Locking-free isogeometric collocation methods for spatial Timoshenko rods

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**Abstract**

In this work we present the application of isogeometric collocation techniques to the solution of spatial Timoshenko rods. The strong form equations of the problem are presented in both displacement-based and mixed formulations and are discretized via NURBS-based isogeometric collocation. Several numerical experiments are reported to test the accuracy and efficiency of the considered methods, as well as their applicability to problems of practical interest. In particular, it is shown that mixed collocation schemes are locking-free independently of the choice of the polynomial degrees for the unknown fields. Such an important property is also analytically proven.

1. Introduction

Isogeometric Analysis (IGA), introduced by Hughes et al.\cite{18,24}, is a powerful analysis tool aiming at bridging the gap between Computational Mechanics and Computer Aided Design (CAD). In its original form IGA has been proposed as a Bubnov-Galerkin method where the geometry is represented by the spline functions typically used by CAD systems and, invoking the isoparametric concept, field variables are defined in terms of the same basis functions used for the geometrical description. This could be therefore viewed as an extension of standard isoparametric Finite Element Methods (FEM), where the computational domain exactly reproduces the CAD description of the physical domain. Moreover, recent works on IGA have shown that the high regularity properties of the employed functions lead in many cases to superior accuracy per degree of freedom with respect to standard FEM (cf., e.g.,\cite{12,19,25,31,32}). Given these unique premises, IGA has been adopted in different fields of Computational Mechanics, and the properties and advantages of this more than promising approach have been successfully tested and analyzed both from the practical and mathematical standpoints (see, among others, [2,5–11,13–15,20,22,26,28–30,33,34]).

The original basic concept of IGA (i.e., the use of basis functions typical of CAD systems within an isoparametric paradigm) can be also exploited beyond the framework of classical Galerkin methods. In particular, isogeometric collocation schemes have been recently proposed in\cite{3}, as an appealing high-order low-cost alternative to classical Galerkin approaches. Such techniques have also been successfully employed for the simulation of elastostatic and explicit elastodynamic problems\cite{4} and their application to many other applications of engineering interest is currently the object of extensive research.

Within this context, a comprehensive study on the advantages of isogeometric collocation over Galerkin approaches is reported in\cite{32}, where the superior behavior in terms of accuracy-to-computational-time ratio guaranteed by collocation with respect to Galerkin is revealed. In the same paper, adaptive isogeometric collocation methods based on local hierarchical refinement of NURBS are introduced and analyzed, as well.

In view of the results briefly described above, isogeometric collocation clearly proposes itself as a viable and efficient implementation of the main IGA basic concepts.

In addition to this, isogeometric collocation has shown a remarkable and, to our knowledge, unique property when employed for the approximation of Timoshenko beam problems. In fact, it has been analytically proven and numerically tested in\cite{16} that mixed collocation schemes for initially straight planar Timoshenko beams are locking-free without the need of any...
compatibility condition between the selected discrete spaces. We highlight that this appealing property is deeply linked to the collocation approach adopted and not only a consequence of the isogeometric paradigm.

Moving along this promising research line, in the present paper, we aim at extending such results to the more interesting case of curved spatial Timoshenko rods of arbitrary initial geometry, limiting the discussion to the case of small displacements and displacement gradients.

The outline of the paper is as follows. In Section 2, we present the mechanical model of spatial Timoshenko rods and introduce the strong form formulation of the problem, in both displacement-based and mixed forms. Section 3 gives a brief review of one-dimensional B-Splines and NURBS as basis for isogeometric analysis. In Section 4, isogeometric collocation is introduced and collocation schemes for spatial Timoshenko rods are explained in detail. The proposed methods are tested on several numerical examples in Section 5, confirming their accuracy and showing possible applications. It is shown that collocation methods based on mixed formulations are locking-free for any choice of polynomial degrees for the different fields. This characteristic is also analytically proven by a theoretical convergence analysis in Section 6. Conclusions are finally drawn in Section 7.

2. The spatial Timoshenko rod equations

In the following we want to introduce a model describing a spatial rod, as, for example, the one reported in Fig. 1, which is clamped on the lower end and subjected to an external distributed load as well as concentrated loads and moments on the upper end. The model is developed under the assumptions of small displacements and displacement gradients, assuming a first-order Timoshenko-like shear deformation and following the approach proposed in [1].

2.1. Geometry description

The rod geometry is described by a spatial curve $\gamma(s)$. The curve parameter $s$ is chosen to be the arc-length parameter and we denote with $[]$ the derivative with respect to the arc-length parameter, i.e., $() = d/ds$. The unit tangent vector of the curve at a point $\gamma(s)$ is defined by

$$
\mathbf{t}(s) = \gamma'(s) = \frac{d\gamma(s)}{ds} \quad \text{for } s \in [0, l],
$$

where $l > 0$ denotes the curve length. Fig. 2 shows a part of the curve of Fig. 1 from $\gamma(0)$ up to an arbitrary location $\gamma(s)$, along with the position vector and the unit tangent vector. In the following, all variables are assumed as functions of the arc-length parameter $s$ (unless otherwise specified) also if we omit to explicitly indicate such a dependency.

2.2. Kinematic equations

Adopting a Timoshenko beam-like model, the rod deformation can be described by the centerline curve $\gamma$, a displacement vector $\mathbf{v}$, and a rotation vector $\varphi$. Accordingly, we may introduce the generalized strains $s$ and $\chi$, respectively defined as the vector of translational (axial and shear) strains and the vector of rotational (torsional and flexural) strains. In particular, the translational strains are obtained by the first derivative of the displacements subtracting the rigid body rotations, whereas the rotational strains are obtained by the first derivatives of the rotations:

$$
\varepsilon = \mathbf{v}' - \varphi \times \mathbf{t},
$$

$$
\chi = \varphi'.
$$

2.3. Constitutive equations

As usual for rod formulations, we introduce a vector $\mathbf{n}$ of internal forces and a vector $\mathbf{m}$ of internal moments. In the following, we assume a linear elastic constitutive relation in the form:

$$
\mathbf{n} = C \varepsilon,
$$

$$
\mathbf{m} = D \chi.
$$

Using an intrinsic basis, i.e., a basis composed by three orthogonal unit vectors with the first one equal to the tangent vector, the material matrices $C$ and $D$ are defined by:

$$
C = \text{diag } [EA, GA_1, GA_2],
$$

$$
D = \text{diag } [\gamma J, EI_1, EI_2],
$$

where $E$ and $G$ are the elastic and the shear modulus, $A$ the cross sectional area, $A_1 = k_1 A$ and $A_2 = k_2 A$ (being $k_1$ and $k_2$ the shear correction factors), $J$ the torsion constant and $I_1$ and $I_2$ the second moments of inertia. Within such a formulation, the components of $\mathbf{n}$ represent the axial force and the two components of the shear force, respectively, while the components of $\mathbf{m}$ represent the torsional moment and the two components of the bending moment, respectively.
2.4. Equilibrium equations

The equations ensuring the equilibrium of external loads with internal forces \( \mathbf{n} \) and moments \( \mathbf{m} \) are to be considered separately for the internal part of the domain and the boundaries. Given a vector \( \mathbf{f} \) describing the external load per unit length acting on the rod, the equilibrium equations on the internal read (see [1]):

\[
\mathbf{n}' + \mathbf{f} = \mathbf{0}, \quad (8) \\
\mathbf{m}' + \mathbf{t} \times \mathbf{n} = \mathbf{0}. \quad (9)
\]

The equilibrium conditions at free boundaries are instead given by:

\[
\mathbf{n} = \mathbf{n}_{E}, \quad (10) \\
\mathbf{m} = \mathbf{m}_{E}. \quad (11)
\]

where \( \mathbf{n} \) and \( \mathbf{m} \) are external boundary forces and moments, respectively, as depicted in Fig. 1.

We remark that in Fig. 1, external boundary forces and moments are applied at the upper end of the rod, whereas the lower end is clamped. Of course, such boundary forces and moments can be applied to both ends, and equations (10) and (11) refer to both ends of the rod. Furthermore, we remark that in case of “mixed” boundary conditions, where only certain components of the displacement variables are free (for example, a simple support), only the corresponding components of Eqs. (10) and (11) apply.

The interested reader is referred to [1] for more details on the derivation of the governing equations for spatial Timoshenko rods.

