

Novel finite particle formulations based on projection methodologies

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SUMMARY

Particle methods are among the most studied meshless approaches, with applications ranging from solid mechanics to fluid-dynamics and thermo-dynamics. The objective of the present paper is to analyze the behavior of finite particle formulations based on projection methodologies, investigating in particular how these approaches behave in approximating 1D and 2D second-order problems. Moreover, the issue of choosing suitable projection functions is discussed and 1D and 2D numerical tests, showing the second-order accuracy of the methods under investigation, are performed. Copyright © 2010 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A large number of numerical methods has been recently proposed in the literature to address advanced mechanical problems, such as those involving rapid deformations, high-intensity forces, large displacement fields. In many of these cases, in fact, classical finite element methods (FEM) suffer from mesh distortion, spurious numerical errors and, above all, mesh sensitivity. Hence, to overcome such issues, a variety of numerical methods, belonging to the family of the so-called meshless techniques, has been widely investigated and applied. The objective of employing these methods is to avoid the introduction of a mesh for the continuum, preferring a particle discretization, with the goal of obtaining an easier treatment of large and rapid displacements. Thus, meshless methods have been widely applied, mainly to fluid dynamics problems, where particle approaches appear to be more feasible. However, recently, a number of researchers have tried to extend meshless methods also to solid mechanics problems.

Among the several meshless numerical methods proposed, particle methods, and in particular smoothed particle hydrodynamics (SPH), have been widely implemented and investigated. Historically, SPH was introduced by Lucy [1] and Gingold and Monagan [2] to treat astrophysics problems, and, then, a variety of formulations has been proposed to apply its principles to different problems, such as incompressible flows [3], elasticity [4], fracture of solids [5, 6], heat conduction [7]. Furthermore, in order to address a number of criticalities and issues, several improvements have been proposed: for instance, Swegle *et al.* [8] highlighted ‘tension instability’, fixing it through

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a viscosity-based procedure, whereas Dyka *et al.* [9] and Randles and Libersky [10] addressed the same problem through a stress point procedure; Johnson and Beissel [11] proposed a method to improve strain calculation and Liu *et al.* [12, 13] introduced the so-called reproducing kernel particle method (RKPM) to overcome deficiencies occurring on the boundary. Nowadays, SPH represents itself as a family of methods, given the large variety of formulations available in the literature stemmed from the original SPH approach.

In this paper, we move from the SPH-inspired particle formulation proposed by Batra and Zhang [14] and Liu *et al.* [15], hereafter referred to as finite particle method (FPM), and in particular we focus on the modified version presented in [16]. We conduct a detailed error analysis in 1D, which can be extended to 2D, showing that such a formulation leads to second-order accuracy when dealing with source second-order problems. We also generalize the method, discussing the choice of the projection functions and proposing in 1D and 2D sets of functions simpler than the classical Gaussian-based projections. We finally present 1D and 2D numerical tests showing the second-order convergence of the method.

2. 1D FORMULATION

In this section, we present the 1D formulation of the original FPM method proposed by Batra and Zhang [14] and Liu *et al.* [15] and of its variant presented in [16]. In particular, following [16], we perform an analysis of the error in the derivative approximation. We then consider the source problem and we show how second-order convergence is attained. Within this context, we discuss the choice of the projection functions, and, in particular, we propose the use of simpler choices than the classical Gaussian function and its derivative.

2.1. Original FPM formulation in 1D: approximation of derivatives

According to the FPM formulation proposed by Batra and Zhang [14] and Liu *et al.* [15], Taylor's formula is projected on a number of independent functions, providing a linear algebraic system for each particle x_i whose unknowns are the approximations at x_i of the function $B(x)$ and its derivatives, up to the order of the Taylor's formula truncation. Hence, in the 1D case, considering a series expansion up to the first-order and employing as projection function a kernel function $W_i(x)$ and its derivative $DW_i(x)$, the following relationships are obtained:

$$B(x_i) \int_{\Omega} W_i(x) d\Omega + DB(x_i) \int_{\Omega} (x - x_i) W_i(x) d\Omega = \int_{\Omega} B(x) W_i(x) d\Omega + e'_1, \quad (1)$$

$$B(x_i) \int_{\Omega} DW_i(x) d\Omega + DB(x_i) \int_{\Omega} (x - x_i) DW_i(x) d\Omega = \int_{\Omega} B(x) DW_i(x) d\Omega + e'_2, \quad (2)$$

where Ω is the problem domain whereas e'_1 and e'_2 represent the errors due to series truncation. We note that, in the literature, the kernel $W_i(x)$ is assumed to be a symmetric Gaussian (or spline) function and, as a consequence, its derivative $DW_i(x)$ is skew-symmetric.

