshpande and Fleck (2000) to describe the plastic behavior of metal foams. It is noted that the limit function is elliptic in the deviatoric-volumetric space, and in the limit \( \kappa = 0 \), it
The Kuhn–Tucker conditions for transformation and plastic domains are independent following equivalent transformation and plastic strain rates

\[ (\ref{21}) \] into equation (16), the equivalent plastic strain is can be derived as

\[ (\ref{14}) \], the effective transformation and plastic strain rates (13), as well as equations (12) and (18) into equation (21). We observe the material parameter \( C_{13} \) corresponds to the maximum effective transformation strain reached at the end of the transformation during a uniaxial test. To satisfy such a constraint, the saturation function \( \varphi_{ul} \) is introduced as

\[ \varphi_{ul}(\|\varepsilon_p^e\|) = \begin{cases} 0 & \text{if } \|\varepsilon_p^e\| \leq \varepsilon_{ul} \\ +\infty & \text{otherwise} \end{cases} \] (26)

Therefore, the constitutive equations and thermodynamic forces introduced in equations (8) may be expressed as

\[
\begin{align*}
\dot{s} = & -2G(e - e^p - e^r) \\
p = & -K[\theta - \theta^p - \theta^r - 3\alpha(T - T_0)] \\
\eta = & -3\alpha\beta(T - T_m)(\varepsilon^p) \\
X_r^p = & -(\frac{1}{\kappa})(\theta^p)(\|\varepsilon^p\|^2 + h^p\theta^p) \\
X_s^p = & -(\frac{1}{\kappa})(\theta^p)(\|\varepsilon^p\|^2 + h^p\theta^p)
\end{align*}
\]

where the variable \( \gamma \) results from the saturation function subdifferential and is defined as

\[
\begin{align*}
\gamma = & \begin{cases} 0 & \text{if } \|\varepsilon_p^e\| < \varepsilon_{ul} \\ \gamma & \text{if } \|\varepsilon_p^e\| = \varepsilon_{ul} \end{cases} \\
\gamma = & \begin{cases} 0 & \text{if } \|\varepsilon_p^e\| < \varepsilon_{ul} \\ \gamma & \text{if } \|\varepsilon_p^e\| = \varepsilon_{ul} \end{cases} \\
\end{align*}
\]

The proposed porous SMA model in the time-continuous frame is summarized in Table 1. We observe that, setting \( R_p = \infty \) and \( \kappa = 0 \), the model...
Table 1. Proposed constitutive model for porous SMAs in the time-continuous frame.

<table>
<thead>
<tr>
<th align="left">External variables:</th>
<th align="left">$e, \theta, T$</th>
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<tbody>
<tr>
<td align="left">Internal variables:</td>
<td align="left">$e^p, \theta^p, e^\rho, \theta^\rho$</td>
</tr>
<tr>
<td align="left">Material parameters:</td>
<td align="left">$K, G, h^\rho, h^p, \beta, T_m, \alpha, \varepsilon, R^\rho, R^p, \kappa$</td>
</tr>
</tbody>
</table>

Constitutive equations and thermodynamic forces:

$$
\begin{align*}
\mathbf{s} &= 2G(e - e^p - e^\rho) \\
\mathbf{p} &= \mathbf{K}[\theta - \theta^p - \theta^\rho - 3\alpha(T - T_0)] \\
\mathbf{X}_d^\rho &= -\frac{\partial \mathbf{s}}{\partial e^\rho} = (1 + \kappa/6) \left( \mathbf{b}(T - T_m) + \gamma \frac{\mathbf{e}^\rho}{\|\mathbf{e}^\rho\|} + h^\rho \mathbf{e}^\rho \right) \\
\mathbf{X}_d^p &= -\frac{\partial \mathbf{p}}{\partial \theta^p} = \mathbf{p} - \left( 1 + \kappa/6 \right) \left( \frac{\mathbf{e}^\rho}{\|\mathbf{e}^\rho\|} (\mathbf{b}(T - T_m) + \gamma) + h^\rho \mathbf{e}^\rho \right) \\
\mathbf{X}_d^\theta &= -\frac{\partial \mathbf{p}}{\partial \theta^\rho} = \mathbf{s} - (1 + \kappa/6) \mathbf{b}^p \mathbf{e}^\rho \\
\mathbf{X}_d^\theta &= \mathbf{p} - \left( 1 + \kappa/6 \right) \mathbf{e}^\rho \mathbf{e}^\rho \\
\end{align*}
$$