2.5. Displacement-based formulation

Using the kinematic and constitutive relations, given in equations (2)-(5), equilibrium equations (8) and (9) can be rewritten in terms of the displacement variables \( \mathbf{v} \) and \( \mathbf{\varphi} \) only, as follows:

\[
\mathbf{C}(\mathbf{v}' - \mathbf{\varphi} \times \mathbf{t} - \mathbf{\varphi} \times \mathbf{t}') + \mathbf{f} = \mathbf{0}, \quad (12) \\
\mathbf{D}\mathbf{\varphi}' + \mathbf{t} \times \mathbf{C}(\mathbf{v}' - \mathbf{\varphi} \times \mathbf{t}) = \mathbf{0}. \quad (13)
\]

with boundary conditions:

\[
\mathbf{C}(\mathbf{v}' - \mathbf{\varphi} \times \mathbf{t}) = \mathbf{n}, \quad (14) \\
\mathbf{D}\mathbf{\varphi}' = \mathbf{m}. \quad (15)
\]

Eqs. (12)-(15) represent the strong form description of the spatially curved Timoshenko rod problem, in a displacement-based formulation.

2.6. Mixed formulation

For the mixed formulation, we introduce the internal force vector \( \mathbf{n} \) as an additional independent variable. As a consequence, the equilibrium of forces remains as formulated in (8), and the equilibrium of moments (9) is rewritten in terms of \( \mathbf{n} \) and \( \mathbf{\varphi} \):

\[
\mathbf{n}' + \mathbf{f} = \mathbf{0}, \quad (16) \\
\mathbf{D}\mathbf{\varphi}' + \mathbf{t} \times \mathbf{n} = \mathbf{0}. \quad (17)
\]

Along with the additional unknown variable, a third equation is added to the system, describing the relation between \( \mathbf{n} \) and the primal variables \( \mathbf{v} \) and \( \mathbf{\varphi} \). This relation is given by the constitutive Eq. (4), expressing the strains \( \varepsilon \) in terms of \( \mathbf{v} \) and \( \mathbf{\varphi} \) via relation (2):

\[
\mathbf{n} = \mathbf{C}(\mathbf{v}' - \mathbf{\varphi} \times \mathbf{t}) = \mathbf{0}. \quad (18)
\]

Eqs. (16) and (17) describe the equilibrium equations, while (18) represents the constitutive law for internal forces.

The boundary terms in the mixed formulation read as:

\[
\mathbf{n} = \mathbf{n}_{E}, \quad (19) \\
\mathbf{D}\mathbf{\varphi}' = \mathbf{m}_{E}. \quad (20)
\]

Eqs. (16)-(20) represent the strong form description of the spatially curved Timoshenko rod problem, in a mixed formulation.

3. NURBS-based isogeometric analysis

For solving the spatial rod equations given in the previous section, we adopt the concept of isogeometric analysis, where both the geometry and the unknown variables are discretized by Non-Uniform Rational B-Splines (NURBS). The aim of this section is to present a short description of B-splines and NURBS, followed by a simple discussion on the basics of isogeometric analysis.

3.1. B-splines and NURBS

B-Splines are smooth approximating functions constructed by piecewise polynomials. A B-spline curve in \( \mathbb{R}^d \) is composed of linear combinations of B-spline basis functions and coefficients (B.). These coefficients are points in \( \mathbb{R}^d \), referred to as control points.

To define such functions we introduce a knot vector as a set of non-decreasing real numbers representing coordinates in the parametric space of the curve

\[
\{\xi_1, \ldots, \xi_{n+p+1} = 1\}, \quad (21)
\]

where \( p \) is the order of the B-spline and \( n \) is the number of basis functions (and control points) necessary to describe it. The interval \( [\xi_i, \xi_{i+p+1}] \) is called a patch. A knot vector is said to be uniform if its knots are uniformly-spaced and non-uniform otherwise; it is said to be open if its first and last knots have multiplicity \( p + 1 \). In what follows, we always employ open knot vectors. Basis functions formed from open knot vectors are interpolatory at the ends of the parametric interval \([0,1]\) but are not, in general, interpolatory at interior knots.

Given a knot vector, univariate B-spline basis functions are defined recursively as follows. For \( p = 0 \) (piecewise constants):

\[
N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\
0 & \text{otherwise} \end{cases}, \quad (22)
\]

for \( p \geq 1 \):

\[
N_{i,p}(\xi) = \begin{cases} \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) & \text{if } \xi_i \leq \xi < \xi_{i+p} \\
\frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) & \text{if } \xi_i \leq \xi < \xi_{i+p+1} \\
0 & \text{otherwise} \end{cases}, \quad (23)
\]

where, in (23), we adopt the convention \( 0/0 = 0 \).
Fig. 4. Quartic NURBS curve with control net (dotted) describing the rod geometry shown in Fig. 1.

In Fig. 3 we present an example consisting of \( n = 9 \) cubic basis functions generated from the open knot vector \( \{0, 0, 0, 0, 1/6, 1/3, 1/2, 2/3, 5/6, 1, 1, 1, 1\} \). If internal knots are not repeated, B-spline basis functions are \( C^{p-1} \)-continuous. If a knot has multiplicity \( m \) the basis is \( C^{m-1} \)-continuous at that knot, where \( k = p - m \). In particular, when a knot has multiplicity \( p \) the basis is \( C^0 \) and interpolates the control point at that location. We define

\[
S = \text{span}(N_{ip}(\xi) , i = 1, \ldots, n)
\]  

(24)

To obtain a NURBS curve in \( \mathbb{R}^3 \), we start from a set \( B^p \in \mathbb{R}^3(i = 1, \ldots, n) \) of control points ("projective points") for a B-spline curve in \( \mathbb{R}^3 \) with knot vector \( \Xi \). Then the control points for the NURBS curve are

\[
[B]_{ik} = \left[B^p\right]_{ik} \omega_i, \quad k = 1, 2, 3
\]  

(25)

where \( [B]_{ik} \) is the \( k \)th component of the vector \( B_i \) and \( \omega_i = [B^p]_{ik} \) is referred to as the \( i \)th weight. The NURBS basis functions of order \( p \) are then defined as

\[
R_p^i(\xi) = \frac{N_p^i(\xi) \omega_i}{\sum_{j=1}^n N_p^j(\xi) \omega_j}
\]  

(26)

The NURBS curve \( x(\xi) \) is defined by

\[
x(\xi) = \sum_{i=1}^n R_p^i(\xi) B_i
\]  

(27)

As an example, in Fig. 4, we present the NURBS model of the rod curve shown in Fig. 1.

As usual, we denote the support of the curve \( x \) by \( \Gamma(x) \), hence \( \Gamma(x) \subset \mathbb{R}^3 \). In addition, we suppose that the map \( x : [0,1] \rightarrow \Gamma(x) \) is smooth and invertible, with smooth inverse denoted by \( x^{-1} : \Gamma(x) \rightarrow [0,1] \).

Note that the NURBS parameterization is here indicated with \( x \) and not with \( y \) since, unlike the map used to describe the model (see Section 2.1), it does not need to be associated to the arc-length coordinate. However, we notice that the supports are equal, i.e., it holds \( \Gamma(x) = \Gamma(y) \).

Following the isoparametric approach, the space of NURBS vector fields on \( \Gamma(x) \) is defined, component by component as the span of the push-forward of the basis functions \( 26 \):

\[
V_n = \text{span}(R_p \circ x^{-1}, i = 1, \ldots, n). 
\]  

(28)

We finally note that the images of the knots through the function \( x \) naturally define a partition of the curve support \( \Gamma(x) \subset \mathbb{R}^3 \), called the associated mesh \( M_n, h \) being the mesh-size, i.e., the largest size of the elements in the mesh.

4. Isogeometric discretization and collocation

To solve the differential equations described in Sections 2.5 and 2.6, the unknown fields are approximated by NURBS functions, and the corresponding control coefficients are determined solving Eqs. (12) and (13) (displacement-based formulation) or (16)–(18) (mixed formulation), which are collocated at the physical images of the Greville abscissae of the unknown field knot vectors as described in the following (cf. [16]).