Thus, defining the matrix \mathbf{A} , whose components are:

$$\begin{aligned} A_{11} &= \int_{\Omega} W_i(x) d\Omega, & A_{12} &= \int_{\Omega} (x - x_i) W_i(x) d\Omega, \\ A_{21} &= \int_{\Omega} DW_i(x) d\Omega, & A_{22} &= \int_{\Omega} (x - x_i) DW_i(x) d\Omega, \end{aligned} \quad (3)$$

Equations (1) and (2) can be rearranged as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B(x_i) \\ DB(x_i) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} B(x) W_i(x) d\Omega \\ \int_{\Omega} B(x) DW_i(x) d\Omega \end{bmatrix} + \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix}. \quad (4)$$

Neglecting error terms e'_1 and e'_2 , these equations represent a linear algebraic system that can be solved with respect to $B(x_i)$ and $DB(x_i)$. To assess the corresponding error, the following considerations can be made.

For particles far from the boundary, the associated kernel functions are negligible on the domain boundary; hence, given that $DW_i(x)$ is skew-symmetric, its second-order momentum vanishes and we have that $e'_2 = \|DW_i\|_{L^1} \cdot O(h^3)$, where h is a measure of the projection function support (i.e. the so-called smoothing length). Being $W_i(x)$ symmetric, we also have that $e'_1 = \|W_i\|_{L^1} \cdot O(h^2)$. Moreover, considering again the symmetry of $W_i(x)$ and the skew-symmetry of $DW_i(x)$, the terms A_{12} and A_{21} vanish and the system yields

$$\begin{bmatrix} B(x_i) \\ DB(x_i) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} B(x)W_i(x) d\Omega/A_{11} \\ \int_{\Omega} B(x)DW_i(x) d\Omega/A_{22} \end{bmatrix} + \begin{bmatrix} e'_1/A_{11} \\ e'_2/A_{22} \end{bmatrix}. \quad (5)$$

Being $A_{11} = C \cdot \|W_i\|_{L^1}$ and $A_{22} = Ch \cdot \|DW_i\|_{L^1}$, the error introduced neglecting e'_1 and e'_2 for the evaluation of $B(x)$ and its first derivative, equal to e'_1/A_{11} and e'_2/A_{22} , respectively, is $O(h^2)$ in both cases.

On the other hand, for particles close to the boundary, both functions $W_i(x)$ and $DW_i(x)$ are not completely developed; we thus have that $e'_1 = \|W_i\|_{L^1} \cdot O(h^2)$ and $e'_2 = \|DW_i\|_{L^1} \cdot O(h^2)$. Moreover, close to the boundary, terms A_{12} and A_{21} do not vanish; hence, the error introduced on $B(x_i)$ and $DB(x_i)$ neglecting e'_1 and e'_2 can be evaluated as

$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix} &\cong \left(\begin{bmatrix} \|W_i\|_{L^1} & 0 \\ 0 & \|DW_i\|_{L^1} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \right)^{-1} \begin{bmatrix} \|W_i\|_{L^1} \cdot O(h^2) \\ \|DW_i\|_{L^1} \cdot O(h^2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & h^{-1} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} \|W_i\|_{L^1}^{-1} & 0 \\ 0 & \|DW_i\|_{L^1}^{-1} \end{bmatrix} \begin{bmatrix} \|W_i\|_{L^1} \cdot O(h^2) \\ \|DW_i\|_{L^1} \cdot O(h^2) \end{bmatrix} \\ &= \begin{bmatrix} O(h^2) \\ O(h) \end{bmatrix}, \end{aligned} \quad (6)$$

where C_{11} , C_{12} , C_{21} and C_{22} represent constant terms. As a consequence, the evaluation of $DB(x_i)$ is affected by an error of order h^2 inside the domain and of order h close to the boundary. Trying to evaluate also the second derivative of the function $B(x)$, the procedure can be in principle reiterated. Unfortunately, this operation causes a propagation of the error, making the formulation not converging.

2.2. Modified FPM formulation in 1D: approximation of derivatives

To overcome the problems highlighted above and to have a converging solution also for the second derivative, the slightly different procedure proposed in [16] is discussed here.

