A study on unfitted 1D finite element methods

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Available online 18 September 2014

\textbf{Keywords:}
Immersed boundary method
Fictitious domain method
Unfitted methods
Finite elements

\textbf{A B S T R A C T}

In the present paper we consider a 1D Poisson model characterized by the presence of an interface, where a transmission condition arises due to jumps of the coefficients. We aim at studying finite element methods with meshes not fitting such an interface. It is well known that when the mesh does not fit the material discontinuities the resulting scheme provides in general lower order accurate solutions. We focus on so-called embedded approaches, frequently adopted to treat fluid–structure interaction problems, with the aim of recovering higher order of approximation also in presence of non fitting meshes; we implement several methods inspired by: the Immersed Boundary method, the Fictitious Domain method, and the Extended Finite Element method. In particular, we present four formulations in a comprehensive and unified format, proposing several numerical tests and discussing their performance. Moreover, we point out issues that may be encountered in the generalization to higher dimensions and we comment on possible solutions.

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\textbf{1. Introduction}

Increasingly enhanced computer performances allow nowadays to tackle very large fluid–structure interaction problems, and state of the art examples, such as parachutes, wind turbines, or biomechanics applications, are now the object of active research (see, e.g., [1]). Furthermore, problems with very large structural deformations are still open to major improvements. In particular, a promising class of methods for such a type of problems belongs to so-called immersed boundary approaches. Many variants of this category of techniques have been proposed in the literature under several names, such as immersed boundary methods, unfitted and embedded methods, and fictitious domain methods.

Accordingly, the goal of the present work is to give highlights of some fundamental issues of immersed approaches by studying a simple 1D problem within the finite element method. In particular, we study some original approaches dating back to the 70–90’s and a more recent one based on the extended finite element method, able to cure some issues of the original methods. For the latter method, we also focus on the issues that may be encountered in higher dimensions, as well as on possible solutions.

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http://dx.doi.org/10.1016/j.camwa.2014.08.018
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We consider problems involving multiple materials, and hence characterized by the presence of an interface, on which constraints have to be imposed. For instance, fluid–structure interaction situations belong to this category of problems, where velocity and stress discontinuities have to be imposed at the fluid–structure interface. From the numerical standpoint, two different approaches may be considered, the first one with a mesh fitting the interface, and the second one with a mesh not fitting the interface. The latter approach, named *unfitted or embedded*, has the advantage of enabling the use of meshes independent of the geometry, and it is the focus of the present paper.

The present problem is also referred in the literature as an *elliptic interface problem*. Typically, solutions of elliptic interface problems are not smooth over the whole domain, but they are smooth away from the interface (see, e.g., [2], [3], and [4]). Earliest error estimates can be traced back to 1970 with the work of Babuška (see [5]) in which the author provided a method based on a penalty approach. An almost optimal order of convergence is recovered for piecewise linear elements in the $H^1$-norm, more precisely $O(h^{3/2})$. In [6], Barrett et al. proposed to enrich the finite element space on elements cut by the interface and to enforce weakly the continuity constraint using a penalty approach. They proved the optimal error estimate in the $H^1$-norm but the estimate in the $L^2$-norm is still suboptimal, i.e., $O(h^{3/2})$. However, these authors show that the optimal error in the $L^2$-norm can be recovered far away of the interface. In [7] a similar method is proposed using the Nitsche method, instead of the penalty method, and the optimal error estimate is obtained in both $H^1$- and $L^2$-norms. In [8], the finite element space is not enriched, but a constraint on the mesh around the interface is added. The constraint consists in defining a “resolution” of the interface by the mesh, and it has to be at least of $O(h^2)$ for piecewise linear elements such that the optimal rate of convergence for the $L^2$- and $H^1$-norms can be attained. However, the development of methods for elliptic interface problems differs from the development of fictitious methods for fluid–structure interaction problems, even if many similarities between the various approaches may be portrayed. In the present paper we consider methods proposed in the literature for fluid–structure interaction problems.

In this context, we discuss four possible schemes: (i) a *one-field Fictitious Domain method*; (ii) a *continuously extended two-field Fictitious Domain method with Boundary Lagrange Multipliers*; (iii) a *continuously extended two-field Fictitious Domain method with Distributed Lagrange Multipliers*; (iv) a *discontinuously extended Fictitious Domain method with boundary Lagrange multipliers*, named herein *two-field Discontinuous Fictitious Domain*. In the following, we briefly discuss the four methods.

The *one-field Fictitious Domain method* (shortly, *one-field FD*) is inspired by the Immersed Boundary method, proposed by Peskin in the 1970s (see [9] and references therein), and it is based on rewriting the problem as a function of a single field defined on the global domain, which is the union of the fluid and the solid domains. In general, the fluid model is extended over the solid domain, and the solid problem acts as a constraint on the fluid extended domain. It follows that the value of the global fluid field naturally describes the fluid in the fluid domain and the solid in the solid region. Since we deal with one field over the whole domain, the continuity at the interface is automatically satisfied, while a discontinuity in the gradient of the global fluid field may occur, with important implications for numerical methods.

The *continuously extended two-field Fictitious Domain method with Boundary Lagrange Multipliers* (shortly, *two-field FD/BLM*) is inspired by the original approach proposed by Glowinski in the 1990s (see [10] and references therein). The method formalizes the problem with two fields: the global fluid and the solid. The global fluid field is defined on the global domain, such that it describes the fluid in the fluid domain and it is non-physical (fictitious) in the solid domain. The continuity between the two fields at the interface is enforced with a boundary Lagrange multiplier, that may introduce a discontinuity in their gradients between the physical fluid domain and the non-physical fluid region. Indeed, the Lagrange multiplier represents the jump in the gradient between the physical and the non-physical fluid domains.

The *continuously extended two-field Fictitious Domain method with distributed Lagrange multipliers* (shortly, *two-field FD/DLM*) is also described in [10] and is based again on two fields, as the previous method, but here the fictitious fluid and the solid fields are constrained to match in the solid domain with a distributed Lagrange multiplier. However, similarly to what happens with the two techniques previously described, a discontinuity in the gradient is introduced in the global fluid field between the physical fluid domain and the non-physical fluid region. Indeed, the distributed Lagrange multiplier imposes that the non-physical fluid behaves as the solid, thus imposing a jump at the interface.

The *discontinuously extended Fictitious Domain method with boundary Lagrange multipliers* (shortly, *two-field DFD/BLM*), inspired by the extended finite element method, has been proposed in [11]. The method is a two-field problem, where the fluid is extended in the fictitious domain by zero, and thus is based on the introduction of a strong discontinuity at the interface between the physical fluid and the fictitious fluid. As a consequence the method differs from all three methods previously described, which are characterized by a continuous global fluid field. Since it is a two-field method we have to enforce the continuity between the two fields at the interface. This operation is performed with a boundary Lagrange multiplier.

In this work, we propose a qualitative and quantitative analysis of the four previously mentioned techniques within the framework of the finite element method. For each scheme we present a variational formulation and its finite element approximation. The focus is on the discrete schemes and their performance. In particular, the variational method is presented formally, also if without any rigorous mathematical analysis, and it serves as a justification of the proposed algorithms. We aim at studying a simple model reproducing the typical characteristics of a fluid–structure problem, that is, a problem with continuity of the primal fields and with a possible discontinuity in the gradient at the interface. In fact, we focus on a steady Poisson problem defined on 1D domains with two different materials, where one surrounds the other; the problem under consideration requires that the continuity of the primal fields and of their fluxes is maintained at the interface. The main reasons for studying primarily a 1D problem are to provide easy and comprehensive formulations of the various methods.
and to analyze features of the methods that are already distinctive in 1D; however, practical implications for extensions to higher dimensions are identified and discussed as well.

We perform numerical tests to analyze the convergence properties in the $L^2$-norm, in the $H^1$-norm, and in the $H^1$-norm far away from the interface, of the four methods with different mesh refinement strategies and various material parameters. In particular, we show that, using linear finite elements, the one-field FD, two-field FD/BLM, and two-field FD/DLM methods are only first-order accurate, while the two-field DFD/BLM method is second-order accurate. We also point out the importance of quadrature over elements cut by the interface, since it is directly related to computational efficiency. As mentioned above, the paper is then completed by a discussion on critical problems in higher dimensions, in particular about the imposition of the continuity constraint. The main interest in this respect is in the construction of second order accurate schemes for the approximation of interface problems with non fitting meshes.