The Greville abscissae related to a spline space of degree \( p \) and knot vector \( \{\xi_1, \ldots, \xi_{np+1}\} \) of the parametric space defined by:

\[
\xi_i = \frac{\xi_{i+1} + \xi_{i+2} + \ldots + \xi_{i+p}}{p}, \quad i = 1, \ldots, n.
\]  

(29)

In the following, we need to construct also the Greville abscissae related to the \( k \)th derivative space, which are defined as:

\[
\xi_i^k = \frac{\xi_{i-1} + \xi_{i+2} + \ldots + \xi_{i+p}}{p - k}, \quad i = 1, \ldots, n - k.
\]  

(30)

4.1. Displacement-based formulation

Within an isogeometric framework, the approximating fields \( \mathbf{v}_h \) (displacements) and \( \mathbf{\varphi}_h \) (rotations) are discretized, independently of each other, by NURBS basis functions as follows:

\[
\mathbf{v}_h(\xi) = \sum_{i=1}^{n_0} R_{ip}(\xi) \mathbf{v}_i,
\]  

(31)

\[
\mathbf{\varphi}_h(\xi) = \sum_{i=1}^{n_0} R_{ip}(\xi) \mathbf{\varphi}_i,
\]  

(32)

where \( \mathbf{v}_i \) and \( \mathbf{\varphi}_i \) are the unknown control variables and \( R_{ip}(\xi) \) are the NURBS basis functions as described in Section 3. As it can be seen, the three components of each field are discretized by the same functions. With these approximations, the equilibrium equations (12)–(15) are collocated at the images of the Greville abscissae \( x(\xi_i^k) \). Since the discretization spaces for displacements and rotations are not necessarily equal, we have two sets of collocation points, \( x(\xi_i^p) \) and \( x(\xi_i^3) \), referring to the displacement and the rotation space, respectively.

Therefore, we adopt the following collocation scheme for the displacement-based formulation:

1. The equation of translational equilibrium is collocated at the images of the Greville abscissae for the displacement space, excluding the boundaries.

2. The equation of rotational equilibrium is collocated at the images of the Greville abscissae for the rotation space, excluding the boundaries.

This can be formally stated as follows:

\[
C(\mathbf{v}_h - \mathbf{\varphi}_h \times \mathbf{t} - \mathbf{\varphi}_h \times \mathbf{t}) + \mathbf{f} = 0 \quad \text{on} \quad x(\xi_i^p), \quad i = 2, \ldots, n_0 - 1,
\]  

(33)

\[
D\mathbf{\varphi}_h + \mathbf{t} \times C(\mathbf{v}_h - \mathbf{\varphi}_h \times \mathbf{t}) = 0 \quad \text{on} \quad x(\xi_i^3), \quad i = 2, \ldots, n_0 - 1.
\]  

(34)

In the case of natural boundary conditions, the boundary equations of equilibrium to be collocated are:

\[
C(\mathbf{v}_h - \mathbf{\varphi}_h \times \mathbf{t}) = \mathbf{n} \quad \text{at free ends},
\]  

(35)

\[
D\mathbf{\varphi}_h = \mathbf{m} \quad \text{at free ends}.
\]  

(36)

In case of essential boundary conditions, these are directly applied to the respective degrees of freedom. In case of "mixed" boundaries,
where only certain degrees of freedom are prescribed (e.g., a simple support), only the respective components of equations (35) and (36) are imposed.

We want to stress that Eqs. (33) and (34) are collocated on the standard Greville abscissae associated to the approximating spaces. Alternatively, collocation can be performed at the Greville abscissae associated with the second derivative spaces. In this case, equations (33) and (34) are collocated at all the images of the Greville abscissae (which are less than the number of unknowns, since the knots used for computing them are smaller than the original knots) and the system is made square by the addition of the equations imposing boundary conditions.

The interested reader is referred to [4,16] for more details on collocation at Greville abscissae and on boundary condition imposition.

4.2. Mixed formulation

Analogously, for the mixed formulation, the three approximating fields $\mathbf{v}_h$ (displacements), $\phi_h$ (rotations), and $n_h$ (internal forces) are discretized, independently of each other, by NURBS basis functions as follows:

$$\mathbf{v}_h(\xi) = \sum_{i=1}^{n_\mathbf{v}} R_{hi}^\mathbf{v}(\xi) \mathbf{v}_i,$$

$$\phi_h(\xi) = \sum_{i=1}^{n_\phi} R_{hi}^\phi(\xi) \phi_i,$$

$$n_h(\xi) = \sum_{i=1}^{n_n} R_{hi}^n(\xi) n_i,$$

where, again, $\mathbf{v}_i, \phi_i,$ and $n_i$ are the unknown control variables and $R_i^\mathbf{v}(\xi)$ are the NURBS basis functions as described in Section 3. As in the displacement-based formulation, the three components of each field are discretized by the same NURBS functions. For the mixed formulation, we apply the following collocation scheme:

1. The equation of translational equilibrium is collocated at the images of the Greville abscissae associated to the first derivatives of the space of internal forces.

2. The equation of rotational equilibrium is collocated at the images of the Greville abscissae associated to the second derivatives of the space of rotations.

3. The constitutive equation of internal forces is collocated at the images of the Greville abscissae associated to the first derivatives of the space of displacements.

As it can be seen, in this collocation scheme the collocation points are chosen from the derivative spaces, such that the order of the derivative space is equal to the highest order of derivative of the variable governing the collocation scheme for the respective equation. This can be formally stated as follows:

$$\mathbf{n}_h + f = 0 \quad \text{on} \quad \mathbf{x}^{(n)}(\xi), \quad \text{for} \quad i = 1, \ldots, n_n - 1,$$

$$D\phi_i + t \times \mathbf{n}_h = 0 \quad \text{on} \quad \mathbf{x}^{(n)}(\xi), \quad \text{for} \quad i = 1, \ldots, n_\phi - 2,$$

$$\mathbf{n}_h - C(\mathbf{v}_i - \phi_h \times t) = 0 \quad \text{on} \quad \mathbf{x}^{(n)}(\xi), \quad \text{for} \quad i = 1, \ldots, n_n - 1,$$

where the sets of collocation points $\mathbf{x}^{(n)}(\xi)$ refer to the images of the Greville abscissae associated to the first derivative space of displacements, the second derivative space of rotations, and the first derivative space of internal forces, respectively.

In the case of natural boundary conditions, the boundary equations of equilibrium to be collocated are:

$$\mathbf{n}_h = \mathbf{n} \quad \text{at free ends},$$

$$D\phi_h = \mathbf{m} \quad \text{at free ends}.$$

In case of essential boundary conditions, these are directly applied to the respective degrees of freedom, and in case of “mixed” boundaries (e.g., a simple support) only the respective components of equations (43) and (44) are imposed.

Remark 4.1. We highlight that, as it is numerically shown in Section 5 and rigorously proven in Section 6, the presented mixed formulation is locking-free for any choice of the discrete spaces for displacements, rotations, and internal forces. The same remarkable property has been recently proven and computationally tested in [16] for the simpler case of initially straight beams.

5. Numerical tests

In this section, different numerical experiments are shown to test the accuracy of the method and to numerically prove the property of Remark 4.1 for the proposed mixed formulation. Also, an
example showing possible applications of the method in a geometrically complicated situation (i.e., an elastic spring) is reported.

### 5.1. Straight cantilever beam

The first example is a simple cantilever beam which is used to test the behavior of the considered formulations under all possible strain modes, i.e., tension, pure bending, bending with shear, and torsion. The beam is clamped on one end and subjected to six different load cases on the other end. Fig. 5 shows the problem setup. The resulting deformation is compared to the analytical solution from linear beam theory (including shear deformations). Since for all six cases the analytical solution is a polynomial of maximum order $p = 3$, cubic NURBS are used for the discretized model. In accordance with linear beam theory, one cubic element yields the exact solution (up to machine precision) for all six load cases, for both the displacement-based and the mixed formulations.

### 5.2. Circular arch with out-of-plane load

In this example, a 90° circular arch, of radius $r$ and thickness $t$, is clamped on one end and subjected to an out-of-plane concentrated load on the other end, as sketched in Fig. 6.

An analytical solution for this example can be obtained by hand calculation. Since the system is statically determinate, stress resultants can be obtained directly from equilibrium considerations. Fig. 7 shows the stress resultants as functions of the angle $\theta$, expressed in terms of an intrinsic basis $[c_1, c_2, c_3]$. The solution for displacements and rotations can be obtained by integration of the stress resultants using the constitutive and kinematic relations described in equations (2)–(5). The analytical solution for the displacements in $z$-direction is therefore:

$$v_z = \frac{F_z r}{2EI} \theta + \frac{F_z r^3}{6EI} (\theta + \cos(\theta) - \sin(\theta)) + \frac{1}{2} \theta \sin(\theta) - \frac{1}{6} \theta^3 \cos(\theta).$$  \hspace{1cm} (45)

This reference solution is used to compute the $L^2$-norm of the error of displacements.

For numerical analysis, the rod is discretized employing both the displacement-based and the mixed formulations, and convergence studies are performed for two different slenderness ratios, namely $t/r = 10^{-1}$ and $t/r = 10^{-2}$, and for polynomial degrees ranging from 3 to 8. The $L^2$-norm of the error of displacements is plotted versus the total number of collocation points in Figs. 8–10.