Limit function:

$$
\begin{align*}
\mathbf{F}^\rho &= \mathbf{X}_d^\rho - R^\rho \\
\mathbf{F}^p &= \mathbf{X}_d^p - R^p - H^p \mathbf{e}^\rho \\
\end{align*}
$$

Evolution equations:

$$
\begin{align*}
\mathbf{e}^\rho &= \mathbf{e}^\rho_{last} + \frac{1 + \kappa/6}{\kappa} \mathbf{X}_d^\rho \\
\theta^\rho &= \theta^\rho_{last} + \frac{1 + \kappa/6}{\kappa} \mathbf{X}_d^\rho \\
\mathbf{e}^\rho &= \mathbf{e}^\rho_{last} + \frac{1 + \kappa/6}{\kappa} \mathbf{X}_d^\rho \\
\theta^\rho &= \theta^\rho_{last} + \frac{1 + \kappa/6}{\kappa} \mathbf{X}_d^\rho \\
\end{align*}
$$

Kuhn–Tucker conditions:

$$
\begin{align*}
\mathbf{c}^\rho &= 0, \quad \mathbf{F}^\rho \leq 0, \quad \mathbf{F}^\rho \mathbf{c}^\rho = 0 \\
\mathbf{c}^p &= 0, \quad \mathbf{F}^p \leq 0, \quad \mathbf{F}^p \mathbf{c}^p = 0 \\
\end{align*}
$$

An implicit backward Euler method. It should be noted that for notation simplicity here, and in the following, we drop the subindex $n + 1$ for all of the variables computed at time $t_{n + 1}$. The discretized version of the constitutive equations takes the following form

$$
\begin{align*}
\mathbf{p} &= \mathbf{K}[\theta - \theta^p - \theta^\rho - 3\alpha(T - T_0)] \\
\mathbf{s} &= 2G(e - e^p - e^\rho) \\
\mathbf{X}_d^\rho &= \mathbf{s} - (1 + \kappa/6) \left( \mathbf{b}(T - T_m) + \gamma \frac{\mathbf{e}^\rho}{\|\mathbf{e}^\rho\|} + h^\rho \mathbf{e}^\rho \right) \\
\mathbf{X}_d^p &= \mathbf{p} - \left( 1 + \kappa/6 \right) \left( \frac{\mathbf{e}^\rho}{\|\mathbf{e}^\rho\|} (\mathbf{b}(T - T_m) + \gamma) + h^\rho \mathbf{e}^\rho \right) \\
\mathbf{X}_d^\theta &= \mathbf{s} - (1 + \kappa/6) h^p \mathbf{e}^\rho \\
\end{align*}
$$

where $\Delta \mathbf{c}^\rho$ and $\Delta \mathbf{c}^p$ are the consistency parameters, time integrated over the interval $[t_n, t_{n+1}]$.

To solve the time-discrete constitutive equations, we use an elastic predictor–inertial corrector procedure implemented in the program MATLAB, utilizing the function `fsolve`. The algorithm assumes the step to be elastic and evaluates an elastic trial state, in which all the internal variables remain constant. The admissibility of the trial function with respect to limit functions, introduced in equations (24)11 and (24)12, is then verified. If the trial state is admissible, the step is elastic; otherwise, the step is inelastic. The algorithm initially assumes the inelastic step to be in a non-saturated condition ($\gamma = 0$), for which three different situations may occur:

- If only equation (29)11 is not satisfied, the transformation strain evolves according to equations (29)7 and (29)8 and the plastic strain remains constant.
- If only equation (29)12 is not satisfied, the plastic strain evolves according to equations (29)9 and (29)10 and the transformation strain remains constant.