Before proceeding with the core of the paper, we wish to emphasize that we do not discuss here other second-order accurate approaches, such as the Fat Boundary method (see [12] and [13] for an analysis with a similar 1D problem) and the Immersed Interface method (see [14]). The Fat Boundary method uses an iterative Dirichlet/Neumann domain decomposition type of approach, while the Immersed Interface method modifies locally (i.e., on elements crossed by the interface) the shape functions such that they represent the interface constraints, introducing physical parameters in the definition of the shape functions. We therefore believe that these methods do not fit within our comparative study.

Moreover, we highlight that a Poisson problem similar to the one treated in the present paper has been used within the framework of the spectral element method in [15]. In particular, Vos et al. investigated the two-field FD/DLM method, the Finite Cell method (see [16]) with boundary Lagrange multipliers, the Fat Boundary method, and a modified formulation of the Fat Boundary method with boundary Lagrange multipliers. We note that the Finite Cell method shares similarities with the two-field DFD/BLM method we consider herein, since the physical parameter for the global fluid is set to a very small value in the solid domain. With this paper, we aim at providing a similar study of various fictitious domain methods with traditional finite elements, and we believe that such a study highlights some fundamental issues that one has to take into account to develop this type of methods.

The paper is organized as follows. In Section 2 we introduce our 1D Poisson model, in both strong and weak forms. In Section 3 to 6 we present the four methods, and in Section 7 we perform the numerical tests. In Section 8 we discuss issues for extension to higher dimensions and possible solutions, in particular for the two-field DFD–BLM method. In Section 9, we draw our concluding remarks.

2. Model problem

We consider a Poisson problem characterized by two distinct materials, such that at the interface only continuity of the primal fields and of the corresponding fluxes have to be guaranteed.

As described in Fig. 1, Material 1 and Material 2 are defined on $\Omega_1$ and $\Omega_2$, respectively, with $\Omega_1 = \{A, B\} \cup \{C, D\}$ and $\Omega_2 = \{B, C\}$. We denote the interface between $\Omega_1$ and $\Omega_2$ by $\Gamma$ (i.e., $\Gamma = \{B, C\}$). The global domain $\Omega$ is the union of $\Omega_1$, $\Omega_2$, and $\Gamma$, that is $\Omega = \{A, D\}$. External boundaries (i.e., $\partial \Omega = \{A, D\}$) are denoted by $\Sigma$.

In the following we introduce classical functional spaces that will be used in the rest of the paper. In particular, $L^2(\Omega)$ is the space of square integrable functions on $\Omega$, $H^1(\Omega)$ is the space of functions defined on $\Omega$ that belong to $L^2(\Omega)$ together with their first derivative, and $H^1_0(\Omega)$ the space of functions belonging to $H^1(\Omega)$ and vanishing on $\partial \Omega$.

The strong formulation for the described problem can be written as follows:

**Problem 1.** Find two functions $u_1 : \Omega_1 \rightarrow \mathbb{R}$ and $u_2 : \Omega_2 \rightarrow \mathbb{R}$ smooth enough such that

$$\begin{cases} -(\alpha_1 u_1') = f_1 & \text{on } \Omega_1, \\ -(\alpha_2 u_2') = f_2 & \text{on } \Omega_2, \\ u_1|\Gamma = u_2|\Gamma, \\ (\alpha_1 u_1')|\Gamma = (\alpha_2 u_2')|\Gamma, \\ u_1|\Sigma = 0, \end{cases}$$

where $\alpha_1 \geq \bar{\alpha} > 0$, $\alpha_2 \geq \bar{\alpha} > 0$, $f_1$, and $f_2$, are given regular functions, and $u|\Gamma$ is the restriction of $u$ on $\Gamma$. 

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**Fig. 1.** A two material 1D Poisson framework.
Remark 1. In Problem 1, we consider for simplicity homogeneous Dirichlet boundary conditions on $\Sigma$ but other boundary conditions can be considered as well.

Problem 2. The standard weak formulation corresponding to Problem 1 can readily obtained as:

Find $u \in H^1_0(\Omega)$ such that

$$\int_{\Omega} \alpha u' v' \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1_0(\Omega),$$

where

$$\alpha = \begin{cases} \alpha_1 & \text{on } \Omega_1, \\ \alpha_2 & \text{on } \Omega_2, \end{cases} \quad \text{and} \quad f = \begin{cases} f_1 & \text{on } \Omega_1, \\ f_2 & \text{on } \Omega_2. \end{cases}$$

We may also split the one-field Problem 2 into:

Problem 3. Find $(u_1, u_2) \in W = \{(v_1, v_2) \in H^1(\Omega_1) \times H^1(\Omega_2); \text{ with } v_1|_{\Sigma} = 0 \text{ and } v_1|_{\Gamma} = v_2|_{\Gamma}\}$ such that

$$\int_{\Omega_1} \alpha_1 u_1' v_1' \, dx + \int_{\Omega_2} \alpha_2 u_2' v_2' \, dx - \int_{\Omega_1} f_1 v_1 \, dx - \int_{\Omega_2} f_2 v_2 \, dx = 0 \quad \forall (v_1, v_2) \in W.$$ 

Remark 2. A discretization with finite elements of Problem 3 requires two partitions, one for $\Omega_1$ and another one for $\Omega_2$. In 1D this implies that the problem is fitted, since the partitions share common nodes at their interfaces. However, in higher dimensions, following the denomination of [17], we distinguish two interface fitted cases: matching and non-matching. We say that an interface fitted problem is matching when all nodes on the interface are shared by both meshes. On the contrary, we say that an interface fitted problem is non-matching when the nodes lying on the boundary are not necessarily common to both meshes.

Remark 3. We add two comments on formulation (3). Firstly, it is clear that if integrals are evaluated exactly, then the problem is equivalent to a standard Galerkin approach. However, when considering multidimensional problems, integration might be a difficult task and we may consider two different meshes for Material 1 and Material 2. Such a strategy results in a different method, that may converge, or may not, as it possibly loses consistency. Such issues are shown in numerical tests. Secondly, the reader may see the one-field FD method as a heuristic way for dealing with more complex problems such as those presented in, e.g., [18] and [19].

3. A one-field fictitious domain method

The one-field Fictitious Domain method (one-field FD) consists in rewriting Problem 3 in terms of a single continuous field $u$ defined over the whole domain $\Omega$, where $u|_{\Omega_1} = u_1$ and $u|_{\Omega_2} = u_2$, with $u|_{\Omega_i}$ denoting the restriction of $u$ on $\Omega_i$. Since we deal with a single continuous field $u$, the continuity constraint at the interface is automatically satisfied, while the continuity of the flux still needs to be enforced.

3.1. Continuum formulation

The strong formulation for the one-field Fictitious Domain problem can be written as follows:

Problem 4. Find one function $u : \Omega \rightarrow \mathbb{R}$ with $u|_{\Sigma} = 0$ such that

$$\begin{cases} -(\alpha u')' - f = 0 & \text{on } \Omega, \\ \|\alpha u'\|_{\Gamma} = 0, \end{cases}$$

where we split $\alpha$ and $f$ defined in Problem 2 such that

$$\alpha = \begin{cases} \alpha_1 & \text{on } \Omega_1, \\ (\alpha_2 - \alpha_f) + \alpha_f & \text{on } \Omega_2, \end{cases} \quad \text{with } \alpha_f \text{ chosen such that } \alpha_f \geq \bar{\alpha} > 0, \text{ and}$$

$$f = \begin{cases} f_1 & \text{on } \Omega_1, \\ (f_2 - f_f) + f_f & \text{on } \Omega_2, \end{cases}$$

The symbol $\|\cdot\|_{\Gamma}$ denotes the jump on $\Gamma$. 
Remark 4. Since we consider an extension of Material 1 over $\Omega_2$ we denote Material 1 over the whole domain $\Omega$ as extended (thus the subscript $e$) and the non-physical part (i.e., the part on $\Omega_2$) as fictitious (thus the subscript $f$). These notations are used hereafter.

Setting

$$\alpha_e = \begin{cases} \alpha_1 & \text{on } \Omega_1 \\ \alpha_f & \text{on } \Omega_2 \end{cases}, \quad \text{and} \quad f_e = \begin{cases} f_1 & \text{on } \Omega_1 \\ f_f & \text{on } \Omega_2 \end{cases},$$  

the weak formulation for Problem 4 can be readily obtained as:

**Problem 5.** Find one function $u \in H^1_0(\Omega)$ such that

$$\int_{\Omega} \alpha_e u' v' \, dx - \int_{\Omega} f_e v \, dx + \int_{\Omega_2} (\alpha_2 - \alpha_f) u' v' \, dx - \int_{\Omega_2} (f_2 - f_f) v \, dx = 0 \quad \forall v \in H^1_0(\Omega).$$  

(6)

It is clear that Problem 5 is equivalent to Problem 3 in the sense that $u|_{\Omega_1} = u_1$ and $u|_{\Omega_2} = u_2$. Moreover in Problem 5 we look for a function $u \in H^1_0(\Omega)$, thus satisfying automatically the continuity of the primal field on $\Gamma$. The continuity of the flux on $\Gamma$ is instead naturally enforced in the weak formulation by continuity of the test function; see also [20,21], and [22].