Fig. 8 shows the convergence plots using the displacement-based method with equal orders for displacements and rotations. Dashed lines are added for comparison with the reference orders of convergence. Dashed lines indicate reference orders of convergence. Fig. 9 shows the convergence plots using the mixed method with equal orders for displacements, rotations, and internal forces. Dashed lines are added for comparison with the reference orders of convergence.
the slenderness. These results confirm the locking-free properties of mixed collocation methods. This characteristic is independent of the choice of spaces for the three fields, i.e., the spaces for displacements, rotations, and internal forces can be chosen freely without any \textit{inf-sup}-like condition to be fulfilled by the spaces (see Remark 4.1 and the theoretical results of Section 6). To further test this, the example is repeated with the following “exotic” choice of polynomial degrees: \( p_v = p_u - 1 = p_n - 2 \). The results are plotted in Fig. 10 and confirm the behavior described above. It should also be noted that in Figs. 9 and 10, zig-zagging of the curves due to converged results is really at machine precision, for both the thick and the thin cases, which proves that the mixed formulation does not cause conditioning problems.

5.3. Elastic spring

In this final example, an elastic spring is clamped on one end and loaded on the other end by a concentrated force. The objective is to show possible applications of the method in a geometrically complicated situation. The spring is made up of 10 coils with a total height of 5 cm and a coil radius of 1 cm. The cross section diameter of the wire (i.e., the rod thickness) is 0.1 cm. The material parameters are chosen as \( E = 10^4 \text{kN/cm}^2 \) and \( v = 0.2 \), and the shear correction factors are set to \( k_1 = k_2 = 5/6 \). The geometry is modeled by a single NURBS curve with \( p = 5,158 \) control points and 153 non-zero knot spans. The problem is solved using the mixed formulation, where the same discretization as for the geometry is used for all involved fields. The total number of degrees of freedom is 1422. The spring is clamped at the bottom end, and subjected to different loadings at the tip. The loading conditions we consider are an axial load (pull) \( F_z = 0.1 \text{kN} \), a lateral force along \( x \), \( F_x = 0.01 \text{kN} \), and a lateral force along \( y \), \( F_y = -0.01 \text{kN} \). The deformed configurations induced by the different loadings are reported in Fig. 11 along with the initial undeformed configuration. A qualitatively good behavior is observed (no analytical solution is available in this case for a quantitative error analysis).

6. Theoretical convergence analysis

In this section we present the theoretical convergence analysis of the proposed method for the \textit{mixed} formulation. As usual in the study of thin structure problems, we consider a suitable scaled problem, which makes it possible to theoretically investigate the possible occurrence of undesirable numerical effects (such as locking phenomena) for the discrete scheme under consideration. Following the approach of [1], the derivation of the scaled model is briefly considered in Section 6.1.

6.1. A scaled model

We here briefly review the scaled model of [1]. First of all, in order to include the NURBS parameterization directly into the analysis, the parameterization used to describe the rod is not assumed to be the curvilinear abscissae anymore. Therefore, we assume that the rod axis is defined by a NURBS curve \( \mathbf{a}(\xi) \), with \( \xi \in [0,1] \), see (27). Accordingly, the tangent vector, indicated directly by \( \mathbf{a}'(\xi) \) relates to \( \mathbf{t}(s) \) by (cf. (1)): 

![Fig. 10. Circular arch with out-of-plane load, solved by a mixed collocation method with \( p_v = p_u - 1 = p_n - 2 \). L^2-norm of displacements error versus number of total collocation points, for \( t/r = 10^{-3} \) (left) and \( t/r = 10^{-4} \) (right). Dashed lines indicate reference orders of convergence.](image)

![Fig. 11. Circular elastic spring subjected to different loadings at the tip. Undeformed configuration (a), load along \( z \), \( F_z = 0.1 \text{kN} \) (b), load along \( x \), \( F_x = 0.01 \text{kN} \) (c), load along \( y \), \( F_y = 0.01 \text{kN} \) (d).](image)
\[ \mathbf{x}(\xi) = \mathbf{t}(\xi) \frac{ds}{ds} \quad \text{if} \quad \mathbf{x}(\xi) = \gamma(\xi). \]  
\( \text{Proposition 6.1.} \) There exists a unique solution \((\mathbf{q}, \mathbf{v}, \mathbf{r}) \in C^0[0,1] \times C^1[0,1] \times C^1[0,1]\) to Problem (47) (and, therefore, also to Problem (48)). Moreover, it holds:

\[ \|\mathbf{q}\|_{W^{2,\infty}} + \|\mathbf{v}\|_{W^{2,\infty}} + \|\mathbf{r}\|_{W^{2,\infty}} \leq C\|\mathbf{q}\|_{L^\infty}. \]  

6.2. Brief review of the proposed collocation scheme

In this section, we review the collocation method for the Timoshenko rod introduced previously, now written in terms of the scaled model and using a slightly different notation, more suitable for the theoretical analysis.

Before proceeding, we need to recall the NURBS space \( \Phi_h \subset C^2[0,1] \) used for the rotation approximation, and associated with the knot vector

\[ \{ \xi^0 = 0, \ldots, \xi^n = 1 \}. \]  

The knot vector (50) will be used for each of the three components of the approximated rotation field. Accordingly, we set (cf. (24) and (28)):

\[ \Phi_h = (V_n)^3. \]  

Analogously, we remind the NURBS space

\[ V_h = (V_n)^3 \subset C^1[0,1] \]  

for the displacement approximation, and associated with the knot vector (used component-wise):

\[ \{ \xi^0 = 0, \ldots, \xi^n = 1 \}. \]  

Finally, we recall the NURBS space

\[ \Gamma_h = (V_n)^3 \subset C^1[0,1] \]  

for the internal force approximation, and associated with the knot vector (used component-wise):

\[ \{ \xi^0 = 0, \ldots, \xi^n = 1 \}. \]  

We notice that it holds

\[ \dim(\Phi_h) = 3n_v; \quad \dim(V_h) = 3n_v; \quad \dim(\Gamma_h) = 3n_r. \]

Remark 6.1. We remark that the three knot vectors above induce, in principle, three different meshes:

\[ \mathcal{M}_{\Phi_h}; \quad \mathcal{M}_{V_h}; \quad \mathcal{M}_{\Gamma_h}. \]

with corresponding mesh-sizes \( h_{\Phi_h}, h_{V_h} \) and \( h_{\Gamma_h} \). In the sequel, we will set \( h = \max(h_{\Phi_h}, h_{V_h}, h_{\Gamma_h}) \). However, we notice that, in practical applications, the three meshes most often coincide.

Remark 6.2. We remark that, in principle, one might also think of using different knot vectors for the different components of the approximated fields. However, this latter choice does not seem to be of practical interest.

In the sequel, we will also use the spaces of first and second derivatives:

\[ \Phi_h^\prime := \{ \mathbf{q}_h : \mathbf{q}_h \in \Phi_h \}; \quad V_h := \{ \mathbf{v}_h : \mathbf{v}_h \in V_h \}; \quad \Gamma_h := \{ \mathbf{r}_h : \mathbf{r}_h \in \Gamma_h \}, \]

whose dimensions are given by \( \dim(\Phi_h^\prime) = 3(n_v - 2); \dim(V_h) = 3(n_v - 1); \) and \( \dim(\Gamma_h) = 3(n_r - 1); \) see (56). Furthermore, we introduce suitable sets of collocation points in \([0,1]\):

\[ \mathcal{N}(\Phi_h^\prime) = \{ x_1, x_2, \ldots, x_{n_{\Phi_h^\prime}} \}; \quad \mathcal{N}(V_h) = \{ y_1, y_2, \ldots, y_{n_{V_h}} \}; \quad \mathcal{N}(\Gamma_h) = \{ z_1, z_2, \ldots, z_{n_{\Gamma_h}} \}. \]
We notice that it holds $3(\#(N'(\mathbf{Q}_s))) = \dim(\mathbf{Q}_0) - 6.3(\#(N'(\mathbf{V}_h))) = \dim(\mathbf{V}_h) - 3$, and $3(\#(N'(\Gamma_h))) = \dim(\Gamma_h) - 3$. Therefore, we have (cf. (48)):

$$3(\#(N'(\mathbf{Q}_s))) + 3(\#(N'(\mathbf{V}_h))) + 3(\#(N'(\Gamma_h))) + (\#(boundary\ conditions)) = \dim(\mathbf{Q}_0) + \dim(\mathbf{V}_h) + \dim(\Gamma_h).$$

(59)

We are now able to present the proposed scheme rewritten in the new setting of this section. Given the finite dimensional spaces defined in (51), (52), and (54), together with the collocation points introduced in (58), the discretized problem reads as follows.

$$\begin{align*}
&\text{Find } (\phi_0, v_h, \tau_h) \in \mathbf{Q}_0 \times \mathbf{V}_h \times \Gamma_h \text{ such that:} \\
&\quad -E'(z_i)\phi(s)(x)_i - E'(z_i)\phi'(s)(x)_i - z'(x)_i \times \tau_0(x)_i = 0, \quad z_i \in N'(\mathbf{Q}_s), \\
&\quad v_h(x)_i - \phi_0(x)_i \times x'(x)_i - d^2 A^{-1}(z)_i \tau_0(y)_i = 0, \quad y_i \in N'(\mathbf{V}_h), \\
&\quad v_h(0)_i = v_h(1)_i = 0, \\
&\quad \phi_0(1)_i = \phi(1)_i = 0.
\end{align*}$$

(60)

Notice that, according with (56) and (59), problem (60) is a linear system of $3(n_x + n_y + n_z)$ equations for $3(n_x + n_y + n_z)$ unknowns.