Since the goal of the present article is to address a novel constitutive model able to reproduce the thermomechanical response of porous SMAs, in the following we sketch a time-discrete version of the model and a simple possible solution algorithm, which is not intended to be general or totally robust. With regard to this, we consider a stress-driven problem which simplifies the solution algorithm since the evolution of transformation and plastic strain is stress-driven mechanisms. Such an algorithm enables us to present some uniaxial and multiaxial tests presented in “Numerical results”.

Assuming we are given the state ($s_n, \mathbf{p}_n, \mathbf{e}^\rho_n, \theta^\rho_n, \mathbf{e}^\rho_n, \theta^\rho_n$) at time $t_n$, the actual deviatoric and hydrostatic stress ($s, \mathbf{p}$) and temperature $T$ at time $t_{n+1}$, the time-discrete values can be computed using degenerates to an SMA model without plastic strain effects similar to the one originally proposed by Souza et al. (1998) and improved by Aurichio and Petrini (2004) and Arghavani et al. (2011a,c).
Table 2. Material parameters for the porous SMA model adopted from Greiner et al. (2005), Lagoudas and Vandygriff (2002), and Köhl et al. (2011) for 16%, 42% and 51% porous NiTi samples, respectively.

<table>
<thead>
<tr>
<th>Porosity (%)</th>
<th>$E$ (GPa)</th>
<th>$h^p$ (MPa)</th>
<th>$h^p$ or $h^0$ (MPa)</th>
<th>$R^p$ (MPa)</th>
<th>$R^0$ (MPa)</th>
<th>$\beta$ (MPaK$^{-1}$)</th>
<th>$T_m$ (K)</th>
<th>$\varepsilon_e$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16%</td>
<td>18</td>
<td>7920</td>
<td>8933</td>
<td>123</td>
<td>299</td>
<td>2.99</td>
<td>269</td>
<td>2.0</td>
</tr>
<tr>
<td>42%</td>
<td>15</td>
<td>7495</td>
<td>3961</td>
<td>44.9</td>
<td>114</td>
<td>1.36</td>
<td>303</td>
<td>2.0</td>
</tr>
<tr>
<td>51%</td>
<td>6.5</td>
<td>2683</td>
<td>761</td>
<td>32.8</td>
<td>110</td>
<td>0.80</td>
<td>269</td>
<td>2.8</td>
</tr>
</tbody>
</table>

1The same temperature as in Köhl et al. (2011) was assumed since not reported by Greiner et al. (2005).

- If both equations (29)$_1$ and (29)$_2$ are not satisfied, both the transformation and plastic strains evolve according to equations (29)$_7$, (29)$_8$, (29)$_9$, and (29)$_{10}$.

Finally, we check the saturation condition (29)$_{15}$: if it is not the case we set $\|\varepsilon^p\| = \varepsilon_e$ and the constitutive equations are solved again with the unknown variable $\gamma$.

It is noted that the proposed algorithm is based on the consideration that, for a given value of stress (as in equations of the time-discrete marching scheme), the evolutions of transformation and plastic strain are decoupled. Moreover, according to equations (29)$_3$ and (29)$_4$, the transformation thermodynamic forces $X^p_j$ and $X^p_\ell$ are not defined for the case of $\|\varepsilon^p\| = 0$. To this end, we use a regularization scheme also used by Auricchio and Petrini (2004) and Arghavani et al. (2011b) to overcome this problem.

**Numerical results**

In this section, we first validate the proposed model by testing its ability to reproduce several experiments presented in “Experimental works on permanent strain effects in porous SMAs”. Also, we show the model capability to describe the main features of porous SMA behavior under cyclic loading. Finally, we present the predictions of the proposed model for more complex thermomechanical proportional as well as non-proportional loadings.

