A metric approach to plasticity via Hamilton-Jacobi equation*

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October 6, 2009

Abstract

Thermodynamical consistency of plasticity models is usually written in terms of the so-called "maximum dissipation principle". In this paper, we discuss constitutive relations for dissipative materials written through suitable generalized gradients of a (possibly non-convex) metric. This new framework allows us to generalize the classical results providing an interpretation of the yield function in terms of Hamilton-Jacobi Equations theory.

Key words: plasticity models, metric associated to Hamilton-Jacobi equation, dissipative metric, energetic formulation

*This work has been partially supported by IMATI-CNR Pavia (Italy)

AMS (MOS) 74C05 (primary), 49L25 (secondary)

1 Introduction

This paper deals with consistency of plasticity models with respect to thermodynamics laws. Indeed, it is known that in a classical formulation of plasticity theory thermodynamically consistency (cf., e.g., [7]) is expressed in terms of the so-called *maximum dissipation principle*.

Classical formulations assume that the elastic region corresponds to a convex set defined in terms of the stress, through the so called *yield function*. The stresses are constrained to belong to this set, whose boundary gives the plasticity surface. Hence, requiring that the evolution of the plastic deformation is normal to this region one gets the maximum dissipation principle, by convex analysis tools, from which the consistency of the model follows. This theory is based on the possibility of finding a convex elastic region described in terms of a convex yield-function.

However, the more frequent use of new materials in engineering applications and the consequent experimental investigations show that more complex and general yield criteria should be accounted for. Moreover, dealing directly with the second principle of thermodynamics we recall that the maximum dissipation principle is a sufficient but not a necessary condition. Thus, one may wonder if we can find more general constitutive relations between the stress and the plastic evolution, which turn out to be consistent with the second principle.

The main novelty of this paper consists in dealing with more general constitutive relations between the stress and the plastic evolution, ensuring thermodynamical consistency of the model. In particular, we deal with more general yield-function (non-convex and possibly depending on the state) defining the set of admissible dissipation for the system. Hence, we introduce the yield function and the set of admissible dissipations by use of the Hamilton-Jacobi theory, in terms of a suitably introduced dissipative metric.

We proceed into two directions, which are at the end equivalent. We first consider a "geometric" and "experimental" approach, prescribing the set of admissible dissipative forces. This set is required to be sufficiently smooth, but not necessary convex and it may depend on the state. Then, exploiting some results of Hamilton-Jacobi equation theory, we show that a "dissipative metric" is naturally associated to this set, in the sense that this metric is a solution of the Hamilton-Jacobi equation associated to the "dissipations" set. In particular this metric is defined in the set of the states and gives a measure of the dissipation produced along a trajectory. It is shown that the function

in terms of which the equation is written corresponds to a (generalized) yield function. On the other hand we can proceed prescribing the yield function is known (sufficiently smooth but not necessarily convex!) and then define the set of admissible dissipative forces (as a sublevel of the yield function). Eventually, the relation between the dissipative forces and the dissipative evolution is written by a generalized sub-gradient of the dissipative metric. Note that this leads to the more classical relation in the case of a convex yield-function not depending on the state.

Before proceeding, let us briefly outline our work. We first introduce the model for plasticity, comparing two different approaches. The first is more classical, while the second one refers to a model introduced in the framework of phase transitions models. In both cases thermodynamical consistency is written by a duality relation between possible dissipative forces in the system and the evolution directions of the states. Thus, in the next part of the paper we mainly focus on constitutive relations for dissipation (and evolution). In Section 3.1 we introduce a geometrical approach to define the set of admissible dissipations. Then, it is proved that this set may be defined in terms of the solution (which is interpreted in terms of a dissipative metric) to an Hamilton-Jacobi equation. Hence, we investigate both the convex and non-convex cases. Eventually, in Section 3.4, we are able to find a suitable constitutive equation for dissipative forces in terms of the above introduced dissipative metric, ensuring (3.1). Finally, in Section 4 we focus on the main results of our analysis as well as some still open questions.

In the following we mainly use notation and results of the theory on Hamilton-Jacobi equation and dissipative distances. The reader may refer to Section 5 for all definitions and notation.

2 The model

In this Section we present the plasticity model we are going to investigate. We first recall the fundamental thermomechanical laws and then discuss constitutive assumptions for the involved physical quantities. We introduce two different approaches: the first mainly refers to a "classical" plasticity theory, as for example described by Lubliner in [7], the second one is less "classical" and it is based on a modelling theory originally introduced by Frémond for solid-solid phase transitions [5]. In this last case we consider internal plasticity variables whose evolution is governed by internal forces which have to be included in the global energy balance of the system. Thus, the principle of virtual power is generalized accounting for the action of internal forces and microscopic velocities.

2.1 Preliminaries and thermomechanical laws

We consider a body located in a (sufficiently smooth) bounded domain Ω, whose boundary is $\Gamma := \partial\Omega$ with $\mathbf{n}$ the outward normal unit vector. Hence, we use the following notation: $\theta > 0$ is the absolute temperature, η the entropy, and e the internal energy; $\mathbf{u}$ the vector of (small) displacements, $\mathbf{v} = \mathbf{u}_t$ the velocity field; $\mathbf{a}$ the acceleration; ρ the mass density. Finally, the symmetric linearized strain tensor $\boldsymbol{\varepsilon}$ is defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[D\mathbf{u} + (D\mathbf{u})^T\right],$$

while $\boldsymbol{\sigma}$ stands for the Cauchy stress tensor (cf. [7]).

We first recall the balance momentum equation, which can be deduced following two different approaches. In the first case it is prescribed a balance of internal and external forces holding in any subdomain $\mathcal{D} \subseteq \Omega$:

$$\int_{\mathcal{D}} \rho\mathbf{b} + \int_{\partial\mathcal{D}} \boldsymbol{\sigma}\cdot\mathbf{n} = 0 \tag{2.1}$$

where $\mathbf{b}$ is a (density) action at a distance, while the traction $\boldsymbol{\sigma}\cdot\mathbf{n}$ is a contact force. As (2.1) holds for any subdomain $\mathcal{D} \subseteq \Omega$, applying the Gauss Theorem, we get the equation

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \mathbf{0} \quad \text{in } \Omega. \tag{2.2}$$

Remark 2.1. Let us comment about the local version (2.2) of the momentum balance. Note that, in (2.1) boundary conditions for $\boldsymbol{\sigma}$ are not directly given. Usually, assuming (2.1) as balance law one has to specify them *a posteriori*. In the following we are showing a different approach in which the traction on the boundary $\partial\Omega$ is directly written in the balance law, which is directly written in the domain Ω but in a variational formulation, i.e. as a balance between the virtual power of interior and exterior forces.