We finally present the following fundamental assumption on the collocation points.

**Assumption 6.1 (Stable interpolation).** There exists a constant $C_{\text{col}}$ independent of the knot vectors, such that the following holds. For all functions $x, w$, and $r$ in $C^0[0,1]^2$ there exist unique interpolating functions $x_0(x)_i = x(x)_i$, $w_0(z)_i = w(z)_i$, $r_0(y)_i = r(y)_i$, with the bounds

$$\|x_0\|_{L^\infty} \leq C_{\text{col}}\|x\|_{L^\infty},$$

$$\|w_0\|_{L^\infty} \leq C_{\text{col}}\|w\|_{L^\infty},$$

$$\|r_0\|_{L^\infty} \leq C_{\text{col}}\|r\|_{L^\infty}.$$  

A discussion on possible practical collocation points satisfying Assumption 6.1 can be found in Section 6.6.

6.3. A useful splitting

Our starting point is the following splitting of the solution $$(\varphi, v, \tau)$$ to problem (48). Indeed, by linearity, we may write

$$\begin{align*}
&\varphi = \varphi_0 + \varphi, \\
v = v_0 + v, \\
\tau = \tau_0 + \tau
\end{align*}$$

(61)

where $$(\varphi_0, v_0, \tau_0)$$ is the solution to the problem:

$$\begin{align*}
&\text{Find } (\phi_0, v_0, \tau_0) \in C^0[0,1] \times C^1[0,1] \times C^1[0,1] \text{ such that:} \\
&\quad -\tau_0(\xi)_0 = q(\xi), \quad \xi \in [0,1], \\
&\quad -E'(\xi)\phi_0'(\xi)_0 - E'(\xi)\phi'(\xi)_0 - x'(\xi)_0 \times \tau_0(\xi)_0 = 0, \quad \xi \in [0,1], \\
&\quad v_0(0)_0 = v_0(1)_0 = 0, \\
&\quad \phi_0(0)_0 = \phi_0(1)_0 = 0.
\end{align*}$$

(62)

and $$(\bar{\varphi}, \bar{v}, \bar{\tau})$$ is the solution to the problem:

$$\begin{align*}
&\text{Find } (\bar{\varphi}, \bar{v}, \bar{\tau}) \in C^0[0,1] \times C^1[0,1] \times C^1[0,1] \text{ such that:} \\
&\quad -\bar{\tau}(\xi) = 0, \quad \xi \in [0,1], \\
&\quad -E(\xi)\bar{\varphi}'(\xi)_0 - E(\xi)\bar{\varphi}'(\xi)_0 - x'(\xi)_0 \times \bar{\tau}(\xi)_0 = 0, \quad \xi \in [0,1], \\
&\quad \bar{v}(0) = -v_0(0), \quad \bar{v}(1) = 0, \\
&\quad \bar{\varphi}(0) = \bar{\varphi}(1) = 0.
\end{align*}$$

(63)

Using the variational approach of [17] and standard regularity results, one gets the following propositions.

**Proposition 6.2.** There exists a unique solution $$(\varphi_0, v_0, \tau_0) \in C^0[0,1] \times C^1[0,1] \times C^1[0,1]$$ to Problem (62). Moreover, it holds:

$$\|\varphi_0\|_{W^{2,\infty}} + \|v_0\|_{W^{2,\infty}} + \|	au_0\|_{W^{1,\infty}} \leq C\|q\|_{L^\infty}.$$  

(64)

**Proposition 6.3.** There exists a unique solution $$(\varphi, v, \tau) \in C^0[0,1] \times C^1[0,1] \times C^1[0,1]$$ to Problem (63). Moreover, it holds:

$$\|\varphi\|_{W^{2,\infty}} + \|v\|_{W^{2,\infty}} + \|	au\|_{W^{1,\infty}} \leq C\|v_0\|_{L^\infty}.$$  

(65)

An analogous splitting holds for the solution $$(\varphi_0, v_0, \tau_0)$$ to problem (60). Indeed, we may write

$$\begin{align*}
&\varphi = \varphi_0 + \varphi, \\
v = v_0 + v, \\
\tau = \tau_0 + \tau
\end{align*}$$

(66)

where $$(\varphi_0, v_0, \tau_0)$$ is the solution to the problem:

$$\begin{align*}
&\text{Find } (\phi_0, v_0, \tau_0) \in \mathbf{Q}_0 \times \mathbf{V}_h \times \Gamma_h \text{ such that:} \\
&\quad -\tau_0(\xi)_0 = q(\xi), \quad \xi \in N'(\mathbf{Q}_s), \\
&\quad -E'(\xi)\phi_0'(\xi)_0 - E'(\xi)\phi'(\xi)_0 - x'(\xi)_0 \times \tau_0(\xi)_0 = 0, \quad \xi \in N'(\mathbf{V}_h), \\
&\quad v_0(0)_0 = v_0(1)_0 = 0, \\
&\quad \phi_0(0)_0 = \phi_0(1)_0 = 0.
\end{align*}$$

(67)

and $$(\bar{\varphi}, \bar{v}, \bar{\tau})$$ is the solution to the problem:

$$\begin{align*}
&\text{Find } (\bar{\varphi}, \bar{v}, \bar{\tau}) \in \mathbf{Q}_0 \times \mathbf{V}_h \times \Gamma_h \text{ such that:} \\
&\quad -\bar{\tau}(\xi) = 0, \quad \xi \in N'(\mathbf{Q}_s), \\
&\quad -E(\xi)\bar{\varphi}'(\xi)_0 - E(\xi)\bar{\varphi}'(\xi)_0 - x'(\xi)_0 \times \bar{\tau}(\xi)_0 = 0, \quad \xi \in N'(\mathbf{V}_h), \\
&\quad \bar{v}(0) = -v_0(0), \quad \bar{v}(1) = 0, \\
&\quad \bar{\varphi}(0) = \bar{\varphi}(1) = 0.
\end{align*}$$

(68)

The proof of the following proposition is postponed to Section 6.4, together with the associated error estimates.

**Proposition 6.4.** For $h$ sufficiently small, Problem (67) admits a unique solution $$(\varphi_0, v_0, \tau_0) \in \mathbf{Q}_0 \times \mathbf{V}_h \times \Gamma_h.$$  

As far as Problem (68) is concerned, we have the following result, whose proof is postponed to Section 6.5.

**Proposition 6.5.** For $h$ sufficiently small, Problem (68) admits a unique solution $$(\varphi_0, v_0, \tau_0) \in \mathbf{Q}_0 \times \mathbf{V}_h \times \Gamma_h.$$  

We now remark that the error quantities $$(\varphi - \varphi_0, v - v_0, \tau - \tau_0)$$ can be split as

$$\begin{align*}
&\varphi = \varphi_0 + \varphi, \\
v = v_0 + v, \\
\tau = \tau_0 + \tau
\end{align*}$$

(69)

Therefore, the error analysis can be performed by estimating the errors arising from the discretizations of problems (62) and (63), respectively.

**Remark 6.3.** We notice that problem (68) can be considered an approximation of problem (63), also because of the presence of the approximated boundary datum $-v_0(0)$ in place of $-v_0(0)$. 


6.4. Discretization error for problem (62)

In the following, we will denote by $k_0, k, k$: the regularity index $k$ introduced in Section 3.1, respectively associated to each space $\mathcal{U}_h, V_h, \Gamma_h$. For simplicity of notation, we assume that such a regularity index is the same for all knots in each of the three knot vectors defining the discrete spaces. We will assume that the loading, the material data $q, E, \lambda$, and the axial force $\alpha$ are (at least) meshwise regular. Therefore also the solution $(\varphi_0, \nu_0, \tau_0)$ can be assumed to share the same (mesh-wise) regularity properties.