**Model validation**

We validate the model with three sets of experimental uniaxial results presented by Greiner et al. (2005), Lagoudas and Vandygriff (2002), and Köhl et al. (2011) and reported in Figures 1 and 2 for 16%, 42% and 51% porous NiTi samples. Moreover, for all three sets of parameters, we assume the thermal expansion coefficient is equal to zero since it has a secondary effect compared to transformation and plastic strains.

We follow the same procedure for material parameter identification and validation of the three sets of experiments. We use only two stress–strain cycles in the procedure described in the following; then, we compare the model predictions for other cycles of stress–strain curves with experiments.

In the procedure, first we use one of the stress–strain cycles and determine the parameters associated with the elastic and transformation regions ($E$, $R^p$, $h^p$, $\beta$). Then, using both stress–strain cycles, we identify the parameters associated with the plastic region. Specifically, the radius of plastic region ($R^0$) and plastic hardening coefficient ($h^p$ or $H^p$) are determined from the values for stress amplitude and maximum plastic strain. Moreover, maximum transformation strain $\varepsilon_e$ is determined from the trend of recoverable strain in Figure 2. It is noted that we should subtract elastic strain from recoverable strain to obtain maximum transformation strain.

Using the proposed procedure for the 16%, 42% and 51% porous NiTi samples, we identify the material parameters reported in Table 2. Note that, since the experiments were only performed under uniaxial compression, the reported stress–strain curves are independent of Poisson’s ratio ($\nu$), porosity parameter ($\kappa$) and type of plastic hardening ($h^p$ or $H^p$). Therefore, such parameters cannot be calibrated and are not reported in Table 2. It is interesting to observe that, for different porous NiTi samples, the parameters related to transformation and plastic limit functions decrease with increasing porosity. This aspect is quite reasonable since the load-bearing capability of porous materials decreases with porosity. Moreover, thanks to the adopted values for $R^p$, $\beta$ and $R^0$, phase transformation evolution starts earlier than plastic strain, in agreement with experimental evidence.

For each porous NiTi material of Table 2, model results are first compared with the two stress–strain curves used in material parameter identification. Then, model predictions for other cycles of loading are compared with experiments. Also, recoverable strain ($\varepsilon^p + \varepsilon^e$) versus maximum strain is compared with experiments for a cyclic loading, while the corresponding stress–strain curve is also provided to show, in more detail, the model results under cyclic loading. In the following, we present and discuss the results for 16%, 42% and 51% porous NiTi samples.

Figure 3 illustrates the model results for 16% porous NiTi which was produced and tested by Greiner et al. (2005). According to Figure 3(a), the model results...
During loading, the slope of the stress–strain curve changes three times due to the start of forward transformation, saturation of transformation strain and start of plastic strain evolution. In fact, since the samples have low porosity, the behavior is mostly similar to dense SMAs. Moreover, Figure 3(b) depicts the model predictions compared with experiments. Two stress–strain curves are presented which predict the phase transformation, saturation and plastic strain evolution during loading and unloading well. Also, in Figure 3(c) recoverable versus maximum strain in cyclic loading shows the same trend as the experimental results. Specifically, the changing rate of recoverable strain decreases when the maximum strain increases during cyclic loading. Stress–strain curves of the cyclic loading are illustrated in Figure 3(d). The results show the plastic hardening behavior in which the plastic strain evolution starts later at higher cycles and therefore transformation and plastic strains evolve separately.

Figure 4 presents the model results for 42% porous NiTi compared with the experiments of Lagoudas and Vandygriff (2002). The experimental data used in material parameter identification are compared with model results in Figure 4(a). Model predictions are also plotted and compared with the corresponding experiment data in Figure 4(b). Good agreement is observed both qualitatively and quantitatively which shows the validity of the model in describing phase transformation and plastic strain evolution of 42% porous NiTi samples. Moreover, recoverable strain–maximum strain and stress–strain curves for cyclic loading are presented in Figure 4(c) and (d) which follow the same trend as
experiments. In comparison with low porous samples, it is deduced that for the 42% porous sample transformation and plastic strains evolve fairly simultaneously. However, the saturation of phase transformation strain is not observed since in the experiments maximum strain loading is limited to 5%.