If the values of $B(x_i)$ do not represent real unknowns, the system can be rearranged to introduce also second derivatives. In fact, introducing Taylor's expansion for $B(x)$ up to the second order, multiplying it once by $W_i(x)$ and once by $DW_i(x)$, and integrating, the following relationships

are obtained:

$$\begin{aligned}
 & \mathbf{D}B(x_i) \left[\int_{\Omega} (x-x_i) W_i(x) \, d\Omega \right] + \frac{1}{2} \mathbf{D}^2 B(x_i) \left[\int_{\Omega} (x-x_i)^2 W_i(x) \, d\Omega \right] \\
 &= \int_{\Omega} [B(x) - B(x_i)] W_i(x) \, d\Omega + e_1'', \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{D}B(x_i) \left[\int_{\Omega} (x-x_i) \mathbf{D}W_i(x) \, d\Omega \right] + \frac{1}{2} \mathbf{D}^2 B(x_i) \left[\int_{\Omega} (x-x_i)^2 \mathbf{D}W_i(x) \, d\Omega \right] \\
 &= \int_{\Omega} [B(x) - B(x_i)] \mathbf{D}W_i(x) \, d\Omega + e_2'', \tag{8}
 \end{aligned}$$

where the terms e_1'' and e_2'' represent the errors due to series truncation. Then, defining

$$\begin{aligned}
 A_{11} &= \int_{\Omega} (x-x_i) W_i(x) \, d\Omega, & A_{12} &= \frac{1}{2} \int_{\Omega} (x-x_i)^2 W_i(x) \, d\Omega, \\
 A_{21} &= \int_{\Omega} (x-x_i) \mathbf{D}W_i(x) \, d\Omega, & A_{22} &= \frac{1}{2} \int_{\Omega} (x-x_i)^2 \mathbf{D}W_i(x) \, d\Omega,
 \end{aligned} \tag{9}$$

Equations (7) and (8) can be rearranged as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{D}B(x_i) \\ \mathbf{D}^2 B(x_i) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} [B(x) - B(x_i)] W_i(x) \, d\Omega \\ \int_{\Omega} [B(x) - B(x_i)] \mathbf{D}W_i(x) \, d\Omega \end{bmatrix} + \begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix}. \tag{10}$$

In this case, conducting a discussion similar to the previous case, it can be observed that, far from the boundary, given that $W_i(x)$ is symmetric, its third-order momentum is null and $e_1'' = \|W_i\|_{L^1} \cdot O(h^4)$, whereas $e_2'' = \|\mathbf{D}W_i\|_{L^1} \cdot O(h^3)$. Moreover, due to the symmetry of $W_i(x)$ and due to the skew-symmetry of $\mathbf{D}W_i(x)$, terms A_{11} and A_{22} vanish, yielding

$$\begin{bmatrix} \mathbf{D}B(x_i) \\ \mathbf{D}^2 B(x_i) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} [B(x) - B(x_i)] \mathbf{D}W_i(x) \, d\Omega / A_{21} \\ \int_{\Omega} [B(x) - B(x_i)] W_i(x) \, d\Omega / A_{12} \end{bmatrix} + \begin{bmatrix} e_2'' / A_{21} \\ e_1'' / A_{12} \end{bmatrix}. \tag{11}$$

Given that A_{12} is proportional to $h^2 \cdot \|W_i\|_{L^1}$ and A_{21} is proportional to $h \cdot \|\mathbf{D}W_i\|_{L^1}$, the error related to the first and the second derivative evaluations is $O(h^2)$.

Instead, close to the boundary, functions $W_i(x)$ and $\mathbf{D}W_i(x)$ are not completely developed and we have that $e_1'' = \|W_i\|_{L^1} \cdot O(h^3)$ and $e_2'' = \|\mathbf{D}W_i\|_{L^1} \cdot O(h^3)$. Terms A_{11} and A_{22} do not vanish and the error introduced neglecting e_1'' and e_2'' can be evaluated as

$$\begin{aligned}
 & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} e_1'' \\ e_2'' \end{bmatrix} \cong \left(\begin{bmatrix} \|W_i\|_{L^1} & 0 \\ 0 & \|\mathbf{D}W_i\|_{L^1} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \|W_i\|_{L^1} \cdot O(h^3) \\ \|\mathbf{D}W_i\|_{L^1} \cdot O(h^3) \end{bmatrix} \\
 &= \begin{bmatrix} h^{-1} & 0 \\ 0 & h^{-2} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} \|W_i\|_{L^1}^{-1} & 0 \\ 0 & \|\mathbf{D}W_i\|_{L^1}^{-1} \end{bmatrix} \begin{bmatrix} \|W_i\|_{L^1} \cdot O(h^3) \\ \|\mathbf{D}W_i\|_{L^1} \cdot O(h^3) \end{bmatrix} \\
 &= \begin{bmatrix} O(h^2) \\ O(h) \end{bmatrix}. \tag{12}
 \end{aligned}$$

Hence, the maximum error related to the first and the second derivative evaluations is of the order of h^2 far from the boundary and of the order of h close to it.