The *Principle of virtual power* is written in the whole body Ω and for any virtual velocity

$$P_i + P_e = P_a \quad \text{in } \Omega, \tag{2.3}$$

where P_i denotes the power of interior forces ($-P_i$ is usually denoted as "power of deformation", see [7]), P_e the power of exterior forces, and P_a the power of accelerations. The power of acceleration forces P_a is given by the time derivative of the kinetic energy K (associated to the virtual velocity $\mathbf{v}$), i.e.

$$K = \frac{1}{2}\int_\Omega \rho\mathbf{v}\cdot\mathbf{v} \quad \text{and} \quad P_a = K_t = \int_\Omega \rho\mathbf{v}\cdot\mathbf{a}. \tag{2.4}$$

Hence, P_e is written for any macroscopic velocity $\mathbf{v}$ as follows

$$P_e = \int_\Omega \rho \mathbf{b} \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v}, \tag{2.5}$$

where now $\mathbf{t}$ is a known traction acting on the boundary (see Remark 2.1). Finally, it is assumed that

$$P_i = -\int_\Omega \boldsymbol{\sigma} \cdot D\mathbf{v}. \tag{2.6}$$

Let us point out that (2.6) reflects the fact that the power of interior forces is zero for any rigid motion.

Now let us comment about the derivation of (2.2) from (2.3). We assume that we are dealing with a quasi-static phenomenon, so that it is $P_a = 0$. As (2.3) has to be satisfied for any virtual velocity $\mathbf{v}$ in Ω, (2.5) and (2.6) eventually lead to the momentum balance equation (2.2) combined with the boundary condition

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \quad \text{on } \partial\Omega. \tag{2.7}$$

We introduce the *First principle of thermodynamics*

$$\frac{d}{dt}(E + K) = P_e + Q \quad \forall \mathcal{D} \subseteq \Omega \tag{2.8}$$

where K is the kinetic energy (cf. (2.4)) considered in a subdomain $\mathcal{D}$, $E(\mathcal{D})$ the (total) internal energy (e is the internal energy density)

$$E(\mathcal{D}) := \int_{\mathcal{D}} \rho e, \tag{2.9}$$

and P_e (cf. (2.5)) is now the actual power of the exterior forces in a subdomain $\mathcal{D}$ (i.e. it is written for the *actual* velocity and in $\mathcal{D}$). Equation (2.8) states that the variation of the (internal and kinetic) energy in any subdomain of the system is given by the actual power of external forces and the heat amount in the body portion, represented by

$$Q := \int_{\mathcal{D}} \rho r - \int_{\partial\mathcal{D}} \mathbf{q} \cdot \mathbf{n}. \tag{2.10}$$

Here, r is a heat density volume source and $\mathbf{q} \cdot \mathbf{n}$ the heat flux through the boundary. Recalling that $P_a = K_t$, the first principle (2.8) can be rewritten by use of (2.3) as follows

$$\frac{d}{dt}E = -P_i + Q \quad \forall \mathcal{D} \subseteq \Omega, \tag{2.11}$$

where P_i is the actual power of interior forces. It is of the form

$$P_i = \int_{\mathcal{D}} p_i.$$

To write a "local energy balance", we exploit the Gauss theorem (cf. (2.10) and (2.9)) and get in Ω

$$\rho e_t + \text{div } \mathbf{q} = \rho r - p_i. \tag{2.12}$$

The *Second principle of thermodynamics* is written in the form of the Clausius-Duhem inequality as follows (we directly write it in Ω for the sake of simplicity)

$$\frac{d}{dt}\Xi \geq \widetilde{Q}, \tag{2.13}$$

where the "total" entropy is

$$\Xi := \int_{\Omega} \rho\eta, \tag{2.14}$$

with η the entropy density. The "entropy amount" $\widetilde{Q}$ is defined in terms of the entropy flux $\mathbf{q}/\theta$, i.e.

$$\widetilde{Q} := \int_{\Omega} \rho\frac{r}{\theta} - \int_{\Gamma} \left(\frac{\mathbf{q}}{\theta}\right) \cdot \mathbf{n}. \tag{2.15}$$

The second principle corresponds to assume that the variation of the entropy of the body is greater than the entropy amount given to the body by heat flux and heat external sources renormalized by the temperature. In fact it requires that some dissipation appears in the system and it is non-negative. As for the internal energy balance (cf. (2.12)) we are allowed to write the local version of the second principle (2.13) by use of the Gauss Theorem applied to (2.15) and (2.14)

$$\rho\eta_t - \rho\frac{r}{\theta} + \text{div } \left(\frac{\mathbf{q}}{\theta}\right) \geq 0. \tag{2.16}$$

2.2 State variables, energy potentials, and constitutive relations

The choice of the state variables, in terms of which the thermomechanical equilibrium of the system is defined, depends on the physical problem to be described. In the following we fix as state variables

$$\mathcal{E} = \{\theta, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\},$$

where the absolute temperature θ and the strain $\boldsymbol{\varepsilon}$ are the external ones, while plastic strain $\boldsymbol{\varepsilon}^p$ is an internal one.

Internal variables are related to the (micro)-structure of the material and, in general, it is not possible to measure them directly, which is the reason why they are called "internal".

Following a theory known as "associated local state" (cf. [6]) we assume that there exists a thermodynamical potential describing the equilibrium of the system at each instant of its evolution (in our case we consider the free energy functional). A system is in thermomechanical equilibrium if the *state variables* remain constant, that is the system has no tendency to change if there is no change in the external controls. Thus, the state of the equilibrium is defined by the stationary points of this potential, which depends only on $\mathcal{E}$ (and not on velocities).

The free energy functional $\Psi(\mathcal{E})$ is related to the internal energy e, the entropy η, and the absolute temperature θ by the Helmhotz relation (cf. Remark 2.2)

$$\Psi = e - \theta\eta \quad \text{with} \quad \eta = -\frac{\partial \Psi}{\partial \theta}. \tag{2.17}$$

Remark 2.2. For the sake of completeness and in order to justify (2.17), let us point out how Ψ is defined.