Hereafter, in order to shorten the exposition, we adopt the following norm and semi-norm notation. For all sufficiently regular scalar and vector functions $f$ on $(0, 1)$, we define

$$
\|f\|_{m, \infty} = \|f\|_{W^{m, \infty}(0, 1)}; \
\|f\|_{m, h} = \max_{E \in \mathcal{M}_h} \|f\|_{W^{m, \infty}(E)} \quad \forall m \in \mathbb{N}.
$$

We will make use of the following interpolation Lemmas, that can be proven using exactly the same techniques as in [15] combined with Assumption 6.1. The details can be found in [16].

**Lemma 6.1.** Let $\tau \in \mathcal{C}^{k-1}(0, 1)$ and such that $\tau_0 \in W^{k+\infty}(E)$. Then for all $0 < m \leq p$, it holds

$$
\|\tau - \tau_0\|_{m, \infty} \leq C \|\tau_0\|_{m, \infty}.
$$

**Lemma 6.2.** Let $\sigma \in \mathcal{C}^{k-2}(0, 1)$ and such that $\sigma|_h \in W^{k+\infty}(E)$. Then for all $0 < m \leq p - 1$, it holds

$$
\|\sigma - \sigma_0\|_{m, \infty} \leq C \|\sigma_0\|_{m, \infty},
$$

where $(\sigma|_h) \in \Phi_h^\sigma$ is defined in Assumption 6.1.

The analysis of the discretization error for problem (62), together with the existence of a unique discrete solution, will be presented very briefly since it follows the same steps for the analogous part in [16].

**Proof of Proposition 6.4 and error estimates.** Comparing (62), and (67), one immediately obtains that $\tau_0$ exists and is unique, since it is determined by $\tau_0 = \tau_0$, and the boundary condition in $0$. Moreover, using Lemma 6.1 and the Poincaré inequality we get the existence of a unique $\tau_0$ with the estimate

$$
\|\tau_0 - \tau_0\|_{1, \infty} \leq C \|\tau_0\|_{1, \infty}
$$

for all $0 < m \leq p$. We now consider Eqs. (62), (67). Note that $\tau$ and $\tau_0$ have been already determined in the previous step and can now be treated as a datum. Therefore, the existence of a unique $\varphi_0$ and an error bound for the discretization of the second order differential Eq. (62) can be derived using the results of [3]. Note that there is also an approximation error deriving from the datum error $\tau - \tau_0$. Such term is handled immediately due to the stability of the considered equation and using (70). Finally, for all $0 < m \leq p$,

$$
\|\varphi_0 - \varphi_0\|_{2, \infty} \leq C \|\tau_0\|_{m, \infty} + h^{k+1} \|\tau_0\|_{m+1, \infty}.
$$

Applying the above estimate with the choice $m = p - 1, m = p$, we obtain:

$$
\|\varphi_0 - \varphi_0\|_{2, \infty} \leq C h^{k+1} \|\tau_0\|_{1, \infty}
$$

with $\gamma = \min(p, p, p - 1)$.

The same argument above can be applied also to the last two equations (62), and (67), where $\varphi_0$ and $\varphi_0$ are now handled as an (approximated) datum. We finally obtain the existence of a unique $\nu_0$ with the error bound

$$
\|\nu_0 - \nu_0\|_{1, \infty} \leq C h^{k+3} \|\tau_0\|_{p+2, \infty} + \|\tau_0\|_{p+2, \infty}
$$

with $\beta = \min(p, p, p - 1)$. Bounds 70, 72, and 73 give the error estimates for problem (62):

**Proposition 6.6.** For $h > 0$ sufficiently small, it holds:

$$
\|\varphi_0 - \varphi_0\|_{2, \infty} + \|\nu_0 - \nu_0\|_{1, \infty} + \|\tau_0 - \tau_0\|_{1, \infty} \leq C h^p,
$$

with $\beta = \min(p, p, p - 1)$. We now rewrite problem (63) in the following way:

$$
\begin{cases}
\text{Find } (\varphi, \nu, \kappa) \in C^2[0, 1] \times C^1[0, 1] \times \mathbb{R}^3 \text{ such that:} \\
-\varepsilon(\zeta) \varphi''(\zeta) - E(\zeta) \varphi(\zeta) = \mathbf{z}(\zeta) \times \mathbf{k}, \\
\nu(\zeta) - \varphi(\zeta) \times \mathbf{z}(\zeta) = -d^2\Lambda^{-1}(\zeta) \mathbf{k} = 0,
\end{cases}
$$

with $\varphi(0) = -\varphi(0); \nu(1) = 0; \kappa(0) = \kappa(1) = 0.$

We now notice that $\mathbf{z}(\zeta) \times \mathbf{k} = \mathbf{M}_\mathbf{k}(\zeta) \mathbf{k}$, where $\mathbf{M}_\mathbf{k}(\zeta)$ is the skew-symmetric matrix given by

$$
\mathbf{M}_\mathbf{k}(\zeta) = \begin{bmatrix}
0 & -\mathbf{z}(\zeta) \cdot \mathbf{e}_2 & -\mathbf{z}(\zeta) \cdot \mathbf{e}_3 \\
-\mathbf{z}(\zeta) \cdot \mathbf{e}_1 & 0 & -\mathbf{z}(\zeta) \cdot \mathbf{e}_2 \\
-\mathbf{z}(\zeta) \cdot \mathbf{e}_3 & -\mathbf{z}(\zeta) \cdot \mathbf{e}_1 & 0
\end{bmatrix}
$$

Therefore, system (75) can be rewritten as:

$$
\begin{cases}
\text{Find } (\varphi, \nu, \kappa) \in C^2[0, 1] \times C^1[0, 1] \times \mathbb{R}^3 \text{ such that:} \\
-\varepsilon(\zeta) \varphi''(\zeta) - E(\zeta) \varphi(\zeta) = \mathbf{M}_\mathbf{k}(\zeta) \mathbf{k}, \\
\nu(\zeta) - \varphi(\zeta) \times \mathbf{z}(\zeta) = -d^2\Lambda^{-1}(\zeta) \mathbf{k} = 0,
\end{cases}
$$

$$
\nu(0) = -\nu(0); \nu(1) = 0; \kappa(0) = \kappa(1) = 0.
$$

Hence, exploiting that $\mathbf{k}$ is a constant vector, we see that

$$
\varphi(\zeta) = \mathbf{S}(\zeta) \mathbf{k}.
$$

Where $\mathbf{S}(\zeta)$ is a $3 \times 3$ function matrix, unique solution to the ODE boundary value problem:

$$
\begin{cases}
\text{Find } \mathbf{S} \in C^1([0, 1], M_{3,3}) \text{ such that:} \\
-\varepsilon(\zeta) \mathbf{S}'(\zeta) - E(\zeta) \mathbf{S}(\zeta) = \mathbf{M}_\mathbf{k}(\zeta), \\
\mathbf{S}(0) = \mathbf{S}(1) = 0.
\end{cases}
$$

Where $M_{3,3}$ denotes the space of $3 \times 3$ real valued matrices.