2.3. Modified FPM formulation in 1D: approximation of source problems

The procedure presented above is then implemented to solve a source problem, that is evaluating a generic function $B(x)$ given its second derivative $D^2B(x)$ and the necessary boundary conditions. This is indeed a more relevant problem in engineering with respect to the approximation of derivatives, since an equation of the type $u(x)'' = f(x)$, along with suitable boundary conditions, may represent either the problem of finding the axial displacement field of an elastic rod or the problem of determining the temperature of a bar at a given location.

Before presenting the approximation properties of the method, we propose here a brief discussion on the choice of the projection functions. As it is traditionally done, in fact, in the previous part of the paper we used the classical Gaussian function along with its derivative. Indeed, this choice derives from two facts, of which one is only a custom and the other is instead fundamental. Indeed, the choice of the Gaussian due to the fact that it can be regarded as an approximation of Dirac's delta is not justified in the FPM method. Instead, the use of a couple of functions that are symmetric and skew-symmetric (as the Gaussian function and its derivative are) is due to the fundamental fact that the projection functions have to generate a non-singular \mathbf{A} matrix (see Equation (9)). We may observe that any other couple of functions leading to a non-singular \mathbf{A} will work, and, as a consequence, any couple of functions, which are, e.g., respectively symmetric and skew-symmetric, constitute in principle a proper choice.

Therefore, to be more general, in all the following FPM equations, we use W^1 (symmetric) and W^2 (skew-symmetric) instead of W and DW , respectively. We also remark that this could have been done in all the previous equations, but we did prefer to present the FPM method in its classical version.

We finally highlight that in this paper we will focus in particular on few simple choices for W^1 and W^2 , selected among the following functions:

- Constant function:

$$\begin{aligned} W_i^1(x) &= 1 \quad \forall x \text{ s.t. } |x - x_i| < h, \\ W_i^1(x) &= 0 \quad \text{otherwise.} \end{aligned} \quad (13)$$

- Linear function:

$$\begin{aligned} W_i^2(x) &= (x - x_i) \quad \forall x \text{ s.t. } |x - x_i| < h, \\ W_i^2(x) &= 0 \quad \text{otherwise.} \end{aligned} \quad (14)$$

- Quadratic function:

$$\begin{aligned} W_i^1(x) &= (x - x_i)^2 \quad \forall x \text{ s.t. } |x - x_i| < h, \\ W_i^1(x) &= 0 \quad \text{otherwise.} \end{aligned} \quad (15)$$

- Cubic function:

$$\begin{aligned} W_i^2(x) &= (x - x_i)^3 \quad \forall x \text{ s.t. } |x - x_i| < h, \\ W_i^2(x) &= 0 \quad \text{otherwise.} \end{aligned} \quad (16)$$

Now, we try to derive an estimate for the error of the source problem.

Introducing $\mathbf{E} = \mathbf{A}^{-1}$, evaluated at the centroid x_i , after discretization (see [16] for details), we may write

$$\begin{bmatrix} DB(x_i) \\ D^2B(x_i) \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} \sum_j [B(x_j) - B(x_i)] W_i^1(x_j) \Delta x_j \\ \sum_j [B(x_j) - B(x_i)] W_i^2(x_j) \Delta x_j \end{bmatrix}, \quad (17)$$

and obtain

$$\begin{aligned} D^2 B(x_i) = E_{21} \cdot \left[\sum_j B(x_j) W_i^1(x_j) \Delta x_j \right] - E_{21} \cdot \left[\left(\sum_j W_i^1(x_j) \Delta x_j \right) \cdot B(x_i) \right] \\ + E_{22} \cdot \left[\sum_j B(x_j) W_i^2(x_j) \Delta x_j \right] - E_{22} \cdot \left[\left(\sum_j W_i^2(x_j) \Delta x_j \right) \cdot B(x_i) \right]. \end{aligned} \quad (18)$$

Writing the previous equation for every centroid x_i ($i = 1, \dots, N$), the following system holds:

$$\mathbf{K} \begin{bmatrix} B(x_1) \\ B(x_2) \\ \vdots \\ B(x_N) \end{bmatrix} = \begin{bmatrix} D^2 B(x_1) \\ D^2 B(x_2) \\ \vdots \\ D^2 B(x_N) \end{bmatrix}, \quad (19)$$

being \mathbf{K} an $N \times N$ matrix whose terms are linear combinations of E_{21} and E_{22} multiplied by W_i^1 and W_i^2 terms. Having assigned the second derivatives, the values of the function $B(x)$ can be therefore determined solving system (19). The related error e'''^i clearly depends on terms $e_2'''^i$ in Equation (12), representing the error (of order h) introduced for the evaluation of the second derivative. In particular, e'''^i can be evaluated as

$$\begin{bmatrix} e'''^1 \\ e'''^2 \\ \vdots \\ e'''^N \end{bmatrix} = \mathbf{K}^{-1} \begin{bmatrix} e_2'''^1 \\ e_2'''^2 \\ \vdots \\ e_2'''^N \end{bmatrix}. \quad (20)$$