In thermodynamics the entropy η can be chosen as state variable in place of the temperature θ. Hence, in this case, the associated energy functional is the internal energy functional $e(\eta, \cdot)$, which is known to be convex w.r.t. the entropy η. Hence, its Legendre transformation, taken w.r.t. η, is defined by

$$e^*(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\varepsilon}^p) = \sup_{\eta}\{\eta\theta - e(\boldsymbol{\varepsilon}, \eta, \boldsymbol{\varepsilon}^p)\}, \tag{2.18}$$

where the temperature

$$\theta = \frac{\partial e}{\partial \eta}$$

is the conjugate variable of the entropy. The free energy functional Ψ is defined as

$$\Psi(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\varepsilon}^p) := -e^*(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\varepsilon}^p), \tag{2.19}$$

so that, in particular, it turns out to be concave w.r.t. the temperature. Then, assuming that the functional e is sufficiently regular, standard convex analysis arguments applied to (2.19) and (2.18) lead to (2.17)

Now, we can rewrite local energy balance (2.12) by use of (2.17), applying

the chain rule. We get

$$\begin{aligned}\rho e_t + \text{div } \mathbf{q} &= \rho \left(\Psi + \theta\eta\right)_t + \text{div } \theta\frac{\mathbf{q}}{\theta} \\ &= \rho\left(\frac{\partial\Psi}{\partial\theta}\theta_t + \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t + \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}{\boldsymbol{\varepsilon}^p}_t + \theta_t\eta + \theta\eta_t\right) \\ &+ \theta\text{div } \frac{\mathbf{q}}{\theta} + \frac{\mathbf{q}}{\theta}\nabla\theta = \rho r - p_i.\end{aligned} \tag{2.20}$$

Then, recalling that

$$\eta = -\frac{\partial\Psi}{\partial\theta}$$

(cf. (2.17)) two terms cancel and, from (2.20), we eventually get

$$\theta\left(\rho\eta_t + \text{div } \frac{\mathbf{q}}{\theta} - \frac{\rho r}{\theta}\right) = -p_i - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}{\boldsymbol{\varepsilon}^p}_t - \frac{\mathbf{q}}{\theta}\nabla\theta. \tag{2.21}$$

Comparing (2.21) with (2.16) we are able to find conditions to ensure thermodynamical consistency. Indeed, (2.16) follows from (2.21) once it is ensured that (recall that $\theta > 0$)

$$-p_i - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}{\boldsymbol{\varepsilon}^p}_t - \frac{\mathbf{q}}{\theta}\nabla\theta \geq 0. \tag{2.22}$$

Our aim is to give constitutive relations for the involved physical quantities and specify some properties required on the energy potential(s) to satisfy (2.22) (and consequently (2.16)).

First, the heat flux law for $\mathbf{q}$ is chosen to satisfy

$$-\frac{\mathbf{q}}{\theta}\nabla\theta \geq 0. \tag{2.23}$$

We do not enter the details of this choice as we are mainly dealing with isothermal case. In the case of the Fourier heat flux law it is $\mathbf{q} = -\lambda\nabla\theta$, $\lambda > 0$.

Now, it remains to ensure that

$$-p_i - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}{\boldsymbol{\varepsilon}^p}_t \geq 0. \tag{2.24}$$

We are presenting two different possibilities. The first idea follows a more "classical" approach, as for example presented in the textbook by Lubliner [7]. The second one follows a somehow less "classical" approach, introduced for (solid-solid) phase transitions by Frémond (cf. [5]). The difference between the two approaches mainly consists in definition of the power of internal forces p_i.

2.3 Approach 1 (cf. [6] and [7])

The power of internal forces p_i is written as in (2.6), for any macroscopic velocity $\mathbf{v}$. Thus, combining (2.6), written for the actual velocity $\mathbf{u}_t$ (so that $D\mathbf{u}_t = \boldsymbol{\varepsilon}_t$), with (2.24) yields

$$\left(\boldsymbol{\sigma} - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\right)\boldsymbol{\varepsilon}_t - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}{\boldsymbol{\varepsilon}^p}_t \geq 0. \tag{2.25}$$

Expression (2.25) highlights the presence of two possible "dissipative" forces, i.e. possible forces which may produce dissipation, defined as

$$\boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}, \quad \mathbf{A} := -\rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}, \tag{2.26}$$

where $\boldsymbol{\sigma}^d$ is the difference between the actual stress and the "non dissipative" stress $\rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}$ while $\mathbf{A}$ stands for a plasticity force. Then, letting $\mathbf{F}^d = (\boldsymbol{\sigma}^d, \mathbf{A})$ we can rewrite (2.25) as $\mathbf{F}^d \cdot (\boldsymbol{\varepsilon}_t, {\boldsymbol{\varepsilon}^p}_t) \geq 0$.

In order to ensure that (2.25) holds one has to find suitable constitutive equations for $\mathbf{F}^d = (\boldsymbol{\sigma}^d, \mathbf{A})$. This problem will be faced in the next Sections. In the general theory for plasticity, as presented by Lubliner, constitutive conditions (2.25) are ensured by an evolution equation written for $\boldsymbol{\varepsilon}^p$ and completing the system (cf. (2.2)).

Remark 2.3. Let us now briefly comment on relation between (2.25) and the classical theory for plasticity. We give some further details on a possible evolution law for $\boldsymbol{\varepsilon}^p$ used in the literature. For simplicity, let now $\rho = 1$ and consider the isothermal case. The energy $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ is usually introduced as

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)\mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \tag{2.27}$$

$\mathbf{E}$ being the elasticity matrix. Then, it is assumed that $\boldsymbol{\sigma}^d = \mathbf{0}$ so that it follows (cf. (2.26))

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \quad \mathbf{A} = -\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p} = \mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma}. \tag{2.28}$$

Thus, by virtue of (2.28), (2.25) simply leads to

$$\boldsymbol{\sigma}\boldsymbol{\varepsilon}^p_t \geq 0. \tag{2.29}$$

In order to ensure (2.29) constitutive relations are recovered by means of a convex yield function f defining the set $\widetilde{K}$ of admissible dissipative forces or, as it is $\mathbf{A} = \boldsymbol{\sigma}$, of admissible stresses,

$$\widetilde{K} := \{\boldsymbol{\sigma} : f(\boldsymbol{\sigma}) \leq 0\}. \tag{2.30}$$