3.2. Discrete formulation

As discussed in Remark 2, we construct meshes with respect to the domain of integration for the integrals involved in the problem formulation. In Problem 5 integrals are defined on $\Omega$ and $\Omega_2$, and, accordingly, we construct partitions for such domains. We consider $\Omega^h$ and $\Omega^k_2$ as partitions for $\Omega$ and $\Omega_2$, respectively, where $h$ and $k$ are the sizes of the largest element in each partition (see Fig. 2(b)).

Given a finite-dimensional space $V^h \subset H^1_0(\Omega)$, the discrete formulation for Problem 5 can be readily obtained as:

**Problem 6.** Find $u^h \in V^h$ such that

$$\int_{\Omega^h} \alpha_e (u^h)' (v^h)' \, dx + \int_{\Omega^k_2} (\alpha_2 - \alpha_f) (u^h)' (v^h)' \, dx = \int_{\Omega^h} f_e v^h \, dx + \int_{\Omega^k_2} (f_2 - f_f) v^h \, dx \quad \forall v^h \in V^h.$$  

(7)

Given the following approximation

$$u^h(x) = N(x) \hat{u},$$

with $N(x)$ being standard piecewise linear shape functions defined on $\Omega^h$ and $\hat{u}$ the primal field nodal value vector, the algebraic formulation corresponding to Problem 6 reads:

$$A \hat{u} = b,$$  

(8)

where

$$A_{ij} = \int_{\Omega^h} \alpha_e N_i^j N_j^i \, dx + \int_{\Omega^k_2} (\alpha_2 - \alpha_f) N_i^j N_j^i \, dx,$$

and

$$b_{i} = \int_{\Omega^h} f_e N_i \, dx + \int_{\Omega^k_2} (f_2 - f_f) N_i \, dx.$$
4. A continuously extended two-field fictitious domain method with boundary Lagrange multipliers

The one-field Fictitious Domain method, described in the previous section, was based on the introduction of a single field $u$ defined over the whole domain $\Omega$. On the contrary, the two-field Fictitious Domain method introduces an extended field for Material 1 over the whole domain $\Omega$, which is fictitious on $\Omega_2$.

In the corresponding discrete formulation we consider unfitted meshes discretizing two fields, and hence we may use a boundary Lagrange multiplier to weakly enforce continuity of the primal fields in both materials at the interface.

In order to obtain a formulation that is consistent with Problem 1 we introduce an extension of $u_1$ on $\Omega_2$ that does not necessarily maintain continuity of the derivatives of the extended $u_1$ on $\Gamma$. As a consequence, we have to consider on which side of the interface (physical or fictitious) we impose continuity of the flux; for convenience, we hereafter denote by $\Gamma_1$ and $\Gamma_2$ the limit to $\Gamma$ approached from $\Omega_1$ and $\Omega_2$, respectively.

4.1. Continuum formulation

The two-field Fictitious Domain formulation is then given by:

**Problem 7.** Find two functions $u_e : \Omega \to \mathbb{R}$ and $u_2 : \Omega_2 \to \mathbb{R}$, with $u_{e|\Sigma} = 0$, such that

\[
\begin{align*}
-(\alpha_e u_e')' &= f_e & \text{on } \Omega, \\
-(\alpha_2 u_2')' + (\alpha_1 u_1')' &= f_2 - f_j & \text{on } \Omega_2, \\
u_{e|\Gamma} &= u_{2|\Gamma}, \\
(\alpha_1 u_1')_{\Gamma_1} &= (\alpha_2 u_2')_{\Gamma_2}.
\end{align*}
\]

(9)

The weak formulation for Problem 7 can be readily obtained as:

**Problem 8.** Find two functions $(u_e, u_2) \in E_\Gamma = \{(u_e, u_2) \in H^1_0(\Omega) \times H^1(\Omega_2); \text{ with } u_{e|\Gamma} = u_{2|\Gamma}\}$ such that for all $(v_e, v_2) \in E_\Gamma$ we have

\[
\int_\Omega \alpha_e u_e' v_e' dx + \int_{\Omega_2} \alpha_2 u_2' v_2' dx - \int_{\Omega_2} \alpha_1 u_1' v_2' dx = \int_\Omega f_e v_e dx + \int_{\Omega_2} (f_2 - f_j) v_2 dx.
\]

(10)

Since the discrete space for the extended Material 1 field may not be interpolatory at the interface, we choose to enforce weakly with a boundary Lagrange multiplier the constraint $u_{e|\Gamma} = u_{2|\Gamma}$, giving rise to the two-field Fictitious Domain with Boundary Lagrange Multipliers (two-field FD/BLM).

**Problem 9.** Find two functions $u_e \in H^1_0(\Omega)$ and $u_2 \in H^1(\Omega_2)$, and the Lagrange multipliers $\lambda_B \in \mathbb{R}$ and $\lambda_C \in \mathbb{R}$ such that

\[
\begin{align*}
\int_\Omega \alpha_e u_e' v_e' dx + \lambda_B u_e(B) - \lambda_C u_2(C) &= \int_\Omega f_e v_e dx \\
\int_{\Omega_2} \alpha_2 u_2' v_2' dx - \int_{\Omega_2} \alpha_1 u_1' v_2' dx - \lambda_B v_2(B) + \lambda_C v_2(C) &= \int_{\Omega_2} (f_2 - f_j) v_2 dx,
\end{align*}
\]

(11)

\[
\begin{align*}
\xi_B(u_e(B) - u_2(B)) &= 0, \\
\xi_C(u_e(C) - u_2(C)) &= 0,
\end{align*}
\]

$\forall v_1 \in H^1_0(\Omega), \forall v_2 \in H^1(\Omega_2), \forall \xi_B \in \mathbb{R}$ and $\forall \xi_C \in \mathbb{R}$. 

**Fig. 3.** Quadrature issue on elements of $\Omega_2^k$ for shape functions with support on a partition for $\Omega$. If we consider the shape function $N_j$ (defined on a partition of $\Omega$ with support on $[x_{i-1}, x_{i+1}]$) admitting a kink on $x_i$ in an element of a partition for $\Omega_2$ ($K_2 = [y_j, y_{j+1}]$), an exact integration on $K_2$ requires to integrate on two sub-elements: $[y_j, x_i]$ and $[x_i, y_{j+1}]$. The blue zone corresponds to the integral of $N_j$ on $[y_j, y_{j+1}]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
Remark 6. We point out that the space of the Lagrange multiplier corresponds to the trace of $H^1(\Omega)$ on $\Gamma$, i.e., $H^{-1/2}(\Gamma)$, the dual of $H^{1/2}(\Gamma)$ since $\Gamma \cap \partial \Omega = \emptyset$. In one dimension, the trace of $H^1$ is $\mathbb{R}$. It can be shown that the Lagrange multipliers can be interpreted as the flux across the interface:

$$\lambda_C = -\alpha_2 u^h_2(C) + \alpha_1 u^h_1(C) \quad \text{and} \quad \lambda_B = -\alpha_2 u^h_2(B) + \alpha_1 u^h_1(B).$$

By introducing a Lagrange multiplier we add a constraint to the system, resulting in a saddle point problem. In order for a saddle point problem to be well-posed an inf–sup condition has to be satisfied (see [24] and references therein). At the discrete level, such an issue is a very difficult task. We discuss the problem in Section 8.

4.2. Discrete formulation

As explained in Section 3.2 for the discrete formulation of the one-field FD method, we consider $\Omega^h$ and $\Omega^k_2$ to be partitions for $\Omega$ and $\Omega_2$, respectively, with mesh sizes $h$ and $k$. Given finite-dimensional spaces $V^h$ and $W^k$ such that $V^h \subset H^1_0(\Omega)$ and $W^k \subset H^1(\Omega_2)$, the discrete formulation for Problem 9 reads:

**Problem 10.** Find two functions $u^h \in V^h$ and $u^k \in W^k$, and Lagrange multipliers $\lambda^h \in \mathbb{R}$ and $\lambda^k \in \mathbb{R}$, such that

$$\begin{aligned}
\int_{\Omega^h} \alpha_e (u^h_e)'(v^h_e)' dx + \lambda^h v^h_e(B) - \lambda^k v^h_k(C) &= \int_{\Omega^h} f_e v^h_e dx, \\
\int_{\Omega^k_2} \alpha_2 (u^k_2)'(v^k_2)' dx - \int_{\Omega^k_2} \alpha_1 (u^k_1)'(v^k_1)' dx - \lambda^h v^k_2(B) + \lambda^k v^k_k(C) &= \int_{\Omega^k_2} (f_2 - f_1) v^k_2 dx \\
\xi_B(u^h_2(B) - u^k_2(B)) &= 0, \\
\xi_k(u^h_2(C) - u^k_2(C)) &= 0,
\end{aligned}$$

$\forall v^h_e \in V^h, \forall v^k_2 \in W^k, \forall \xi_B \in \mathbb{R}$ and $\forall \xi_k \in \mathbb{R}$.