Figure 5 compares the model results for 51% porous NiTi with the experiment data of Köhl et al. (2011) and Bram et al. (2011). Two sets of stress–strain data used in material parameter identification are compared with model results in Figure 5(a). Moreover, model predictions for two cycles are presented in Figure 5(b). The model can predict plastic strain evolution precisely. However, since phase transformation and elastic deformation are quite mixed in highly porous SMAs, the predictions for phase transformation are not as precise as for the plastic strain region. Moreover, the recoverable strain–maximum strain curve obeys the same trend as the experiment results and the curve has an asymptotic trend (Figure 5(c)). Also, cyclic stress–strain curves up to 20% maximum strain are plotted in Figure 5(d). Since strain loading is far above the maximum transformation strain, saturation is observed. Meanwhile, due to plastic hardening, the simultaneous transformation and plastic strains change to separate regions in higher cycles of loading.

Model predictions for uniaxial loading

This section presents several model predictions for tests including both uniaxial cyclic and thermomechanical loading, in order to show that the model includes the
features discussed in the introduction. With regard to this, we adopt the parameters identified for the porous NiTi sample, reported in Table 2.

First, we present the results performed at $T = 310$ K under two types of cyclic loading, that is, increasing and decreasing. Figure 6 illustrates model results for the 51% porous SMA sample under an increasing cyclic loading. In particular, the maximum stress increases from 100 to 275 MPa during eight cycles of loading. Figure 6(a) shows the obtained stress–strain curve and Figure 6(b) presents the evolution of inelastic strain ($e^{ir} + e^p$) and plastic strain ($e^p$). As can be observed, plastic strain is always increasing and it evolves during loading but not unloading of each cycle. In addition, plastic strain evolution continues even after completion of phase transformation. Similarly, Figure 7 shows the results for the 51% porous SMA sample under a decreasing cyclic loading. In such a case, the maximum stress decreases from 200 to 100 MPa during six cycles of loading. While phase transformation occurs in all of the cycles, Figure 7(a) and (b) show that inelastic strain increases in the first cycle and remains constant in the subsequent cycles with lower maximum stress. Such a phenomenon is in accordance with the experiments observing pseudoelastic behavior even after considerable accumulation of plastic strain in the sample (Köhl et al., 2011; Scalzo et al., 2009).

Moreover, we demonstrate the model capability of predicting the shape memory effect of porous SMAs. Similar to dense SMAs, porous SMAs show shape memory effect at low temperatures (Scalzo et al., 2009;
Figure 6. Model predictions for increasing cyclic loading of the 51% porous NiTi sample at \(T = 310\) K: (a) axial stress–strain curve, (b) inelastic and plastic strain vs time curves.

Figure 7. Model predictions for increasing cyclic loading of the 51% porous NiTi sample at \(T = 310\) K: (a) axial stress–strain curve, (b) inelastic and plastic strain curves.

Shishkovsky, 2012a). Figure 8 depicts the shape memory effect for different porous SMA samples under a thermomechanical loading consisting of a uniaxial mechanical loading at low temperatures, followed by an increasing and decreasing thermal loading. During mechanical loading, both transformation and plastic strains evolve in the samples, while during unloading, partial strain recovery is observed. By increasing temperature above the reference temperature \(T_m\), transformation strain is recovered completely. Decreasing the temperature to the reference temperature has no effect on the stress and strain states. However, considerable plastic strain remains in the sample at the end of the thermomechanical loading.

**Model predictions for multiaxial loadings**

In this section, we demonstrate model predictions for the behavior of porous SMAs under non-proportional multiaxial loading. For this purpose, we need the material parameters reported in Table 2 as well as the values for other parameters: Poisson’s ratio \((\nu)\) and the porosity parameter \((\kappa)\). Due to lack of experimental data, we utilize a computational simulation method described in...
Table 3. A typical set of porous SMA material parameters used for the multiaxial loading test.