The order of the matrix \mathbf{K} is related to the order of terms E_{21} , E_{22} , W_i^1 and W_i^2 . In particular, the order of the matrix \mathbf{E} can be obtained as

$$\begin{aligned} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} &\cong \left(\begin{bmatrix} \|W_i^1\|_{L^1} & 0 \\ 0 & \|W_i^2\|_{L^1} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h^2 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} h^{-1} & 0 \\ 0 & h^{-2} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} \|W_i^1\|_{L^1}^{-1} & 0 \\ 0 & \|W_i^2\|_{L^1}^{-1} \end{bmatrix}, \end{aligned} \quad (21)$$

yielding

$$E_{21} \cong h^{-2} \cdot c_1 \cdot \|W_i^1\|_{L^1}^{-1}, \quad (22)$$

and

$$E_{22} \cong h^{-2} \cdot c_2 \cdot \|W_i^2\|_{L^1}^{-1}, \quad (23)$$

being c_1 and c_2 constant values. In matrix \mathbf{K} , the terms E_{21} and E_{22} are multiplied by W_i^1 and W_i^2 , whose order is h . Hence, it can be written that

$$\mathbf{K} = h^{-1} \cdot \mathbf{T}, \quad (24)$$

being \mathbf{T} a constant matrix.

From Equation (20), we can then evaluate the errors e'''^i as

$$\begin{bmatrix} e'''^1 \\ e'''^2 \\ \vdots \\ e'''^N \end{bmatrix} = h \cdot \mathbf{T}^{-1} \begin{bmatrix} O(h) \\ O(h) \\ \vdots \\ O(h) \end{bmatrix} = \begin{bmatrix} O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \end{bmatrix}. \quad (25)$$

We may then conclude that, whereas the FPM procedure guarantees just a first-order error to evaluate the second derivative of a given function, it is second-order accurate in the associated source problem, that is, in the evaluation of the function given its second derivative.

3. 1D NUMERICAL TESTS

The aim of these tests is to address the static problem for an elastic rod (assumed to have unitary length and axial stiffness), determining the axial displacement u at particle positions given an axial body load f , and comparing the obtained results with the exact solution. The second-order differential equation governing the problem is

$$-u(x)'' = f(x), \quad x \in (0, 1), \quad (26)$$

with appropriate boundary conditions.

The tests are carried out on the domain $\Omega = [0, 1]$, with a uniformly spaced particle discretization and a parameter h such that every projection function spans three particles. In particular, we consider two examples. In the first one, we assume $f = \sin(\pi x)$, and clamped ends for $x = 0$ and for $x = 1$. Hence, we impose $u(0) = u(1) = 0$ and the corresponding exact solution is $u = \sin(\pi x)/\pi^2$. In the second example, we assume $f = e^x$, a clamped end for $x = 0$ and a traction-free condition for $x = 1$. Hence, we impose $u(0) = u'(1) = 0$ and the corresponding exact solution is $u = -e^x + ex + 1$.

To evaluate convergence orders, the infinity norm of the difference between the exact and the numerical solutions is computed for an increasing number of particles discretizing the domain. The following combinations of symmetric and skew-symmetric functions, guaranteeing the non-singularity of the matrix \mathbf{A} , are considered (see Equations (13)–(16)):

- constant + linear;
- constant + cubic;
- linear + quadratic;
- quadratic + cubic.

The obtained numerical results are reported for the two proposed examples in Figures 1 and 2, respectively, where it can be observed that the approximation procedure presents a second-order convergence for all projection function choices.

4. 2D FORMULATION

The new FPM formulation, presented in the previous sections for the one-dimensional case, can be easily extended to higher dimensions. In this section we limit for simplicity to the 2D case and, in particular, we discuss and study the formulation for the approximation of the Poisson problem, which is of particular interest in engineering, since it models, among others, the elastic membrane and the heat diffusion problems.