Given the following approximations

$$u^h_e(x) = N(x)\hat{u}_e, \quad u^k_2(x) = M(x)\hat{u}_2,$$

where $N(x)$ and $M(x)$ are standard piecewise linear shape functions, $\hat{u}_e$ and $\hat{u}_2$ are the primal field nodal value vectors, and $\hat{\lambda} = \{\lambda_B, \lambda_C\}^T$, the algebraic formulation corresponding to Problem 10 reads

$$\begin{bmatrix}
A_{e2} & 0 & L^T_e \\
-A_{e2} & A_2 & -L^T_2 \\
L_e & -L_2 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u}_e \\
\hat{u}_2 \\
\hat{\lambda}
\end{bmatrix}
= \begin{bmatrix}
f^e \\
f^k \\
0
\end{bmatrix}.$$  \hspace{1cm} (13)

The components of the system matrix are given by

$$\begin{aligned}
A_{eij} &= \int_{\Omega^h} \alpha_e N_i^e N_j^e dx, \\
A_{e2ij} &= \int_{\Omega^k_2} \alpha_2 M_i^e N_j^e dx, \\
A_{2ij} &= \int_{\Omega^k_2} \alpha_2 M_i^k M_j^k dx, \\
L_{eij} &= (\delta_i N_j)^e_\Gamma, \\
L_{2ij} &= (\delta_i M^e_j)^k_\Gamma,
\end{aligned}$$

where $\delta_{1|B} = 1$, $\delta_{1|C} = 0$, $\delta_{2|B} = 0$, $\delta_{2|C} = 1$, and those of the right hand side by

$$\begin{aligned}
f_{eij} &= \int_{\Omega^h} f_e N_i dx, \\
f_{2ij} &= \int_{\Omega^k_2} (f_2 - f_1) M_i dx.
\end{aligned}$$

**Remark 7.** In system (13) the terms $A_{e2}$ is difficult to compute since the functions $N_i$ are not defined on $\Omega^k_2$ but on $\Omega^h$ (see Remark 5 for a discussion on the implementation).
5. A continuously extended two-field Fictitious Domain method with distributed Lagrange multipliers

In the two-field Fictitious Domain method, described in Section 4, we considered two fields \( u_e \) and \( u_2 \), where \( u_e \) was \( u_1 \) extended continuously over \( \Omega_2 \). The coupling between \( u_e \) and \( u_2 \) at the interface was enforced with a boundary Lagrange multiplier. Since \( u_e \) is fictitious on \( \Omega_2 \) (i.e., it has no physical meaning) we may constrain the extension of \( u_1 \) on \( \Omega_2 \) such that \( u_{e|\Omega_2} = u_2 \) (see [10]). Since the nodes of the meshes for \( \Omega_1 \) and \( \Omega_2 \) are not necessarily common to both meshes, we may choose to enforce weakly the constraint \( u_{e|\Omega_2} = u_2 \) with a distributed Lagrange multiplier, giving rise to the two-field Fictitious Domain with Distributed Lagrange Multipliers (two-field FD/DLM).

5.1. Continuum formulation

Analogously to the case of the two-field FD/BLM method, the weak formulation for the two-field Fictitious Domain with a strong enforcement of the constraint \( u_{e|\Omega_2} = u_2 \) is given by:

**Problem 11.** Find \((u_e, u_2) \in E_{\Omega_2} = \{(u_e, u_2) \in H_0^1(\Omega) \times H^1(\Omega_2); \text{ with } u_{e|\Omega_2} = u_2\}\) such that

\[
\int_{\Omega} \alpha_e u_e' v'_e \, dx + \int_{\Omega_2} \alpha_2 u_2' v_2' \, dx - \int_{\Omega_2} \alpha_f u_f' v_2' \, dx = \int_{\Omega} f_e v_e \, dx + \int_{\Omega_2} (f_2 - f_f) v_2 \, dx, \quad \forall (v_e, v_2) \in E_{\Omega_2}. \tag{14}
\]

Enforcing weakly the constraint \( u_{e|\Omega_2} = u_2 \), it follows that Problem 11 with distributed Lagrange multipliers reads:

**Problem 12.** Find two functions \( u_e \in H_0^1(\Omega) \), \( u_2 \in H^1(\Omega_2) \), and a Lagrange multiplier \( \lambda \in L^2(\Omega_2) \) such that

\[
\begin{align*}
\int_{\Omega} \alpha_e u_e' v'_e \, dx + \int_{\Omega_2} \lambda v_e \, dx &= \int_{\Omega} f_e v_e \, dx, \\
\int_{\Omega_2} \alpha_2 u_2' v_2' \, dx - \int_{\Omega_2} \alpha_f u_f' v_2' \, dx &= \int_{\Omega_2} (f_2 - f_f) v_2 \, dx, \quad \forall v_e \in H_0^1(\Omega), \forall v_2 \in H^1(\Omega_2), \text{ and } \forall \xi \in L^2(\Omega_2). \tag{15}
\end{align*}
\]

**Remark 8.** We note that the two-field FD/DLM method is asymmetric. It is possible to obtain a symmetric formulation by replacing \( u_e \) by \( u_2 \) in the second equation of (15), since they are equal on \( \Omega_2 \). At the continuous level both formulations are identical, but that is not the case at the discrete level (see [25] for more details on the asymmetric formulation).

5.2. Discrete formulation

As explained in Section 3.2 for the discrete formulation of the one-field FD method we consider \( \Omega^h \) and \( \Omega_2^k \) to be partitions for \( \Omega \) and \( \Omega_2 \), respectively, with mesh sizes \( h \) and \( k \).

Given finite-dimensional spaces \( V^h \) and \( W^k \) such that \( V^h \subset H_0^1(\Omega) \) and \( W^k \subset H^1(\Omega_2) \), the discrete formulation of Problem 12 reads:

**Problem 13.** Find two functions \( u_e^h \in V^h \), \( u_2^k \in W^k \), and the Lagrange multiplier \( \lambda^k \in W^k \) such that

\[
\begin{align*}
\int_{\Omega^h} \alpha_e (u_e^h)' (v_e^h)' \, dx + \int_{\Omega_2^k} \lambda^k v_e^h \, dx &= \int_{\Omega^h} f_e v_e^0 \, dx, \\
\int_{\Omega_2^k} \alpha_2 (u_2^k)' (v_2^k)' \, dx - \int_{\Omega_2^k} \alpha_f (u_f^k)' (v_2^k)' \, dx - \int_{\Omega_2^k} \lambda^k v_2^k \, dx &= \int_{\Omega_2^k} (f_2 - f_f) v_2^0 \, dx, \\
\int_{\Omega_2^k} \xi^k (u_e^h - u_2^k) \, dx &= 0, \quad \forall u_e^h \in V^h, \forall v_2^k \in W^k, \text{ and } \forall \xi^k \in W^k. \tag{16}
\end{align*}
\]

**Remark 9.** In Problem 13, we choose to use continuous finite elements for the Lagrange multiplier. However, since the distributed Lagrange multiplier is only in \( L^2(\Omega_2) \) we may use discontinuous finite elements, as well.