<table>
<thead>
<tr>
<th>Material parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>6.5</td>
<td>GPa</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.26</td>
<td>–</td>
</tr>
<tr>
<td>$H^{tr}$</td>
<td>2683</td>
<td>MPa</td>
</tr>
<tr>
<td>$H^p$</td>
<td>761</td>
<td>MPa</td>
</tr>
<tr>
<td>$H^p$</td>
<td>100</td>
<td>MPa</td>
</tr>
<tr>
<td>$R^{tr}$</td>
<td>32.8</td>
<td>MPa</td>
</tr>
<tr>
<td>$R^p$</td>
<td>110</td>
<td>MPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.65</td>
<td>–</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.80</td>
<td>MPaK$^{-1}$</td>
</tr>
<tr>
<td>$T_m$</td>
<td>269</td>
<td>K</td>
</tr>
<tr>
<td>$\epsilon_L$</td>
<td>2.8</td>
<td>%</td>
</tr>
</tbody>
</table>

detail by Panico and Brinson (2008) and Ashrafi et al. (2015). In this, values of $\nu$ and $\kappa$ for a 51% porous SMA sample are determined by simulation under axial and hydrostatic loading. In combination with the parameters for the 51% porous sample in Table 2, we use a set of material parameters reported in Table 3 and present the model results under several loading profiles.

First, we focus on results for two non-proportional loading tests performed at $T = 310$ K. Figures 9 and 10 show the predicted curves for an axial-shear triangle-shaped and an axial-shear L-shaped loading profile, respectively. Although there is no available experimental data for porous SMAs under such a non-proportional loading, Sittner et al. (1995) performed such experiments on dense SMAs. Compared to those experiments, model results seem reasonable. However, due to the plastic strain effects considered in the present model, the strain response is not completely reversible. Although experiments should be performed to validate the predictions of the model under general multiaxial loading, the model works effectively under such loadings and exhibits reasonable results.

Finally, we show the pressure-dependent behavior of the proposed porous SMA model. The results are now discussed for a hydrostatic–shear loading at $T = 310$ K. The square-shaped loading profile ABCDEFGH is presented in Figure 11(a) for different amplitudes. Such loadings with different amplitudes are considered for better comparison. Shear strain–volumetric strain outputs are illustrated for the three loadings in Figure 11(b) showing the model’s capability to predict material behavior under non-proportional multiaxial loadings. The results show a coupling between shear and volumetric responses, which increases with loading amplitude. Specifically, shear
loading induces volumetric strain in paths BC, DE and FG, while hydrostatic loading induces shear strain in paths CD and EF. Moreover, both volumetric and deviatoric inelastic strains are induced in the porous samples. Plastic strain evolution (volumetric and deviatoric) can be clearly observed at the end of each loading profile which is much more significant for higher amplitude loadings.

**Summary**

In this study, we proposed a three-dimensional constitutive model for porous SMAs which is capable of predicting pressure-dependent behavior and taking into account plasticity effects. As a phenomenological model, it significantly reduces the computational cost comparing to micromechanical models. The model predicts the simultaneous evolution of transformation and plastic strain observed in several experiments on porous NiTi samples. It has been observed that plastic strain becomes dominant as porosity increases. Good agreement between numerical predictions and experimental results for a wide range of porosity is observed under uniaxial loading. For example, under 10% strain loading, the model predicts 6% and 5% recoverable strain for porous samples with low (16%) and high (51%) porosity.
porosity, respectively. Through several examples, the model capability of describing the shape memory effect and pressure-dependent behavior of porous SMAs under general thermomechanical loadings has been evaluated. Regarding simplicity together with accuracy, the proposed model can be used for the design and simulation of porous SMA-based structures.

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**Conflict of interest**

None declared.

**References**


Shishkovsky IV (2012b) Functional design of porous drug delivery systems based on laser assisted manufactured nitinol. In: *MRS proceedings*.