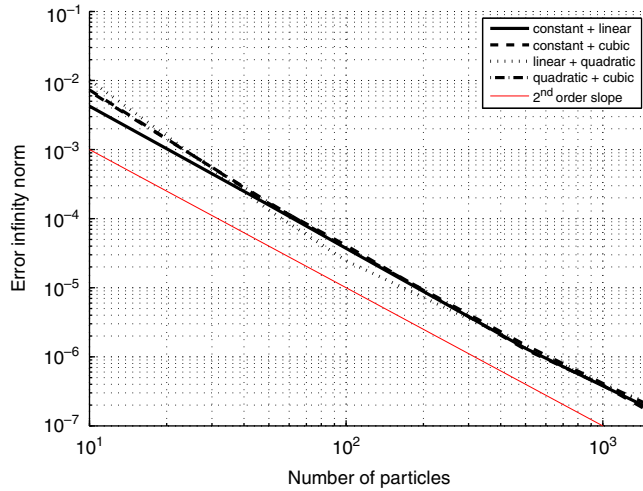


Figure 1. 1D second-order problem: infinity norm error versus number of particles for a sinusoidal load, with Dirichlet homogeneous boundary conditions.

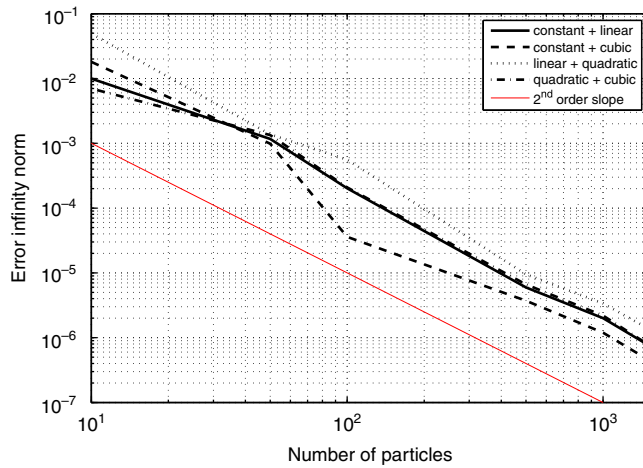


Figure 2. 1D second-order problem: infinity norm error versus number of particles for an exponential load, with a Dirichlet and a Neumann homogeneous boundary conditions.

4.1. Modified FPM formulation in 2D: approximation of derivatives

Considering a Taylor’s expansion for a generic 2D function $B(x, y)$ up to the second order, the following equation holds:

$$\begin{aligned}
 B(x, y) = & B(x_i, y_i) + (x - x_i)D_x B(x_i, y_i) + (y - y_i)D_y B(x_i, y_i) + \frac{1}{2}(x - x_i)^2 D_x^2 B(x_i, y_i) \\
 & + \frac{1}{2}(y - y_i)^2 D_y^2 B(x_i, y_i) + (x - x_i)(y - y_i)D_{xy}^2 B(x_i, y_i) + O(|x - x_i|^3). \quad (27)
 \end{aligned}$$

Multiplying both sides of the equation by five projection functions W_i^k ($k = 1, \dots, 5$) defined on a compact support and integrating over the domain, we obtain

$$\begin{aligned}
 D_x B(x_i, y_i) \left[\int_{\Omega} (x - x_i) W_i^k(x, y) dx dy \right] + D_y B(x_i, y_i) \left[\int_{\Omega} (y - y_i) W_i^k(x, y) dx dy \right] \\
 + D_x^2 B(x_i, y_i) \left[\int_{\Omega} \frac{1}{2} (x - x_i)^2 W_i^k(x, y) dx dy \right] + D_y^2 B(x_i, y_i) \left[\int_{\Omega} \frac{1}{2} (y - y_i)^2 W_i^k(x, y) dx dy \right]
 \end{aligned}$$

$$\begin{aligned}
 & +D_{xy}^2 B(x_i, y_i) \left[\int_{\Omega} (x-x_i)(y-y_i) W_i^k(x, y) dx dy \right] \\
 = & \left[\int_{\Omega} (B(x, y) - B(x_i, y_i)) W_i^k(x, y) dx dy \right] + e'_k, \quad k=1, \dots, 5, \tag{28}
 \end{aligned}$$

where terms e'_k represent the error related to series truncation. Similar to the 1D approach, Equations (28) can be rearranged as

$$\mathbf{A} \begin{bmatrix} D_x B(x_i, y_i) \\ D_y B(x_i, y_i) \\ D_x^2 B(x_i, y_i) \\ D_y^2 B(x_i, y_i) \\ D_{x,y}^2 B(x_i, y_i) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^1(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^2(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^3(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^4(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^5(x, y) dx dy \end{bmatrix} + \begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \\ e'_4 \\ e'_5 \end{bmatrix}, \tag{29}$$

where \mathbf{A} is a 5×5 matrix whose terms represent the first- and second-order momenta of the W_i^k functions. Then, neglecting the e'_k errors and inverting the resulting system, the values of the first and the second derivatives of $B(x, y)$ are obtained as

$$\begin{bmatrix} D_x B(x_i, y_i) \\ D_y B(x_i, y_i) \\ D_x^2 B(x_i, y_i) \\ D_y^2 B(x_i, y_i) \\ D_{x,y}^2 B(x_i, y_i) \end{bmatrix} = \mathbf{E} \begin{bmatrix} \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^1(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^2(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^3(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^4(x, y) dx dy \\ \int_{\Omega} [B(x, y) - B(x_i, y_i)] W_i^5(x, y) dx dy \end{bmatrix}, \tag{30}$$

with $\mathbf{E} = \mathbf{A}^{-1}$.