Given the following approximations

\[
u_e^\Delta(x) = N(x) \hat u_e, \quad u_2^\Delta(x) = M(x) \hat u_2, \quad \lambda^\Delta(x) = M(x) \hat \lambda,
\]
where $\mathbf{N}(x)$ and $\mathbf{M}(x)$ are standard piecewise linear shape functions, $\mathbf{\hat{u}}_1, \mathbf{\hat{u}}_2,$ and $\mathbf{\hat{\lambda}}$ are the primal field and Lagrange multiplier nodal value vectors, the algebraic formulation corresponding to Problem 13 reads

$$
\begin{bmatrix}
\mathbf{A}_1 & \mathbf{0} & \mathbf{L}_1^T \\
-\mathbf{A}_2 & \mathbf{A}_2 & -\mathbf{L}_2^T \\
\mathbf{L}_1 & \mathbf{L}_2 & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\hat{u}}_1 \\
\mathbf{\hat{u}}_2 \\
\mathbf{\hat{\lambda}}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{f}_1 \\
\mathbf{f}_2 \\
\mathbf{0}
\end{bmatrix}.
$$

(17)

The components of the system matrix are given by

$$
\begin{align*}
A_{ij} & = \int_{\Omega^h} \alpha_e N_i^e N_j^e \, dx, \\
A_{ij} & = \int_{\Omega^k} \alpha_1 M_i^1 M_j^1 \, dx, \\
A_{ij} & = \int_{\Omega^k} \alpha_2 M_i^2 M_j^2 \, dx, \\
L_{ij} & = \int_{\Omega^2} \mathbf{N}_i \, dx, \\
L_{ij} & = \int_{\Omega^2} \mathbf{M}_i \, dx,
\end{align*}
$$

and those of the right hand side by

$$
\begin{align*}
f_{ij} & = \int_{\Omega^h} \mathbf{f}_i \mathbf{N}_j \, dx, \\
f_{ij} & = \int_{\Omega^k} \{j_2 - j_1\} \mathbf{M}_i \, dx.
\end{align*}
$$

6. A discontinuously extended two-field Fictitious Domain method with boundary Lagrange multipliers

In all previously presented methods, an extension of Material 1 is constructed on $\Omega_2$ such that the extended formulation is continuous over $\Omega$. In the next approach we also consider a two-field method but we extend $u_1$ on $\Omega_2$ such that $u_1|\Omega_2 = 0$. Therefore, the extended $u_1$ is discontinuous over the interface. The continuity of the primary fields at the interface is enforced with a boundary Lagrange multiplier defined on the physical sides of Material 1. We define the obtained method as the discontinuously extended two-field Fictitious Domain method with boundary Lagrange multipliers (two-field DFD/BLM).

6.1. Continuum formulation

In the previously described methods we considered $\alpha_e$ and $f_e$ given by Eq. (5) with the condition that $\alpha_e \geq \bar{\alpha} > 0$. Here we consider the following extension:

$$
\alpha_e = \begin{cases} 
\alpha_1 & \text{on } \Omega_1, \\
\bar{\alpha}_1 & \text{on } \Omega_2,
\end{cases} \quad \text{and} \quad f_e = \begin{cases} 
f_1 & \text{on } \Omega_1, \\
0 & \text{on } \Omega_2,
\end{cases}
$$

(18)

where $\bar{\alpha}_1 \geq \bar{\alpha} > 0$.

Let us introduce the space of discontinuous $u_e$ on $\Gamma$,

$$
D = \{u_e \in L^2(\Omega) : \text{ with } u_e|\Omega_1 \in H^1(\Omega_1), u_e|\Omega_2 \in H^1_0(\Omega_2), \text{ and } u_e|\Sigma = 0\}.
$$

Then, since $u_e$ admits a discontinuity on $\Gamma$ we have to consider on which side of $\Gamma$ we impose the continuity constraint. A weak formulation for the two-field DFD/BLM technique is given by the following statement:

**Problem 14.** Find $(u_e, u_2) \in D_F = \{(u_e, u_2) \in D \times H^1(\Omega_2) : \text{ with } u_e|\Gamma_1 = u_2|\Gamma_2\}$ such that

$$
\int_{\Omega} \alpha_e u'_e v'_e \, dx + \int_{\Omega_2} \alpha_2 u'_2 v'_2 \, dx - \int_{\Omega} f_e v_e \, dx - \int_{\Omega_2} f_2 v_2 \, dx = 0, \quad \forall (v_e, v_2) \in D_F.
$$

(19)

We note that using the definitions of $\alpha_e$ and $f_e$ we recover **Problem 3**.

In the following discrete formulation, partitions for $\Omega$ and $\Omega_2$ may not be fitted, and hence the standard shape functions defined on a partition of $\Omega$ may not be interpolatory on the interface. As a consequence, we choose to enforce weakly the constraint $u_e|\Gamma_1 = u_2|\Gamma_2$. For this purpose a boundary Lagrange multiplier is here employed, obtaining the following weak formulation:
Problem 15. Find two functions \( u_1, u_2 \in D \), and Lagrange multipliers \( \lambda_B \in \mathbb{R} \) and \( \lambda_C \in \mathbb{R} \) such that

\[
\begin{align*}
\int_{\Omega} \alpha_e u_1^e v_1^e \, dx + \lambda_B u_1(B_1) - \lambda_C v_1(C_1) &= \int_{\Omega} f_v \, dx, \\
\int_{\Omega_2} \alpha_2 u_2^h v_2^h \, dx - \lambda_B v_2(B_2) + \lambda_C v_2(C_2) &= \int_{\Omega_2} f_2 \, v_2 \, dx, \\
\xi_B(u_1(B_1) - u_2(B_2)) &= 0, \\
\xi_C(u_1(C_1) - u_2(C_2)) &= 0,
\end{align*}
\]

(20)

\( \forall v_1 \in D, \forall v_2 \in H^1(\Omega_2) \), and \( \forall \xi_B \in \mathbb{R} \) and \( \forall \xi_C \in \mathbb{R} \), whereby \( B_1 \) mean \( B \) approached from \( \Omega_1 \), etc.

6.2. Discrete formulation

As explained in Section 3.2 for the discrete formulation of the one-field FD method, we consider \( \Omega^h \) and \( \Omega^k_2 \) to be partitions for \( \Omega \) and \( \Omega_2 \), respectively, with mesh sizes \( h \) and \( k \). We also assume that \( \Omega_2 \) contains at least one element of \( \Omega^h \). It implies that we associate with each Lagrange multiplier on \( B \) and \( C \) at least one degree of freedom in the fictitious domain (denoted “free” node), otherwise the system is overconstrained. This can be overcome considering the extended finite element method on elements crossed by the interface as presented, for instance, in [26], such that the system has enough “free” nodes with respect to the number of Lagrange multipliers. We point out that all degrees of freedom of the field of Material 1 that are associated to elements without support on \( \Omega_1 \) are eliminated from the linear system of equations. Moreover an extension of the DFD/DLM method to higher dimensions is not trivial due to locking issues, as further discussed in Section 8.

Given finite-dimensional spaces \( V^h \) and \( W^k \) such that \( V^h \subset H^1_0(\Omega) \) and \( W^k \subset H^1(\Omega_2) \), the discrete formulation of Problem 15 reads:

Problem 16. Find \( u^h_1 \in V^h_{\Omega_1}, u^k_2 \in W^k \), and Lagrange multipliers \( \tilde{\lambda}_B \in \mathbb{R} \) and \( \tilde{\lambda}_C \in \mathbb{R} \), such that

\[
\begin{align*}
\int_{\Omega^h} H_{\Omega^h}(x) \alpha_e(u^h_1)'(v^h_1)' \, dx + \tilde{\lambda}_B u^h_1(B) - \tilde{\lambda}_C v^h_1(C) &= \int_{\Omega^h} f^h_1 \, v^h_1 \, dx, \\
\int_{\Omega^k_2} \alpha_2(u^k_2)'(v^k_2)' \, dx - \tilde{\lambda}_B v^k_2(B) + \tilde{\lambda}_C v^k_2(C) &= \int_{\Omega^k_2} f^k_2 \, v^k_2 \, dx, \\
\xi^h_B u^h_1(B) - \xi^h_C u^h_1(C) &= 0, \\
\xi^k_C u^k_2(B) - \xi^k_C u^k_2(C) &= 0,
\end{align*}
\]

(21)

\( \forall v_1^h \in V^h_{\Omega_1}, \forall v_2^k \in W^k, \forall \xi^h_B \in \mathbb{R} \) and \( \forall \xi^k_C \in \mathbb{R} \),

where \( H_{\Omega^h}(x) \) is the Heaviside function, that is 1 on \( \Omega_1 \) and 0 otherwise. Given the following approximations

\[
u^h_1(x) = N(x) \hat{u}_1, \quad u^k_2(x) = M(x) \hat{u}_2,
\]

where \( N(x) \) and \( M(x) \) are standard piecewise linear finite elements, \( \hat{u}_1 \) and \( \hat{u}_2 \) are the primal field nodal value vectors, and \( \hat{\lambda} = [\lambda_B, \lambda_C]^T \), the algebraic formulation to Problem 16 reads

\[
\begin{bmatrix}
A_{11} & 0 & L_{1e}^T \\
0 & A_{22} & -L_{2e}^T \\
-L_{e} & -L_{2} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{\lambda}
\end{bmatrix}
= \begin{bmatrix}
f^h_1 \\
f^k_2 \\
0
\end{bmatrix}.
\]

(22)

The components of the system matrix are given by

\[
\begin{align*}
A_{11} &= \int_{\Omega^h} \alpha_e N_i^e N_j^e \, dx, \\
A_{22} &= \int_{\Omega^k_2} \alpha_2 M_i^k M_j^k \, dx, \\
L_{1e} &= (\delta N_j^e)_{i,r}, \\
L_{2e} &= (\delta M_j^k)_{i,r},
\end{align*}
\]

where \( \delta_{1|B} = 1, \delta_{1|C} = 0, \delta_{2|B} = 0, \delta_{2|C} = 1 \), and those of the right hand side by

\[
\begin{align*}
f^h_1 &= \int_{\Omega^h} f_1 \, N_i \, dx, \\
f^k_2 &= \int_{\Omega^k_2} f_2 \, M_i \, dx.
\end{align*}
\]
Table 1

<table>
<thead>
<tr>
<th>Material</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1. Material parameters definitions.