As f is convex it follows that $\widetilde{K}$ is a convex set. Hence, it is prescribed the following constitutive relation between velocities ${\boldsymbol{\varepsilon}^p}_t$ and dissipative pasticity force $\mathbf{A} = \boldsymbol{\sigma}$ (cf. (2.28))

$$\boldsymbol{\varepsilon}^p_t \in \partial I_{\widetilde{K}}(\boldsymbol{\sigma}), \tag{2.31}$$

where $I_{\widetilde{K}}$ is the indicator function of the convex $\widetilde{K}$ and the subdifferential is intended in the sense of convex analysis (so that it is in particular a maximal monotone graph). For the sake of completeness, we recall that the domain of the subdifferential operator is $\widetilde{K}$ and $\partial I_{\widetilde{K}}(\boldsymbol{\sigma}) = 0$ if $\boldsymbol{\sigma}$ belongs to the interior of $\widetilde{K}$ while $\partial I_{\widetilde{K}}$ is the normal to the boundary if $\boldsymbol{\sigma}$ is on the boundary of the convex set. Thus, by definition of the subdifferential (cf. [2]), it follows

$$\boldsymbol{\varepsilon}^p_t(\mathbf{y} - \boldsymbol{\sigma}) \leq 0 \quad \forall \mathbf{y} \in \widetilde{K}, \tag{2.32}$$

which is known as the "maximum dissipation principle". Note that in the case $\mathbf{0} \in \widetilde{K}$, (2.32) implies that (2.29) holds. Moreover, let us point out that relation (2.31) represents a constraint on $\mathbf{A}$, and thus on $\boldsymbol{\sigma}$, as it implies $\boldsymbol{\sigma} \in \widetilde{K}$.

Hence, one could introduce a dissipative functional Φ (which is usually called pseudo-potential of dissipation, see the subsequent Approach 2 for further details) depending only on the velocity ${\boldsymbol{\varepsilon}^p}_t$ (recall that $\boldsymbol{\sigma}^d = \mathbf{0}$). More precisely, the functional Φ is the convex conjugate function of the indicator of the set $\widetilde{K}$, defined by (2.30), $I_{\widetilde{K}}(\sigma)$. It is

$$\Phi({\boldsymbol{\varepsilon}^p}_t) = I^*_{\widetilde{K}}({\boldsymbol{\varepsilon}^p}_t) = \sup_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}{\boldsymbol{\varepsilon}^p}_t - I_{\widetilde{K}}(\boldsymbol{\sigma})) = \sup_{f(\boldsymbol{\sigma})\leq 0} \boldsymbol{\sigma}{\boldsymbol{\varepsilon}^p}_t, \tag{2.33}$$

where f is the yield function defining the set of admissible stresses $\widetilde{K}$. It is known from convex analysis that

$$\partial\Phi = (\partial I_{\widetilde{K}})^{-1}. \tag{2.34}$$

Thus (2.31) can be rewritten as

$$\mathbf{A} = \boldsymbol{\sigma} = \partial\Phi({\boldsymbol{\varepsilon}^p}_t) = (\partial I_{\widetilde{K}})^{-1}({\boldsymbol{\varepsilon}^p}_t). \tag{2.35}$$

2.4 Approach 2

We now extend to plasticity a different approach, originally proposed by Frémond to describe solid-solid phase transitions. In this theory, providing a macroscopic description of the phenomenon, the forces which are responsible for (microscopic) changes in the structure of the material have to be included in the global balance of the energy of the thermomechanical system itself (cf.

(2.12)). This leads to a generalized definition of p_i including the effects of micro-forces and velocities (related to the evolution of internal variables). One directly recovers two balance equations: the classical momentum balance and a new balance equation for internal variables. In this paper we present a new application of this theory to plasticity.

Now, let us make precise our approach. We consider two different virtual velocities: a macroscopic velocity $\mathbf{v}$ and a microscopic velocity $\boldsymbol{\gamma}$ (the corresponding actual velocities are $\mathbf{u}_t$ and ${\boldsymbol{\varepsilon}^p}_t$, respectively). The external power is defined as in (2.5), i.e. it is assumed that there are not exterior forces acting at a microscopic level. The power of interior forces $\mathcal{P}_i$ is now written for any $\mathbf{v}$ and $\boldsymbol{\gamma}$. We have

$$P_i = -\int_\Omega \boldsymbol{\sigma} \cdot D\mathbf{v} - \int_\Omega \mathbf{B} \cdot \boldsymbol{\gamma}. \tag{2.36}$$

Here, $\mathbf{B}$ can be interpreted as an internal force responsible for the evolution of $\boldsymbol{\varepsilon}^p$.

Remark 2.4. Note that (2.36) is in accordance with the axiom of rigid motions. Indeed for a rigid motion ($D\mathbf{v} = \mathbf{0}$) no changes occur in the microstructure of the system and consequently internal variables do not evolve, yielding $\boldsymbol{\gamma} = \mathbf{0}$.

As (2.36) and (2.5) are written for any $\boldsymbol{\gamma}$ and $\mathbf{v}$, we recover two balance equations: the momentum balance (2.2) and the following new balance equation

$$\mathbf{B} = \mathbf{0}. \tag{2.37}$$

Hence, we combine (2.22) (and (2.23)) with (2.36), written in Ω and for the actual velocities $\mathbf{u}_t$ (and $\boldsymbol{\varepsilon}_t$) in place of $\mathbf{v}$ (and $D\mathbf{v}$) and ${\boldsymbol{\varepsilon}^p}_t$ in place of $\boldsymbol{\gamma}$. We get the condition for thermodynamical consistency (cf. (2.24))

$$\left(\boldsymbol{\sigma} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}\right)\boldsymbol{\varepsilon}_t + \left(\mathbf{B} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p}\right){\boldsymbol{\varepsilon}^p}_t \geq 0. \tag{2.38}$$

Equation (2.38) highlights the presence of two possible dissipative forces, i.e. possible forces which amy produce dissipation, defined as

$$\boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{B}^d := \mathbf{B} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p}$$

i.e. as the difference between $\boldsymbol{\sigma}$ and $\mathbf{B}$ and non dissipative forces $\boldsymbol{\sigma}^{nd}$ and $\mathbf{B}^{nd}$, where

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{nd} + \boldsymbol{\sigma}^d, \quad \boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\sigma}^{nd} := \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \tag{2.39}$$

and

$$\mathbf{B} = \mathbf{B}^{nd} + \mathbf{B}^{d}, \quad \mathbf{B}^{d} := \mathbf{B} - \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{p}}, \quad \mathbf{B}^{nd} := \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{p}}. \tag{2.40}$$

Thus, (2.38) is rewritten as

$$\mathbf{F}^{d} \cdot (\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^{p}{}_t) \geq 0, \quad \mathbf{F}^{d} := (\boldsymbol{\sigma}^{d}, \mathbf{B}^{d}). \tag{2.41}$$