Inserting (78) into the second equation of (75), we obtain the equation for $\nu(\zeta)$:

$$
\nu(\zeta) = \mathbf{S}(\zeta) \mathbf{k} \times \mathbf{z}(\zeta) + d^2\Lambda^{-1}(\zeta) \mathbf{k} \quad \zeta \in [0, 1].
$$

Integrating and using the skew-symmetry of the vector product together with the boundary condition $\nu(1) = 0$, we get

$$
\nu(\zeta) = \int_0^\zeta (\mathbf{z}(\rho) \times (\mathbf{S}(\rho) \mathbf{k}) - d^2\Lambda^{-1}(\rho) \mathbf{k}) d\rho.
$$

Using now the boundary condition $\nu(0) = -\nu(0)$, we infer that it holds
\[-\mathbf{v}_0(0) = \int_0^1 \left[ \mathbf{z}'(\rho) \times (S(\rho)k) - d^2 A^{-1}(\zeta)k \right] d\rho. \tag{82} \]

Recalling (76), Eq. (82) can be written as
\[-\mathbf{v}_0(0) = \left[ \int_0^1 (M_x(\rho)S(\rho) - d^2 A^{-1}(\rho)) d\rho \right] k. \tag{83} \]

Similar computations can be performed for the discrete problem (68). More precisely, problem (68) can be written as:

\[
\begin{align*}
\text{Find } & (\varphi_\delta, v_h, k_h) \in \Phi_h \times V_h \times \mathbb{R}^3 \text{ such that: } \\
- \mathcal{R}(x)\varphi_\delta'(x) - \mathcal{R}(x)\varphi_\delta(x) & = M_x(x)k_h, \quad x \in N(\Phi_h) \\
V_h(y_h) - \varphi_\delta(y_h) & = A^{-1}(y_h)k_h = 0, \quad y_h \in N(V_h) \\
\varphi_\delta(0) = -v_{\delta h}(0) &, \quad v_h(1) = 0, \quad \varphi_\delta(0) = 0.
\end{align*}
\tag{84} \]

From the first equation of (84), exploiting that \(k_h \in \mathbb{R}^3\) is a constant vector, we see that
\[
\varphi_\delta(\xi) = S_h(\xi)k_h,
\tag{85}
\]
where \(S_h(\xi)\) is a \(3 \times 3\) function matrix, unique solution to the discrete ODE boundary value problem:

\[
\begin{align*}
\text{Find } & S_h \in \mathcal{M}_{3 \times 3}(\mathbb{R}^3) \text{ such that: } \\
- \mathcal{R}(x)S_h'(x) - \mathcal{R}(x)S_h(x) & = M_x(x), \quad x \in N(\Phi_h) \\
S_h(0) = S_h(1) & = 0.
\end{align*}
\tag{86} \]

Above, \(\mathcal{M}_{3 \times 3}(\mathbb{R}^3)\) denotes the space of \(3 \times 3\) matrices, whose entries are functions in the NURBS space \(\mathcal{V}_{\alpha h}\), see (51).

**Remark 6.4.** We highlight that, using the techniques of [3], it can be proven that (86) admits a unique solution only for \(h\) sufficiently small. However, in practical computations this restriction does not appear.

Inserting (85) into the second equation of (84), we obtain the equation for \(v_h(\xi)\):
\[
v_h(\xi) = \left( (S_h(\xi)k_h) \times (\varphi_\delta'(\xi)) \right)_\| + d^2 A^{-1}(\xi)k_h, \quad \xi \in [0, 1]. \tag{87} \]

Integrating and using the skew-symmetry of the vector product together with the boundary condition \(v_h(1) = 0\), we get
\[
v(\xi) = \int_0^1 \left[ \left( \mathbf{z}'(\rho) \times (S_h(\rho)k_h) \right)_\| - d^2 A^{-1}_m(\rho)k_h \right] d\rho. \tag{88} \]

Using now the boundary condition \(v_h(0) = -v_{\delta h}(0)\), we infer that it holds
\[-v_{\delta h}(0) = \int_0^1 \left[ \left( \mathbf{z}'(\rho) \times (S_h(\rho)k_h) \right)_\| - d^2 A^{-1}_m(\rho)k_h \right] d\rho. \tag{89} \]

Recalling (76), Eq. (89) can be written as
\[-v_{\delta h}(0) = \left[ \int_0^1 (M_xS_h(\rho))_\| - d^2 A^{-1}_m(\rho)) d\rho \right] k_h. \tag{90} \]

**Remark 6.5.** In Eqs. (87) and (90), we have introduced the matrix-valued interpolated functions \(A^{-1}_m(\rho)\) and \((M_xS_h)_\|\), respectively. These quantities, with a little abuse of notation, should be intended as the corresponding row-wise interpolated vectorial functions, using the interpolation operator introduced in Assumption 6.1.

The following Lemma is useful for what follows.

**Lemma 6.3.** Referring to (83) and (90), it holds:
\[
\left\| (M_xS - d^2 A^{-1}) - \left( (M_xS_h)_\| - d^2 A^{-1}_m \right) \right\| \leq C h^\delta,
\tag{91} \]
where \(\delta = \min(p_x p_y - 1).\)

**Proof.** We first notice that the triangle inequality gives:
\[
\left\| (M_xS - d^2 A^{-1}) - \left( (M_xS_h)_\| - d^2 A^{-1}_m \right) \right\| \leq \left\| (M_xS - (M_xS_h)_\|) \right\| + d^2 \left\| A^{-1} - A^{-1}_m \right\|.
\tag{92} \]

Furthermore, it holds:
\[
\left\| (M_xS - (M_xS_h)_\|) \right\| \leq \left\| M_xS - (M_xS) \right\|_\| + \left\| (M_xS) \right\|_\| - \left\| (M_xS_h) \right\|_\|. \tag{93} \]

Using also Lemma 6.1 and the stability Assumption 6.1, we get:
\[
\left\| M_xS - (M_xS_h)_\| \right\| \leq C h^p |M_xS|_{p_x, p_y} + \|M_x\| \|S - S_h\|_\|^p. \tag{94} \]

Furthermore, comparing (79) and (86), using a Poincaré inequality and the results in [3], yields
\[
\|S - S_h\|_\|^p \leq C \|S - S_h\|_\|^{p_x + p_y} \leq C h^{p_x + p_y} |\varphi|_{p_x + p_y}. \tag{95} \]

Combining (94) and (95), we obtain:
\[
\left\| M_xS - (M_xS_h)_\| \right\| \leq C h^\delta, \tag{96} \]
where \(\delta = \min(p_x p_y - 1).\) In addition, Lemma 6.1 gives:
\[
d^2 \left\| A^{-1} - A^{-1}_m \right\| \leq C h^\delta \|A^{-1} \|_{p_x, p_y}. \tag{97} \]

Estimate (91) now follows from (96) and (97). □

We now prove the following propositions.

**Proposition 6.7.** The linear operator \(\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2\) defined by (cf. (83)):
\[
\mathcal{L} w : = \left[ \int_0^1 (M_x(\rho)S(\rho) - d^2 A^{-1}(\rho)) d\rho \right] w \tag{98} \]

is an isomorphism.

**Proof.** By contradiction. Suppose there exists \(w^* \neq 0\) such that \(\mathcal{L}w^* = 0\). Now, set \((\varphi(\xi), v'(\xi), \tau'(\xi))\) as:
\[
\begin{align*}
\varphi(\xi) & = S(\xi)w^* \\
v'(\xi) & = \int_0^1 \left[ \mathbf{z}'(\rho) \times (S(\rho)w^*) - d^2 A^{-1}(\rho)w^* \right] d\rho \\
\tau'(\xi) & = w^*.
\end{align*}
\tag{99} \]

where \(S(\xi)\) is defined by (79). Using \(w^* \neq 0\) together with \(\mathcal{L}w^* = 0\), one can see that \((\varphi, v', \tau')\) is a non-vanishing regular solution to the problem (cf. (47)):
\[
\begin{align*}
\text{Find } & (\varphi, v, \tau) \in C^2[0, 1] \times C[0, 1] \times C[0, 1] \text{ such that: } \\
- \tau'(\xi) & = 0, \quad \xi \in [0, 1] \\
- E(\xi)(\varphi'(\xi) - \mathbf{z}'(\xi)) \times \tau(\xi) & = 0, \quad \xi \in [0, 1] \\
v'(\xi) - \varphi'(\xi) \times \mathbf{z}'(\xi) - d^2 A^{-1}(\xi)\tau(\xi) & = 0, \quad \xi \in [0, 1] \\
v(0) = v(1) & = 0, \quad \varphi(0) = \varphi(1) = 0.
\end{align*}
\tag{100} \]

Recalling Proposition 6.1, we obtain that problem (100) has the unique trivial solution \((\varphi, v, \tau) = (0, 0, 0)\), which provides the contradiction. □
Proposition 6.8. There exists $h_0 > 0$ such that, for every $h$ with $0 < h < h_0$, the linear operator $L_h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by (cf. (90)):

$$L_h w := \left[ \int_0^1 (\langle M_x S_h \rangle (\rho) - d^2 A_{\text{un}}^{-1}(\rho)) d\rho \right] w$$

is an isomorphism. Moreover, it holds:

$$\| L - L_h \| \leq C h^3,$$

(102)

where $\delta := \min(p_0, p_\rho, -1)$, $\| \|$ denotes a given operator norm, and $C$ depends on the chosen norm, but it is independent of $h$.

Proof. Using Lemma 6.3, we get that $L_h \to L$. Since the linear operator $L$ is an isomorphism (cf. Proposition 6.7), it follows that also $L_h$ is an isomorphism for $h$ sufficiently small. Estimate (102) is an immediate consequence of Lemma 6.3. □

We are now ready to provide the proof of Proposition 6.5.