Fig. 4. Analytical solutions for the numerical test with $f_1 = 1$ on $[A, B \cup C, D]$, $f_2 = 1$ on $[B, C]$, for the different material parameters reported in Table 1.

7. Numerical tests

To test the performance of the four different methods discussed in Sections 3–6, we study the model in Section 2, considering a $h$-refinement strategy with piecewise linear finite elements and different material parameters (see Table 1). Additional tests are performed with two more combinations of material parameters and they are given in the Appendix. The additional tests reported in the Appendix confirm the trends observed with the numerical tests of this section.

As discussed in Remark 5, many integrals involve terms that are not global polynomials on elements of the mesh of $\Omega_2$, but are polynomials on sub-elements of the mesh of $\Omega_2$ (as depicted in Fig. 3). As a consequence, we have to integrate exactly with 2 Gauss points per sub-element (see again Fig. 3), and we denote this integration strategy as \textit{exact} quadrature scheme. Also, since integrations schemes over sub-elements may be expensive, we perform integration using a standard Gauss quadrature over the element of all meshes (2 Gauss points per element in our test problems), and we denote this integration strategy as \textit{approximated} quadrature scheme.

7.1. Test problems

For all methods we consider the following geometry: $A = 0, B = e, C = 1 + \pi, D = 6$ (see Fig. 1(a) for a description of the geometry). Interfaces $B$ and $C$ are such that the problem remains unfitted for all refinement steps (see Tables 2 and 3), and, to accomplish this goal easily, we select irrational numbers for $B$ and $C$ and rational numbers for $A$ and $D$. The material parameters for Material 1 ($\alpha_1$) and Material 2 ($\alpha_2$) are chosen constant on $[A, B \cup C, D]$ and $[B, C]$, respectively, and we select constant loads $f_1 = 1$ on $[A, B \cup C, D]$ and $f_2 = 1$ on $[B, C]$. The extension of Material 1 for the one-field FD and two-field FD methods over $\Omega_2$ is chosen such that $\alpha_f = \alpha_1$ and $f_f = f_1$. For the two-field FDF method $\alpha_e$ and $f_e$ are defined in Eq. (18).

The different set of material parameters are given in Table 1 with the corresponding analytical solutions reported in Fig. 4. All simulations are performed using piecewise linear finite elements to approximate all unknown fields, including the discrete distributed Lagrange multipliers which are defined on the mesh of $\Omega_2$.

7.2. Refinement ratios

Recalling that $h$ and $k$ denote, respectively, the sizes of the largest element in $\Omega^h$ and $\Omega^k$, we denote by $h_r = h/k$ the mesh ratio, by $L$ the length of $[A, D]$ (i.e., $\Omega$), and by $L_2$ the length of $[B, C]$ (i.e., $\Omega_2$). We consider two uniform $h$-refinement strategies such that $h_r < 1$ (i.e., $h_r = L/L_2 \times 1/8 = 6/(1 + \pi - e) \times 1/8 \approx 1/2$, that is, $h$ is “roughly” twice smaller than $k$) and $h_r > 1$ (i.e., $h_r = L/L_2 \times 1/2 = 6/(1 + \pi - e) \times 1/2 \approx 2$, that is, $h$ is “roughly” twice larger than $k$). Such mesh refinement strategies are presented for each mesh in Tables 2 and 3.

For the one-field FD with an exact quadrature scheme the choice of a partition for $\Omega_2$ has no impact on the solution since integration is performed exactly. On the contrary, with the approximated quadrature scheme the integration error

---

1. Since the two-field FD/DLM method requires at least 2 Gauss points per element to integrate exactly integrals involving the distributed Lagrange multiplier, we choose to use a 2 Gauss–Legendre point rule for all integral terms for all methods.
Table 2
Mesh refinement strategy for a mesh ratio \( h_r \approx 2. \)

<table>
<thead>
<tr>
<th>( h/L )</th>
<th>1/24</th>
<th>1/48</th>
<th>1/96</th>
<th>1/192</th>
<th>1/384</th>
<th>1/768</th>
<th>1/1536</th>
<th>1/3072</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k/L_2 )</td>
<td>1/12</td>
<td>1/24</td>
<td>1/48</td>
<td>1/96</td>
<td>1/192</td>
<td>1/384</td>
<td>1/768</td>
<td>1/1536</td>
</tr>
</tbody>
</table>

Table 3
Mesh refinement strategy for a mesh ratio \( h_r \approx 1/2. \)

<table>
<thead>
<tr>
<th>( h/L )</th>
<th>1/24</th>
<th>1/48</th>
<th>1/96</th>
<th>1/192</th>
<th>1/384</th>
<th>1/768</th>
<th>1/1536</th>
<th>1/3072</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k/L_2 )</td>
<td>1/3</td>
<td>1/6</td>
<td>1/12</td>
<td>1/24</td>
<td>1/48</td>
<td>1/96</td>
<td>1/192</td>
<td>1/384</td>
</tr>
</tbody>
</table>

Fig. 5. The one-field FD method.

depends on the partition for \( \Omega_2 \), and thus on the mesh ratio \( h_r \). However, since the mesh ratio has no impact on the rate of convergence of the method with an exact quadrature scheme and that, with an approximated one, the mesh ratio has only an impact on the integration error, we pick only one refinement strategy (i.e., \( h_r > 1 \) presented in Table 2, that is the one that leads to the best quadrature). Such a choice implies that care should be taken to generalize the results for different mesh ratios with an approximated quadrature scheme.

For two-field methods, boundary and distributed Lagrange multipliers are different since, for a 1D problem, a boundary Lagrange multiplier is defined on a set of discrete points and a distributed Lagrange multiplier on a segment. The boundary Lagrange multiplier is thus identically defined for every choice of meshes for \( \Omega_2 \). Its definition is only affected by the choice of a mesh for \( \Omega \) (because the evaluation of the shape functions \( M(x) \) at the boundary of the domain \( \Omega_2 \) is always 1, which is not the case for shape functions \( N(x) \) at the boundary of the domain \( \Omega_2 \)). Thus, the convergence rate is not affected by a change in the mesh ratio. We therefore pick an arbitrary mesh ratio (i.e., \( h_r > 1 \) as presented in Table 2) for problems with boundary Lagrange multipliers. For distributed Lagrange multipliers we choose to use a mesh of \( \Omega_2 \) with piecewise linear finite elements, and thus the method with distributed Lagrange multipliers depends on the mesh ratio \( h_r \). We test the two-field FD/DLM method with both mesh ratios \( h_r > 1 \) and \( h_r < 1 \), as presented in Tables 2 and 3.
Remark 10. In [8] it is pointed out that when computing the $H^1$-norm away from the interface the optimal convergence rate in the $H^1$-norm can be obtained, precisely when using the error measurement:

$$E_{1, \Omega \setminus \Gamma_e} \text{ with } \Gamma_e = \{ x \in \Omega : \text{dist}(x, \Gamma) < \epsilon \}.$$  

In the numerical tests we choose $\epsilon = h$. Such a choice is discussed in Remark 10.

**Remark 10.** In [8] a constraint for the construction of the mesh is added. It is required that the mesh is “$\epsilon$-resolved” near the interface, i.e., there must not be an element that overlaps $\Gamma_e$. In that work, it is proved that for $\epsilon = \Theta(h^3)$ the method has the optimal rate of convergence in both $L^2$- and $H^1$-norms. However, in our numerical experiments the mesh is not $\epsilon$-resolved for $\epsilon = \Theta(h^3)$ but it is for $\epsilon = \Theta(h)$. It justifies our choice of $\epsilon = h$. In the present numerical tests we show that we do not have the optimal rate of convergence for the $H^1(\Omega)$- and $L^2(\Omega)$-norms, at the exception of the DFD/BLM method, but we may attain it using the $H^1(\Omega \setminus \Gamma_e)$-norm.