Now, let us discuss constitutive relations for $\boldsymbol{\sigma}^{d}$ and $\mathbf{B}^{d}$ ensuring (2.41). In the theory for phase transitions by Frémond (2.41) is ensured assuming the existence of pseudo-potential of dissipation Φ depending on velocities $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^{p}{}_t)$. This functional (cf. [14]) is convex, lower semicontinuous, and assumes its minimum 0 when $(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^{p}{}_t) = (\mathbf{0}, \mathbf{0})$. Indeed, prescribing the constitutive relation

$$\mathbf{F}^{d} = (\boldsymbol{\sigma}^{d}, \mathbf{B}^{d}) \in \partial\Phi(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^{p}{}_t) = \left(\frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_t}, \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}^{p}{}_t}\right) \tag{2.42}$$

where the subdifferential $\partial\Phi$ is intended in the sense of convex analysis (it is a possible multivalued operator), (2.42) ensures that (2.41) holds by definition of the subdifferential. Due to (2.39), (2.40), and constitutive relation (2.42), $\boldsymbol{\sigma}$ and $\mathbf{B}$ in (2.2) and (2.37) are eventually defined by

$$\boldsymbol{\sigma} = \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_t}, \quad \mathbf{B} = \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{p}} + \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}^{p}{}_t}. \tag{2.43}$$

In the following of the paper, we are going to generalize the theory. Our idea is to find dissipative functionals (in place of Φ in (2.43) and (2.42)) which are not necessarily convex but leading to (2.41).

Remark 2.5. Let us now comment on the relation between the above approach and the more classical model for plasticity, as that detailed in Remark 2.3. Let $\rho = 1$ and fix the isothermal case. Assume (2.27), so that (cf. (2.39) and (2.40))

$$\boldsymbol{\sigma}^{nd} = \mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{p}), \quad \mathbf{B}^{nd} = -\mathbf{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{p}) = -\boldsymbol{\sigma}^{nd}. \tag{2.44}$$

Hence, let the pseudo-potential Φ do not depend on $\boldsymbol{\varepsilon}_t$, so that $\boldsymbol{\sigma}^{d} = \mathbf{0}$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{nd}$ (see (2.42) and (2.43)). In particular, a possible choice for Φ is (cf. Remark 2.3) is the convex conjugate function of the indicator of the set $\widetilde{K}$, defined by (2.30), $I_{\widetilde{K}}(\sigma)$. Thus, as it is (cf. (2.42))

$$\mathbf{B}^{d} = \partial\Phi(\boldsymbol{\varepsilon}^{p}{}_t) = (\partial I_{\widetilde{K}})^{-1}(\boldsymbol{\varepsilon}^{p}{}_t), \tag{2.45}$$

using (2.44), (2.43) (see (2.40) and (2.42)), and (2.45) in (2.37), we get the balance equation

$$(\partial I_{\widetilde{K}})^{-1}(\boldsymbol{\varepsilon}^{p}{}_t) - \boldsymbol{\sigma} \ni \mathbf{0}, \tag{2.46}$$

from which (2.31) is recovered just inverting the graph $(\partial I_{\widetilde{K}})^{-1}$, i.e.

$$\boldsymbol{\varepsilon}^p{}_t \in \partial I_{\widetilde{K}}(\boldsymbol{\sigma}).$$

Remark 2.6. In both the previous approaches (2.22) is ensured through some duality relation between dissipative forces and velocities. However this duality is guaranteed in the classical situation in the framework of convex analysis.

We are going to generalize this argument, dealing with "dissipative" potentials (as Φ) defined in terms of yield functions which are not necessarily convex.

3 Ensuring thermodynamical consistency: a new idea

Thermodynamical consistency of our model is ensured once the second-law of thermodynamics turns out to be satisfied. More precisely, our aim is to find suitable constitutive relations for $\mathbf{F}^d$ in order to ensure that

$$\mathbf{F}^d \cdot (\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^p{}_t) \geq 0 \tag{3.1}$$

which leads to (2.41) or (2.38), depending on the approach we choose. Hence, we let the density be constant and $\rho = 1$. Finally, we restrict our analysis to the isothermal case, just for the sake of simplicity.

Our aim is to generalize the convex framework in which (3.1) has been considered in the literature (see Remarks 2.3 and 2.5). In particular, we will charaterize dissipation through a dissipative metric enhancing the model to be consistent with respect to thermodynamics. We will discuss later in which sense this metric is related with a (generalized) yield function and defines the set of admissible dissipations. Let us point out that we are able to weak assumptions on the "yield" function and, consequently, on the set of admissible dissipations (cf. (2.30)), including less smooth and possible non-convex relations. Secondly, we account for local effects in the model, that is we prescribe that the admissible dissipation depends on the actual state of the material.

Remark 3.1. For the sake of completeness, we give some comments on possible relations between our approach and the well-known *Energetic Formulation models* by Mielke (cf., e.g., [8], [9], [10], [11], [12], [13] and references therein), which are based on some "dissipative metric". Actually, we characterize the dissipative metric as the (unique viscosity) solution of an Hamilton-Jacobi equation, involving the yield function (in our approach a

given datum). In particular, we do not introduce it through some minimization problem, as it is done by Mielke. The advantage of our approach is more evident looking at the non-convex case. Indeed, in the convex case the two different approaches lead exactly to the same result. On the contrary, in the nonconvex case, in generale, the minimization problem yields some difficulties while we have still a solution (and a representation formula) to the Hamilton-Jacobi equation. Hence, although the Energetic Formulation models do not directly concern with the definition of the dissipative metric, which is given only for the convex case, some properties can be recovered in its abstract formulation. These properties are fulfilled in the case one introduces the dissipative metric by our approach.

3.1 Formal derivation: a geometrical approach

Let us first introduce for any state $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ the set $Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ of admissible dissipative forces $\mathbf{F}^d$ (let us now take $\mathbf{F}^d = (\boldsymbol{\sigma}^d, \mathbf{B}^d)$.

Remark 3.2. Let us point out that $Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ plays the role of $\widetilde{K}$ (cf. (2.30)) as it represents a constraint on the dissipative forces (recall that in the more classical approach dissipative forces coincide with $\boldsymbol{\sigma}$). However, note in particular that Z may depend on the state $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ and it is not necessarily convex. Moreover, we are not a priori assuming that $\boldsymbol{\sigma}^d = \mathbf{0}$.

Concerning the assumptions required on Z, we let the following hypotheses hold (for the physical meaning see the subsequent Remark 3.3)

(Z1) $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \mapsto Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ is a continuous mapping (with respect to the Hausdorff distance) in the set of the states;

(Z2) $(\mathbf{0}, \mathbf{0}) \in \operatorname{int}\big(Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)\big)$;

(Z3) $Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ is a compact set, w.r.t. the topology induced by the states.