Following now an approach similar to that presented in the previous section for the 1D case, it can be shown that in Equation (30) the evaluation of the first derivatives is affected by an $O(h^2)$ error, whereas the evaluation of the second derivatives is affected by an $O(h)$ error.

4.2. Modified FPM formulation in 2D: approximation of source problems

Equations (30) can be rearranged to solve a source problem, that is, to evaluate a function $B(x, y)$ given $P(x, y) = QB(x, y)$, where Q is a generic linear differential operator depending on the first and the second derivatives of $B(x, y)$.

As an example, the Laplace operator $\Delta = D_x^2 + D_y^2$ is here considered and the formulation to solve the Poisson problem is investigated, that is, given $P(x, y)$ in all particles and suitable boundary conditions, we look for a function $B(x, y)$ satisfying

$$\Delta B(x, y) = P(x, y). \tag{31}$$

To write our FPM approximation of (31), the third and fourth equations of system (30), expressing $D_x^2 B(x_i, y_i)$ and $D_y^2 B(x_i, y_i)$, respectively, have to be discretized and summed up, yielding, for the i th particle

$$\begin{aligned}
 & E_{31} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^1(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{32} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^2(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{33} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^3(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{34} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^4(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{35} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^5(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{41} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^1(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{42} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^2(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{43} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^3(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{44} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^4(x_j, y_j) \Delta x_j \Delta y_j \\
 & + E_{45} \sum_j [B(x_j, y_j) - B(x_i, y_i)] W_i^5(x_j, y_j) \Delta x_j \Delta y_j = P(x_i, y_i), \quad i = 1, \dots, N. \quad (32)
 \end{aligned}$$

Computing Equation (32) for every particle, the following linear algebraic system is obtained:

$$\mathbf{K} \begin{bmatrix} B(x_1, y_1) \\ B(x_2, y_2) \\ \vdots \\ B(x_N, y_N) \end{bmatrix} = \begin{bmatrix} P(x_1, y_1) \\ P(x_2, y_2) \\ \vdots \\ P(x_N, y_N) \end{bmatrix}, \quad (33)$$

being \mathbf{K} an $N \times N$ matrix, whose terms depend on the matrix \mathbf{E} and on the functions W^k ($k = 1, \dots, 5$). System (33) can be finally solved to obtain the values of the function $B(x, y)$ in each particle. Also in this case, in a way similar to that used in the 1D case, it can be proved that this formulation for the evaluation of $B(x, y)$ is affected by an $O(h^2)$ error.

A remark is in order about the choice of the five projection functions W^k . The only condition to be respected is that \mathbf{A} matrix has to be non-singular for each particle and it can be easily proved that the following set of functions does respect this condition:

- Linear function in x -direction:

$$\begin{aligned}
 W_i^1(x, y) &= (x - x_i) \quad \forall(x, y) \quad \text{s.t. } |x - x_i| < h \quad \text{and} \quad |y - y_i| < h, \\
 W_i^1(x, y) &= 0 \quad \text{otherwise.}
 \end{aligned} \quad (34)$$

- Linear function in y -direction:

$$\begin{aligned}
 W_i^2(x, y) &= (y - y_i) \quad \forall(x, y) \quad \text{s.t. } |x - x_i| < h \quad \text{and} \quad |y - y_i| < h, \\
 W_i^2(x, y) &= 0 \quad \text{otherwise.}
 \end{aligned} \quad (35)$$

- Quadratic function in x -direction:

$$\begin{aligned} W_i^3(x, y) &= (x - x_i)^2 \quad \forall(x, y) \quad \text{s.t. } |x - x_i| < h \quad \text{and} \quad |y - y_i| < h, \\ W_i^3(x, y) &= 0 \quad \text{otherwise.} \end{aligned} \quad (36)$$

- Quadratic function in y -direction:

$$\begin{aligned} W_i^4(x, y) &= (y - y_i)^2 \quad \forall(x, y) \quad \text{s.t. } |x - x_i| < h \quad \text{and} \quad |y - y_i| < h, \\ W_i^4(x, y) &= 0 \quad \text{otherwise.} \end{aligned} \quad (37)$$

- Bilinear function:

$$\begin{aligned} W_i^5(x, y) &= (x - x_i)(y - y_i) \quad \forall(x, y) \quad \text{s.t. } |x - x_i| < h \quad \text{and} \quad |y - y_i| < h, \\ W_i^5(x, y) &= 0 \quad \text{otherwise.} \end{aligned} \quad (38)$$

We finally highlight that this set is just one of the possible choices leading to a non-singular \mathbf{A} matrix.