For two-field methods we only measure errors in physical domains (i.e., $\Omega_1$ and $\Omega_2$). Our error measurement (in $L^2$-norm) is given by

$$E_{0, \Omega} = \frac{\left(\|u_1 - u_1^h\|^2_{L^2(\Omega_1)} + \|u_2 - u_2^h\|^2_{L^2(\Omega_2)}\right)^{1/2}}{\left(\|u_1\|^2_{L^2(\Omega_1)} + \|u_2\|^2_{L^2(\Omega_2)}\right)^{1/2}}.$$  

where $\| \cdot \|_{L^2}$ is defined as in (23) and $u_1$ and $u_2$ are the analytical solutions of the problem over $\Omega_1$ and $\Omega_2$, respectively.

In the same fashion, the $H^1$-seminorm is defined by

$$E_{1, \Omega_1, \Omega_2} = \frac{\left(\|u_1 - u_1^h\|^2_{L^2(\Omega_1)} + \|u_2 - u_2^h\|^2_{L^2(\Omega_2)}\right)^{1/2}}{\left(\|u_1\|^2_{L^2(\Omega_1)} + \|u_2\|^2_{L^2(\Omega_2)}\right)^{1/2}},$$  

while our measurement in the $H^1(\Omega \setminus \Gamma_e)$-seminorm is given by

$$E_{1, \Omega \setminus \Gamma_e, \Omega_2}.$$  

**Fig. 6.** Errors for Test 1 ($\alpha_1/\alpha_2 = 1/4$) with the one-field FD method with exact quadrature (i.e., standard Galerkin). The dots symbolize the position of the nodes, while the red lines the position of the interface. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
7.4. Results

In order to emphasize the impact of the quadrature schemes, we first present the results with an exact quadrature scheme and then with an approximated quadrature. Finally, we discuss the conditioning of the various methods.

7.4.1. Finite elements with exact quadrature

For the one-field FD method we observe in Fig. 5(a) and Fig. 5(b) that the rates of convergence oscillate, but averagely a convergence of order 1 is attained in $L^2$-norm and of order 1/2 in $H^1$-norm. Instead, in the $H^1(\Omega \setminus \Gamma_\epsilon)$-seminorm a linear order of convergence is achieved. We recall that assuming an exact integration this method is equivalent to the standard Galerkin method.

Remark 11. We can observe in Fig. 6(a) that the error $u - u_h$ at the interface propagates to the whole domain, preventing a possible optimal convergence rate in the $L^2$-norm away from the interface. On the contrary, for the $H^1$-norm, we can observe (see Fig. 6(b)) that the error in the derivatives clearly converges linearly away from the interface, even showing a super-convergence property at the middle of the elements not cut by the interface. Differently to $u - u_h$, the quantity $u' - u'_h$ does not appear to converge on the interface, but here the support of the large error values is limited to the elements crossed by the interface. It follows that the optimal rate of convergence would be obtained if the error is integrated only on elements not crossed by the interface. This example also shows that if the mesh is $\epsilon$-resolved with $\epsilon = O(h^2)$, then we may obtain the optimal rate of convergence in the $H^1$-norm for a smaller $\epsilon$, i.e., $\epsilon < h$, as observed in [8].

For the two-field FD/BLM method, we also observe in Fig. 7(a) and Fig. 7(b) that the rates of convergence also oscillate but averagely the method has a convergence of order 1 in the $L^2$-norm and of order 1/2 in the $H^1$-norm. On the contrary to the one-field FD method, the error in the $H^1(\Omega \setminus \Gamma_\epsilon)$-norm appears to be optimal only with $\alpha_1/\alpha_2 < 1$ while the error for $\alpha_1/\alpha_2 > 1$ in the $H^1(\Omega \setminus \Gamma_\epsilon)$-norm is equivalent to the error in the $H^1$-norm. Indeed, the norm for the two-field problems is defined as a combination of the errors in both fields. Results from the one-field method show that the error is concentrated at the interface, and thus in case of a two-field problem the error on the interface is “transmitted” from the first field to the second field. It follows that a suboptimal $H^1(\Omega \setminus \Gamma_\epsilon)$ rate of convergence is obtained. In fact, further tests show that for $\alpha_1/\alpha_2 < 1$ the $H^1(\Omega \setminus \Gamma_\epsilon)$ convergence rate is almost optimal, indicating an almost optimal rate of convergence in the $H^1(\Omega_2)$-norm.
Remark 12. The two-field FD/BLM may be second-order accurate if \( u'_1 \) is continuous over \( \Omega \), as described in [27]. But such a case is unlikely with our definition of the extension over the whole domain \( \Omega \) of the load \( f_1 \). We note that in order to obtain an optimal method we may seek for an extension \( f_2 \) on \( \Omega_2 \) such that \( u_e \in H^2(\Omega) \) (see for instance the work of [28]).

For the two-field FD/DLM method we observe in Fig. 9 that for a mesh ratio \( h_r \approx 1/2 \) the method has a linear convergence behavior in \( L^2 \). This result is due to the poor approximation of the Lagrange multipliers. That issue also occurs for fitted meshes (see Remark 13). On the contrary, for a mesh ratio \( h_r \approx 2 \) the method converges with oscillatory rates of convergence, but averagely the rate of convergence is linear in \( L^2 \), as depicted in Fig. 8. In general, the \( H^1 \) convergence rate appears closer to 1/2 rather than 1, showing a suboptimal behavior. On the contrary, for the \( H^1(\Omega \setminus \Gamma_\epsilon) \) rate of convergence, identical conclusions can be drawn from the two-field FD/BLM method.

Finally, the two-field DFD/BLM method is second-order accurate, as depicted in Fig. 10. We note that the error is slightly dependent of the material ratio.

To summarize, the fact that the first three methods are not second-order accurate is not surprising, since they are all characterized by a discontinuity in the gradient at the interface; as a consequence, their corresponding solutions are not in \( H^2(\Omega) \) and the rate of convergence cannot be optimal. The solution depends on the position of the interface with respect to the mesh of \( \Omega \), which varies arbitrarily with a \( h \)-refinement, and hence the rates of convergence are not constant. However, results confirm that away from the interface the \( H^1 \) rate of convergence may be optimal, i.e., in the \( H^1(\Omega \setminus \Gamma_\epsilon) \)-norm. Comparing one-field and two-field methods, there is a clear advantage in using a one-field approach since the optimal \( H^1(\Omega \setminus \Gamma_\epsilon) \) convergence rate is recovered on both \( \Omega_1 \) and \( \Omega_2 \). However, for two-field methods the optimal rate of convergence in the \( H^1(\Omega \setminus \Gamma_\epsilon) \)-norm is recovered only restricting it to \( \Omega_1 \) but the rate of convergence is almost optimal when \( \alpha_1/\alpha_2 < 1 \), a case which represents a large class of applications. On the contrary to the previous methods, the two-field DFD method explicitly takes into account the discontinuity at the interface in the finite element space and thus it attains a second-order rate of convergence.

Remark 13. With fitted meshes all methods are second-order accurate, except the two-field FD/DLM method which is only first-order accurate with a mesh ratio strictly lower than 1. The fact that the methods are second-order accurate with fitted meshes results from our finite element basis that allows jumps in the gradient at vertices. In the case of the two-field FD/DLM method with fitted meshes and a mesh ratio strictly lower than 1, the method is not second-order accurate since the Lagrange multiplier space is not rich enough (see Fig. 11).
7.4.2. Finite elements with approximated quadrature

The one-field FD method with approximate quadrature converges with a similar rate of convergence as with an exact quadrature scheme only if the material ratio \( \alpha_1/\alpha_2 \) is lower than one, as depicted in Fig. 5(c), while it simply does not converge otherwise, as depicted in Fig. 5(d). Another noticeable result is that the \( H^1(\Omega \setminus \Gamma_\epsilon) \) rate of convergence is suboptimal with a rate of 1/2. For the one-field FD method the mesh ratio and the quadrature rule are a very important factor. Indeed, for a finer partition of \( \Omega_2 \), or a more precise quadrature rule, the quadrature error is reduced and thus the method may converge even for a large material ratio \( \alpha_1/\alpha_2 \), since the method converges with an exact integration. Nevertheless, the relation between the quadrature rule and the material ratio remains unclear, and thus the approximated quadrature for the one-field FD method requires specific care.

The two-field FD/BLM and two-field FD/DLM methods with approximate quadrature converge for all cases with similar convergence rates as with an exact quadrature, as depicted for the FD/BLM method in Fig. 7, and for the FD/DLM method in Figs. 8 and 9. Such a result follows from the two-field structure of the methods, which implies that as long as the extended Material 1 may converge then the convergence for both fields is maintained. Similar results as with the exact integration cases are found for the errors in the \( H^1(\Omega) \) and \( H^1(\Omega \setminus \Gamma_\epsilon) \) seminorms.