A stronger version of (Z2) is assumed in some situations (we will specify later the advantages of choosing this condition on the set Z)

(Z4) there exist $a, b > 0$ such that $B(0, a/(|(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)| + b)) \subset Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$.

Remark 3.3. Let us briefly comment on the physical meaning of the above assumptions. $(Z1)-(Z4)$. In (Z1) we are requiring that the material does not present discontinuous reactions to stress, in the sense that if we start from two near initial states (w.r.t. the metric defined in the set of the states), the allowed dissipations corresponding to each of them are not too much different. In (Z2) we require that the material is in some sense stable w.r.t

small dissipative processes. We include in the set of admissible dissipations those with $|\mathbf{F}^d|$ sufficiently small, and in particular $\mathbf{F}^d = (\mathbf{0}, \mathbf{0})$.
In its stronger version (Z4), we have a local uniform estimate on this small values.
Assumption (Z3) is related to the fact that no infinite dissipative forces can be allowed.

Under assumptions (Z1)-(Z4) we are in the position to deduce that there exists a continuous Hamiltonian function H such that:

$$\begin{aligned} H((\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p), \mathbf{F}^d) \leq 0 \quad &\text{if} \quad \mathbf{F}^d \in Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p), \\ H((\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p), \mathbf{F}^d) > 0 \quad &\text{if} \quad \mathbf{F}^d \notin Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p). \end{aligned}$$

We can explicitely construct this hamiltonian H and then find a naturally associated metric $\mathcal{M}$, defined on the state variables and related to the "amount of actual dissipation" occurring to join two states. More precisely, once an initial reference state $(\boldsymbol{\varepsilon}_0, {\boldsymbol{\varepsilon}^p}_0)$ is fixed, the function $u(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) =: \mathcal{M}((\boldsymbol{\varepsilon}_0, {\boldsymbol{\varepsilon}^p}_0), (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p))$ is a viscosity solution in $\mathbb{R}^{2n} \setminus \{(\boldsymbol{\varepsilon}_0, {\boldsymbol{\varepsilon}^p}_0)\}$ of the Cauchy problem

$$H((\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p), Dv(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)) = 0, \qquad v(\boldsymbol{\varepsilon}_0, {\boldsymbol{\varepsilon}^p}_0) = 0 \tag{3.2}$$

where Dv denotes the gradient if v. Note that, as a consequence, if $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \neq (\boldsymbol{\varepsilon}_0, {\boldsymbol{\varepsilon}^p}_0)$ there holds

$$D^+u(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \subset Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p),$$

where $D^+u(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ denotes the superdifferential of u at $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$, see also Definition 5.8.

In the case when the set $Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \equiv Z$ does not depend on the state $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$, from (2.30) follows that H can be read as the yield function of our thermomechanical system. Thus, also in the case of a set Z not depending on the state, we are actually generalizing the notion of yield function, as we are not prescribing any convexity assumptions, but only suitable properties on the sublevels of this function. Of course, we could proceed in the opposite direction, introducing a yield function H whose sublevels, satisfying $(Z1) - (Z3)$ (or $(Z4)$), play the role of the dissipative set Z.

In the following two sections we investigate both the case in which Z is assumed to be convex (but not necessarily H is required to be convex, see Remark 3.8) or not.

3.2 The convex case

This section concerns with the case when the set $Z(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ is assumed to be convex, providing an explicit description of the associated metric. More

precisely, throughout this section, we let $(Z1)-(Z3)$ hold as well as

$$\forall\,(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)\in\mathbb{R}^{2n}\text{ the set }Z(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)\text{ is convex.}$$

Now, we are in the position to introduce a metric which is naturally associated to the dissipative set Z. Before proceeding, to simplify notation let us denote by x vectors $\mathbf{x}$ and in particular the couple of tensors $(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)\in\mathbb{R}^{2n}$.

Definition 3.4. For every pairs $x_1,x_2\in\mathbb{R}^{2n}$, we define the following function:

$$L(x_1,x_2):=\inf\left\{\int_0^1\delta_{Z(\xi^{x,\zeta}(t))}(\zeta(t))\,dt,\zeta\in B_{x_1,x_2}\right\}. \tag{3.3}$$

where B_{x_1,x_2} stands for the set of velocities of curves joining x to y, $\xi^{x,\zeta}:[0,1]\to\mathbb{R}^{2n}$ for the position on the curve (see Definition 5.13), and for a given convex set C, $\delta_C(\cdot)$ is the support function to C (see Definition 5.6).

In the following Proposition (the proof is given in [16], Prop. 1.1 and 1.2) some properties of the function L are given. In particular it turns out that L is a metric.

Proposition 3.5. *The function $L:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to[0,+\infty[$ satisfies the following properties:*

1. *L is a Finsler metric on $\mathbb{R}^{2n}$;*
2. *L is locally equivalent to the Euclidean norm: for any compact $K\subset\mathbb{R}^{2n}$ there are positive constants $R>r$ such that $r|x-y|<L(x,y)\leq R|x-y|$ for every $x,y\in K$;*
3. *if $(Z4)$ holds, the metric L is complete and for any $x_0\in\mathbb{R}^{2n}$ we have*

$$\lim_{|x|\to\infty}L(x_0,x)=+\infty.$$

Then, after introducing an Hamiltonian function associated to Z it is shown that there is a relation between the metric L and the resulting Hamilton-Jacobi equation.

Definition 3.6. Define the following Hamiltonian function $H:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$

$$H(x,p):=\gamma_{Z(x)}(p)-1,$$

where, for a given convex set C, $\gamma_C(\cdot)$ is the gauge function of C (see Definition 5.6).

Theorem 3.1. *Let $(Z1)-(Z3)$ hold, Z to be convex and H defined by Def. 3.6. Then, for every $x_0 \in \mathbb{R}^{2n}$ the function $u(x) := L(x_0, x)$ is a viscosity supersolution of $H(x, Du(x)) = 0$ in $\mathbb{R}^{2n} \setminus \{x_0\}$ and a viscosity subsolution in $\mathbb{R}^{2n}$.*

Moreover, if $(Z4)$ is satisfied, L turns out to be the unique metric with $\lim_{|x|\to\infty} L(x_0, x) = +\infty$ solving the equation and such that $L(x_0, x_0) = 0$.

Proof. See Theorem 2.1 and Proposition 2.2 in [16]. For a definition of viscosity super- and sub- solution, see Section 5. □

Note that we could construct H and L following a reversed direction. Indeed, we can start from an Hamilton-Jacobi equation, and define Z and L in order to ensure that Theorem 3.1 is satisfied.