5. 2D NUMERICAL TESTS

With the modified FPM approach outlined above and using the projection functions in (34)–(38), we now tackle the following 2D Poisson problem defined on the domain $\Omega = [0, 1] \times [0, 1]$:

$$\begin{aligned} -\Delta u(x, y) &= f(x, y) \quad \forall x \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \quad (39)$$

with

$$f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y), \quad (40)$$

such that the exact solution is

$$u(x, y) = \sin(\pi x) \sin(\pi y). \quad (41)$$

To implement the modified FPM formulation, the domain Ω is discretized with equally spaced particles in x - and y -directions, as shown, e.g., in Figure 3 (left). Moreover, to test the robustness of the method, a (substantial) random perturbation of maximum amplitude $\pm\delta/4$ (where δ is the distance, along x or y , between two consecutive particles in the uniform case) is applied to the particle distribution, as shown, e.g., in Figure 3 (right). To evaluate the convergence order, problem (39) is then solved in both cases of uniform and perturbed particle distributions and the infinity norms of the difference between the exact and the numerical solutions are computed. These values are plotted in Figure 4 in logarithmic scale, as a function of the number of particles per direction. As it is expected the method shows a second-order convergence, also in the case where non-uniform particle distributions are adopted.

6. CONCLUSIONS

In this work, moving from the particle formulation proposed in [14, 15], we have focused on the modified FPM method presented in [16]. In particular, we have conducted an error analysis in 1D, which can be extended to 2D, showing the second-order accuracy of the formulation when applied to source second-order problems. We have also discussed the choice of the projection functions and have proposed in 1D and 2D sets of functions simpler than the classical Gaussian-based projections. We have finally presented 1D and 2D numerical tests highlighting the second-order convergence of the method.

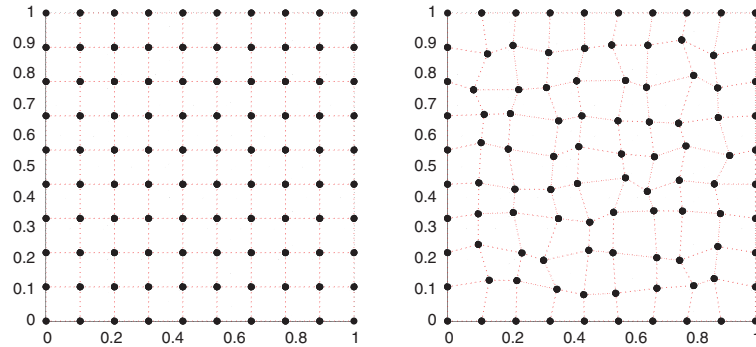


Figure 3. 10×10 particle distribution over the domain $\Omega = [0, 1] \times [0, 1]$ in the uniform (left) and randomly perturbed (right) cases. The random perturbation has a maximum amplitude of $\pm\delta/4$, where δ is the distance, along x or y , between two consecutive particles in the uniform case.

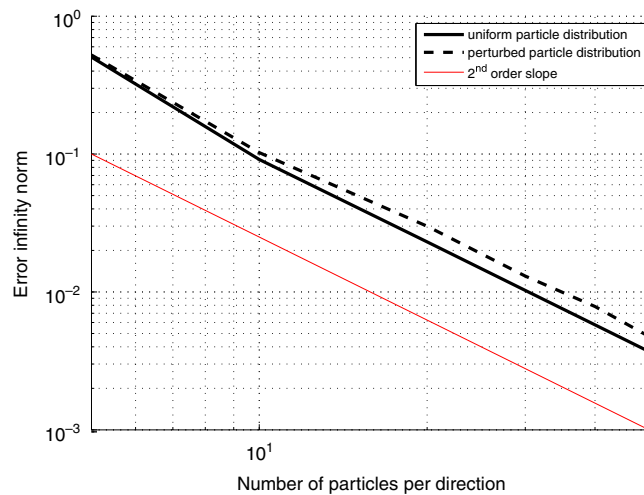


Figure 4. 2D Poisson problem: infinity norm error versus number of particles per direction for a sinusoidal load, with Dirichlet homogeneous boundary conditions, for both cases of uniform and non-uniform particle distributions.

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