Proof of Proposition 6.5. We prove that Problem (68) (or, equivalently, Problem (84)) has the unique trivial solution $(\tilde{\varphi}, \tilde{v}, \tilde{t}_h) = (0, 0, 0)$ when $v_{0, h}(0) = 0$.

To do so, we first notice that Eq. (90) may written as:

$$k_h = -L_h v_{0, h}(0).$$

(103)

Set now $v_{0, h}(0) = 0$ in Problem (84), and therefore in (103). From Proposition 6.8 we infer that, for $h$ with $0 < h < h_0$, it holds $k_h = 0$, which means $t_h = 0$. Hence, considering Problem (84) with $k_h = 0$ and using again the results of [3], we get $(\tilde{\varphi}, \tilde{v}) = (0, 0)$, which ends the proof.

We now give an estimate for $\tilde{t} - \tilde{t}_h = k - k_h$ (cf. (75) and (84)). We have the following lemma.

Lemma 6.4. For $h$ sufficiently small, it holds:

$$\| \tilde{t} - \tilde{t}_h \|_{L^\infty} = |k - k_h| \leq C (\|v_{0, h}(0)\| - \|L - L_h\|).$$

(104)

Proof. We first denote with $\text{inv}(\mathbb{R}^3)$ the set of invertible linear operators from $\mathbb{R}^3$ to itself. Now, we notice that the function $\varphi : \text{inv}(\mathbb{R}^3) \to \text{inv}(\mathbb{R}^3)$, defined by $\varphi(L) = L^{-1}$, is differentiable at every point $L$, hence it is locally Lipschitz. As a consequence, it is immediate to check that it exists positive constants $C$ independent of $h$ such that

$$\|L^{-1} - L_h^{-1}\| \leq C (\|L - L_h\|, \|L_h^{-1}\|).$$

Therefore, first recalling 83, 98, 90 and 101, then using the above bounds and some trivial algebra, we get:

$$|k - k_h| = \|L^{-1}v_{0, h}(0) - L^{-1}v_0(0)\|
\leq \|L^{-1}v_{0, h}(0) - L_h^{-1}v_{0, h}(0)\|
+ \|L_h^{-1}v_0(0) - L^{-1}v_0(0)\|
\leq \|L^{-1}\| \|v_{0, h}(0) - v_0(0)\| + \|L_h^{-1} - L^{-1}\| \|v_{0, h}(0)\|
\leq C (\|v_{0, h}(0) - v_0(0)\| + \|L - L_h\|),$$

(105)

where we included $|v_{0, h}(0)|$ in the constant. □

The Corollary below follows immediately combining Lemma 6.4 with bound (73) and Proposition 6.8.

Corollary 1. For $h$ sufficiently small, it holds:

$$\| \tilde{t} - \tilde{t}_h \|_{L^\infty} = |k - k_h| \leq C h^3,$$

(106)

where $\beta := \min(p_0, p_\rho, p_\varphi - 1)$.

Finally, we have the following results for the rotation and the displacement variables.

Lemma 6.5. For $h$ sufficiently small, it holds:

$$\| \varphi - \varphi_h \|_{L^\infty} + \| v - v_h \|_{L^\infty} \leq C h^\beta,$$

(107)

where $\beta := \min(p_0, p_\varphi, p_\varphi - 1)$.

Proof. We first prove that it holds:

$$\| \varphi - \varphi_h \|_{L^\infty} \leq C (h^\beta + |k - k_h|),$$

(108)

$$\| v - v_h \|_{L^\infty} \leq C (h^\beta + |k - k_h|),$$

(109)

with $\delta := \min(p_0, p_\rho, -1)$. In fact, recalling the identities (78) and (85), a triangle inequality and bound (95) give

$$\| \varphi - \varphi_h \|_{L^\infty} \leq \|S - S_h\|_{L^\infty} \|k - k_h\| \leq C (h^\beta + |k - k_h|),$$

where we included the terms $|k|$ and $\|S_h\|_{L^\infty}$ in the constant $C$, independent of $h$.

The estimate for the error $v - v_h$ follows recalling that, by (81) and (88), it holds

$$v_0(\xi) = -\left[ \int_0^1 (\langle M_x S(\varphi) - d^2 A_{\text{un}}^{-1}(\varphi) \rangle d\rho \right] k, k_h,$$

(110)

As a consequence, simple algebraic manipulations combined with Lemma 6.3 yield

$$\| v - v_h \|_{L^\infty} \leq C \left( \|S - S_h\|_{L^\infty} \|k - k_h\| \right),$$

(108)

with $\delta := \min(p_0, p_\rho, -1)$. Estimate (107) now follows from (108), (109) and Corollary 1. □

From Corollary 1 and Lemma 6.5, we obtain the error estimate for Problem 48:

Proposition 6.9. For $h$ sufficiently small, it holds

$$\| \tilde{t} - \tilde{t}_h \|_{L^\infty} + \| \varphi - \varphi_h \|_{L^\infty} + \| v - v_h \|_{L^\infty} \leq C h^\beta,$$

(111)

where $\beta := \min(p_0, p_\varphi, p_\varphi - 1)$.

6.6. Discretization error for the rod problem (48)

It is now straightforward to obtain the following error estimate for the error of the proposed collocation method.

Theorem 6.1. Let $(\varphi, v, \tau)$ and $(\varphi_h, v_h, \tau_h)$ represent the solutions of problem (48) and (60), under Assumption 6.1 on the collocation points. Then it holds

$$\| \varphi - \varphi_h \|_{L^\infty} + \| v - v_h \|_{L^\infty} + \| \tau - \tau_h \|_{L^\infty} \leq C h^\beta,$$

(112)

with $\beta = \min(p_0, p_\varphi, p_\varphi - 1),$$

(113)

and where the constant $C$ is independent of the knot vectors and the thickness parameter $d$.

Proof. The proof immediately follows recalling (69) and by combining Proposition 6.6 with Proposition 6.9. □
We remark that the theoretical results establish error estimates in the $W^{2,q^0}_0$-norm while the convergence plots in Section 5 are reported in terms of $L^2$-norm errors, which are more relevant in engineering applications. However, we point out that, since the $L^2$-norm is bounded from above by the $W^{2,q^0}_0$-norm, our theoretical error estimates hold for the $L^2$-norm as well.

One could extend the above results to the case of less regular loads $q$, obtaining a lower convergence rate $\beta$. Moreover, approximation results in higher order norms can also be derived by using inverse estimates. We do not provide here the details of these rather simple extensions. We also remark that Eq. (113) should not be intended as a recipe to find the optimal balancing among $p_{U,V}$ and $p_1$ since the provided estimates are not sharp.

The optimal selection of points for interpolation of one-dimensional splines is addressed in various papers. The only choice proven to be stable (i.e., satisfying Assumption 6.1) for any mesh and degree are the so-called Demko abscissae, see for instance [21,23]. A different approach proposed in the engineering literature [27] is to collocate at the Greville abscissae. We refer to [3] for a deeper investigation and comparison between the Demko and Greville choices.

**Remark 6.6. Theorem 6.1** yields a converge estimate, uniform in the thickness parameter, without requiring any particular compatibility condition among the three discrete spaces $\Phi_h V_h$ and $V_h$. Therefore, the proposed method is locking-free regardless of the chosen polynomial degrees and space regularities. Even different meshes can be adopted among the three spaces. Such result is surprising, at least in comparison with Galerkin schemes. Indeed, in typical Galerkin approaches the discrete spaces $\Phi_h V_h$ must be carefully chosen, in order to avoid the locking phenomenon and the occurrence of spurious modes.

### 7. Conclusions

In this work we have presented the application of isogeometric collocation techniques to the solution of spatial Timoshenko rods. After introducing the strong form equations of the problem in both displacement-based and mixed forms, we have considered their discretization via NURBS-based isogeometric collocation at the images of Greville abscissae.

The obtained collocation schemes have been then numerically tested on several examples in order to assess their accuracy and efficiency, as well as their possible application to problems of practical interest. In particular, it is interesting to highlight that the considered mixed formulations appeared to be locking-free for any choice of the discrete spaces for displacements, rotations, and internal forces; such a remarkable behavior has also been analytically proven in the second part of the paper. The same property of isogeometric collocation was analytically proven and computationally tested in [16] in the context of the simpler case of initially straight Timoshenko beams.

These results propose isogeometric collocation methods as a viable and efficient alternative to standard approximation methods for curved beams. They moreover constitute a first fundamental step towards the development of novel efficient locking-free approaches for the simulation of bidimensional thin structures, such as plates and shells. In particular, extension to Reissner-Mindlin plates is currently under investigation.

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### References