Proposition 3.7. *Let $H : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous and such that*

$$\begin{aligned} H(x,0) &< 0, \\ \liminf_{|p|\to\infty} H(x,p) &> 0. \end{aligned}$$

Then, define Z such that for every $x \in \mathbb{R}^{2n}$ there holds

$$\begin{aligned} Z(x) &:= \{p \in \mathbb{R}^{2n} : H(x,p) \le 0\} \text{ is convex,} \\ \partial\mathrm{int}(Z(x)) &= \{p : H(x,p) = 0\}. \end{aligned}$$

Then, (Z1)-(Z3) hold. Moreover, if we assume that there exist $a, b > 0$ such that $H(x,p) < 0$ for any $x \in \mathbb{R}^{2n}$ and $|p| < a/(|x| + b)$ then $(Z4)$ holds, too.

Proof. See Proposition 2.1 in [16]. □

Remark 3.8. Notice that, in order to ensure the convexity of $Z(x)$ we are not forced to take H convex in both variables, but we just require that for every x the 0-sublevel of $H(x, \cdot)$ is convex.

Remark 3.9. In order to recover a metric interpretation of the yield function (that is to justify the fact that H actually plays the role of the yield function), we consider the special case in which $H = H(p)$ i.e. H does not depend on the state. It follows that Z does not depend on the state and $Z = \{p : H(p) \le 0\}$. According to the definition of yield function and recalling the definition of the set of admissible stresses $\widetilde{K}$ (cf. (2.30)) we have $\widetilde{K} = Z$. In a more general situation, this fact allow us to consider the Hamiltonian H as a *generalized yield function* possibly depending also on the state.

Now, let us investigate the infinitesimal behavior of the metric L.

Lemma 3.10. *Let $\psi \in AC([0,1],\mathbb{R}^{2n})$, we will denote by $\ell_E(\psi)$ the Euclidean length of ψ, namely*

$$\ell_E(\psi) := \int_0^1 |\dot\psi(t)|\,dt.$$

Assume that the set-valued map $x \mapsto Z(x)$ is convex valued and satisfies assumptions (Z1)-(Z4), and let L be the associated metric defined by (3.3). Then:

1. *The only loop with intrinsic L-length zero is the constant one. More precisely, for all $\varepsilon > 0$, $y \in \mathbb{R}^{2n}$, we have:*

$$\inf_{\xi\in C_y(\varepsilon)} \int_0^1 \delta_{Z(\xi(t))}(\dot\xi(t))\,dt > 0, \qquad \inf_{\xi\in C_y(\varepsilon)} \int_0^1 \delta_{-Z(\xi(t))}(\dot\xi(t)) > 0,$$

   where $C_y(\varepsilon) = \{\xi \in AC([0,1],\mathbb{R}^{2n}) : \xi(0) = \xi(1) = y, \ell_E(\xi) \geq \varepsilon\}$ and $-Z(x) = \{p \in \mathbb{R}^{2n} : -p \in Z(x)\}$ for all $x \in \mathbb{R}^{2n}$.

2. *For every $y_0 \in \mathbb{R}^{2n}$, $\varepsilon > 0$ there is $\delta_\varepsilon \in]0,\varepsilon[$ such that if $x_0 \in B_E(y_0,\delta_\varepsilon)$ then*

$$L(y_0,x_0) = \inf\left\{\int_0^1 \delta_{Z(\xi(t))}(\dot\xi(t)) : \xi \in AC([0,1],\mathbb{R}^{2n}),\ \dot\xi \in B_{y_0,x_0},\ \ell_E(\xi) < \varepsilon\right\}.$$

Proof. 1. Let $\psi \in C_y(\varepsilon)$. By the nondegeneracy (Z4) and the continuity (Z1) assumptions on the set valued map $x \to Z(x)$, there exist $\rho,\delta > 0$ such that $B(0,\rho) \subseteq Z(x)$ for every $x \in B(y,\delta)$. Let

$$I = \{t \in [0,1] : \psi(t) \in B(y,\delta)\}.$$

As ψ and $d_E^\sharp(\cdot, B(y,\delta))$ are continuous and $\psi(0) = y$, we have that I has positive measure, (here $d_E^\sharp(\cdot,\cdot)$ is the signed distance, see Section 5). Moreover, owing to the fact that $\delta_{Z(x)}(p) > 0$ for all $x,p \in \mathbb{R}^{2n}$ and $(Z2)$ we deduce

$$\begin{aligned}\int_0^1 \delta_{Z(\psi(t))}(\dot\psi(t))\,dt &> \int_I \delta_{Z(\psi(t))}(\dot\psi(t))\,dt = \int_I \sup_{p\in Z(\psi(t))} \langle p,\dot\psi(t)\rangle\,dt \\ &\geq \int_I \sup_{p\in B(0,\rho)} \langle p,\dot\psi(t)\rangle\,dt = \rho\int_I |\dot\psi(t)|\,dt.\end{aligned}$$

Note that if $I = [0,1]$, we have that the last term is exactly $\rho\ell_E(\psi) = \rho\varepsilon > 0$. On the contrary, there exists $\bar t \in [0,1]$ such that $|\psi(\bar t) - y| = \delta$, yielding

$$\rho\int_I |\dot\psi(t)|\,dt > \rho\int_{[0,\bar t]} |\dot\psi(t)|\,dt \geq \rho|\psi(\bar t) - y| = \rho\delta > 0.$$

Let $\alpha := \min\{\rho\varepsilon, \rho\delta\}$. It results that

$$\int_0^1 \delta_{Z(\psi(t))}(\dot{\psi}(t))\, dt \geq \alpha > 0.$$

Analogously proceeding for $-Z(x)$ (which of course satisfies $(Z1) - (Z4)$) we are allowed to deduce

$$\int_0^1 \delta_{-Z(\psi(t))}(\dot{\psi}(t))\, dt \geq \alpha > 0,$$

which concludes our proof.