The two-field DF/DLM method loses its second order property in the \( L^2 \)-norm and it appears to converge at most linearly, as depicted in Fig. 10. We show the importance of an exact quadrature for the two-field DF/DLM method in another numerical test by integrating with a higher number of Gauss points on elements crossed by the interface. The results in Fig. 12 show that a clear quadratic rate of convergence is recovered with 400 Gauss points per element crossed by the interface. Notice that when using approximated quadrature the \( H^1(\Omega \setminus \Gamma_\epsilon) \) convergence rate remains optimal and it is almost optimal in the \( H^1(\Omega) \) seminorm.

The integration scheme is an important point to be taken into account since it influences the cost of the method. A first-order accurate method is interesting if it is much faster in terms of computational time than a second-order accurate method. We might therefore use a standard integration scheme with the first-order accurate methods since their convergence rates are not drastically changed with respect to an exact integration scheme (with special care for the one-field FD method). On the contrary, for the two-field DF/DLM method it is mandatory to integrate exactly or a drastic loss in the rate of convergence in the \( L^2 \)-norm has to be expected, but approximated integration appears to have a small impact on the \( H^1 \) and \( H^1(\Omega \setminus \Gamma_\epsilon) \) rates of convergence.
Fig. 10. The two-field DFD/BLM method.

Fig. 11. Convergence rates of the two-field FD/DLM problem for a fitted case and different mesh ratios, with $A = 0, B = 3, C = 4.5, D = 6$. The coefficients are given by $\alpha_1 = 1$ and $\alpha_2 = 4$ (similar results can be obtained with different material ratios). It can be noticed that, for this method, $h_r < 1$ results in first-order convergence also for the fitted case.

7.4.3. Conditioning

The condition numbers for the various methods are presented in Fig. 13. Clearly, the one-field method as well as the methods using boundary Lagrange multipliers show a standard $O(h^{-2})$ conditioning. Instead, the method using the distributed Lagrange multipliers shows a much higher conditioning of order $O(h^{-4})$; this result appears to be independent of the mesh ratio.

8. Discussion on the extension to higher dimensions, with a focus on the discontinuously extended two-field Fictitious Domain method with boundary Lagrange multipliers

The methods discussed in this paper represent a stepping stone for multiple dimensional problems. For instance, the one-field method of Section 3 can be seen as an extremely simplified configuration with respect to more complex fluid–structure interaction problems studied in [18] or [19]. The order of accuracy of these methods is limited by the regularity of the solution.
Fig. 12. Importance of the quadrature scheme for the two-field DFD/BLM method using a standard quadrature with a different number of Gauss points on elements cut by the interface. Since the method is independent of the material ratio, we perform only one test, Test 1 (see Table 1), and the $h$-refinement strategy is described in Table 2. It can be observed that a clear quadratic rate of convergence is recovered with 400 Gauss points.

Fig. 13. Condition numbers of the global linear system of the various methods. The problem under consideration is Test 1 with exact quadrature. However, similar results were obtained for different material parameters with exact and approximated integrations.

and by the fact that the computational mesh does not fit with the interface. The focus of this paper is to compare different approaches in order to identify the numerical schemes that can achieve higher order of convergence also in presence of material discontinuities.

Two-field methods, like the ones discussed in Sections 4 and 5 are saddle point problems and thus they require that an inf–sup condition is satisfied. Interested readers are referred to, e.g., [27] for Boundary Lagrange Multipliers and to [29] or [25] for distributed Lagrangian multiplier. Since these methods are only first-order accurate we choose not to give here a detailed account about their extension, and we focus instead on the two-field DFD/BLM method which, under the assumptions presented in this paper, is second-order accurate.

A simple approach for the Lagrange multiplier (e.g., by constructing a piecewise linear Lagrange multiplier space where the nodes are the intersection points of the interface with the extended mesh) in 2D or 3D results in locking (see for instance [30]). Such a problem occurs since there are more constraints by the Lagrange multiplier than “free” nodes associated with the elements cut by the interface.

We mention three possible strategies to solve the problem:

1. Adding degrees of freedom in the primary field, such as bubbles (see [31]);
2. Lowering the dimension of the Lagrange multiplier space (see [32]);
3. Using stabilization techniques such as Nitsche (see [30]) or Barbosa–Hughes (see [33]).

The first strategy, if used with a piecewise constant Lagrange multiplier, consists in enriching elements cut by the interface with a bubble function (the bubble can be eliminated at the element level by static condensation). It is also possible to apply static condensation a second time to eliminate the Lagrange multiplier, and thus the method shares similarities with the Nitsche method (as shown in the work of [31]). An important feature of the method is that it does not depend on user parameters, unlike the third approach.

The second strategy has been applied to 2D problems in [32] and to 3D problems in [34]. The method consists in properly selecting intersection vertices of the interface with the extended mesh and taking the trace on the interface of the shape functions of the global mesh to build the Lagrange multiplier finite element space. Such a method is called the “Lagrange Multiplier Vital Vertices”. It has been proven in [32] that it satisfies an inf-sup condition for 2D problems. We note that the
main drawback of lowering the dimension of the Lagrange multiplier space is that it reduces the accuracy of the flux at the interface.

Concerning the third strategy, we have to say that the Nitsche method was introduced for interface problems in [7]. However, it has been shown in [30] that this method does not achieve satisfactory results for specific situations, but that a slight modification of the method introducing a second user parameter cures the problem. This method is known as the $\gamma$-Nitsche method. We point out that in [7] the first Nitsche parameter is given by geometrical consideration, and is thus defined locally on each intersected element, while the parameter $\gamma$ may more likely depend on material parameters. Alternatively, in [33], a Barbosa–Hughes stabilization technique is proposed. We note that it has been shown in [35] that such a stabilization technique can be derived from the Nitsche method. Both approaches are dependent on user parameters which depend heavily on how the interface cuts the element. An inappropriate choice of parameters might result in a dramatic loss of accuracy. For instance, specific techniques have to be employed when sub-elements are very small compared to their global elements (see for instance [33]). However, a promising path following [36] on the use of stabilization by projection has been introduced for boundary Lagrange multipliers in, e.g., [37], avoiding the computation of a stabilization parameters. These kinds of approaches constitute currently an active area of research (see, e.g., [38] and [39]).

9. Conclusion

In this paper, we aimed at giving highlights of some fundamental issues of immersed approaches. In particular, within the finite element method, we analyzed various embedded approaches in order to tackle a 1D Poisson problem with different materials in a unified framework. We focused on four embedded methods inspired by the Immersed Boundary, the Fictitious Domain, and the Extended Finite Element method.

Detailed results showed that, in unfitted cases, the first three studied methods are only first-order accurate since they consider a continuous extension using standard finite elements, while the method inspired by the extended finite element method is second-order accurate because the irregularity of the solution at the interface was explicitly taken into account. Moreover, since it seems that for the latter method it is not straightforward conserving the optimal second-order of convergence in higher dimensions while imposing essential constraints inside elements, we also commented on possible extension strategies of such a method to 2D/3D.

We note that one of the main issues regarding the efficiency of the method inspired by the extended finite element method, besides the imposition of essential boundary conditions on the interface, lies in the need to integrate correctly over sub-elements to obtain second-order accuracy. In other words, an important work is required in order to compute the intersection of the interface with the crossed elements. On the contrary, a much cheaper and easier to implement, in particular in 2D/3D, integration scheme may be used for first-order accurate methods without corrupting their rate of convergence. Therefore, a trade-off between computational cost and accuracy has to be always considered when dealing with immersed approaches.

Acknowledgments

This work is funded by: the Cariplo Foundation through the Project no. 2009.2822; the European Research Council through the ERC Starting Grant Project no. 259229; Ministero dell’Istruzione, dell’Università e della Ricerca through the PRIN Project no. 2010BFXRHS.

Appendix. Additional numerical experiments with $\alpha_1/\alpha_2 = 1/100$ and $\alpha_1/\alpha_2 = 100$

See Figs. A.14–A.19

Fig. A.14. Analytical solutions for the numerical test with $f_1 = 1$ on $\mathcal{A}$, $\mathcal{B} \cup \mathcal{C}$, $\mathcal{D}$, $f_2 = 1$ on $\mathcal{B}$, $\mathcal{C}$, for the different material parameters.
Fig. A.15. The one-field FD method.

Fig. A.16. The two-field FD/BLM method.
Fig. A.17. The two-field FD/DLM method with $h_r \approx 2$.

Fig. A.18. The two-field FD/DLM method with $h_r \approx 1/2$. 
Fig. A.19. The two-field DFD/BLM method.

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