2. The proof follows the arguments exploited in [4, Lemma 5.5]. We proceed by contradiction. Assume that there exists $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to y_0 such that $\forall n$

$$L(y_0, x_n) < \inf\left\{\int_0^1 \delta_{Z(\xi(t))}(\dot{\xi}(t)) : \xi \in AC([0,1], \mathbb{R}^{2n}),\, \dot{\xi} \in B_{y_0,x_0},\, \ell_E(\xi) < \varepsilon\right\}.$$

It follows that $L(y_0, x_n) \to 0$, so that there exist geodesics $\xi_n \in AC([0,1], \mathbb{R}^{2n})$ joining $\xi_n(0) = y_0$, $\xi_n(1) = x_n$, with $\ell_E(\xi_n) > \varepsilon$ and

$$\int_0^1 \delta_{Z(\xi_n(t))}(\dot{\xi}_n(t))\, dt \to 0.$$

Thus, considering ξ_n and a geodesic joining x_n and y_0, we can construct a curve η_n, parametrized in $[0,1]$, such that $\eta_n(0) = \eta_n(1) = y_0$, and $\ell_E(\eta_n) \geq \varepsilon$. Thus, $\eta_n \in C_{y_0}(\varepsilon)$ but

$$\int_0^1 \delta_{Z(\eta_n(t))}(\dot{\eta}_n(t))\, dt \to 0,$$

which is a contradiction.

□

Proposition 3.11. *Assume that the set-valued map $x \mapsto Z(x)$ is convex valued and satisfies assumptions (Z1)-(Z4), let L be the associated metric, and set $H(x,p) = \gamma_{Z(x)}(p) - 1$. Then for every $y \in \mathbb{R}^{2n}$, we define $L_y : \mathbb{R}^{2n} \to [0, +\infty[$ by setting $L_y(x) = L(y,x)$ for all $x \in \mathbb{R}^{2n}$. There holds*

$$\partial L_y(y) = D^- L_y(y) = Z(y),$$

where ∂L_y and $D^- L_y$ denote respectively the Clarke's generalized gradient and the subgradient of L_y (see Definition 5.8).

Proof. Let $p_0 \in \mathrm{int}(Z(y))$ (cf. $(Z2)$). By the continuity of $x \to Z(x)$, for every $\rho > 0$ there exists $\varepsilon > 0$ such that:

$$d_E^\sharp(p_0, Z(x)) < -\rho \text{ for all } x \in B(y, \varepsilon).$$

Lemma 3.10 implies that there exists $0 < \delta_\varepsilon < \varepsilon$ such that:

$$L(y_0, x) = \inf\left\{ \int_0^1 \delta_{Z(\xi(t))}(\dot\xi(t)) : \xi \in AC([0,1], \mathbb{R}^{2n}),\, \dot\xi \in B_{y_0,x},\, \ell_E(\xi) < \varepsilon \right\}.$$

for every $x \in B(y, \delta_\varepsilon)$. Letting $x \in B(y, \delta_\varepsilon)$, $x \neq y$, every curve $\xi \in AC([0,1], \mathbb{R}^{2n})$, with $\ell_E(\xi) < \varepsilon$ and joining y with x, is contained in $B(y, \varepsilon)$. It follows that

$$\langle p_0, x - y\rangle = \int_0^1 \langle p_0, \dot\xi(t)\rangle\, dt < \int_0^1 \delta_{Z(\xi(t))}(\dot\xi(t))\, dt - \rho|x - y| < L(y, x).$$

Recalling that $L(y, y) = 0$, the map $x \to \langle p_0, x - y\rangle$ is subtangent to $L(y, \cdot)$ at y and $p_0 \in D^- L_y(y)$. Moreover, since $D^- L_y(y)$ is closed and $Z(y)$ is convex (hence $Z(y) = \overline{\mathrm{int}(Z(y))}$), we have $Z(y) \subseteq D^- L_y(y)$.

By $(Z3)$ the subsolutions of $H(x, Du) = 0$ are Lipschitz continuous. Moreover, as the sublevels of H are convex, by the notion of subsolution and a.e. defined subsolution are equivalent (see [1]). As a consequence u is a subsolution of $H(x, Du) = 0$ if and only if $H(x, p) \leq 0$ for all $x \in \mathbb{R}^{2n}$ and $p \in \partial u(x)$.
Since L_y is a subsolution on the whole of $\mathbb{R}^{2n}$, we have in particular $\partial L_y(x) \subseteq Z(x)$ and we eventually conclude that

$$\partial L_y(y) \subseteq Z(y) \subseteq D^- L_y(y) \subseteq \partial L_y(y),$$

(for a similar procedure cf. also [4, Proposition 5.4]). □

3.3 The nonconvex case

In this section, we aim to deal with a more general situation in which Z is not necessarily convex. Note that in this case we cannot proceed as in Def. 3.4, and the construction of the metric turns to be much more difficult. Thus, for the sake of simplicity, let us now prescribe an Hamiltonian function H and then recover the set Z (cf. Prop. 3.7.

Condition 3.2. Let $H : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous, and define

$$Z(x) := \{p \in \mathbb{R}^{2n} :\, H(x, p) \leq 0\}.$$

Assume that for every $x \in \mathbb{R}^{2n}$ we have

1. $H(x,0) < 0$;

2. $\partial \mathrm{int}(Z(x)) = \{p : H(x,p) = 0\}$;

Note that, also in this case $(Z1) - (Z3)$ hold, but the convexity of Z is not prescribed. However, for the sake of completeness, let us recall a stronger version of $(Z4)$ which can be assumed in some special situations to tackle the possibly lack of connectedness of $Z(x)$.

(Z5) for all $K \subset \mathbb{R}^{2n}$, K compact, exists $R > 0$ such that $\inf\{H(x,p) : |p| > R, x \in K\} > 0$.

In particular, $(Z5)$ requires some local uniformity in the boundedness of $Z(x)$.

Now, we are in the position of introducing a metric S which plays the same role of L but in this more general setting.

Definition 3.12. Assume that Condition 3.2 holds. For every pair $x, y \in \mathbb{R}^{2n}$ we define (see Def. 5.11 and 5.12)

$$S(y,x:) = \inf_{\gamma \in \Gamma_{x,y}} \sup_{\eta \in B} \left\{ \int_0^1 [\langle -\gamma[\eta](t), \eta(t) \rangle - |\gamma[\eta](t)| d_E^\sharp(\eta(t), Z(\xi^{x,\gamma[\eta]}(t)))] \, dt \right\}.$$

The following proposition ensures that the metric S satisfies the same properties of L (see also Prop. 3.5).

Proposition 3.13. *Under the above assumptions, the function $S : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to [0, +\infty[$ satisfies the following properties:*

1. *S is a Finsler metric on $\mathbb{R}^{2n}$.*

2. *S is locally equivalent to the Euclidean norm: for any compact $K \subset \mathbb{R}^{2n}$ there are positive constants $R > r$ such that $r|x-y| < S(x,y) \leq R|x-y|$ for every $x, y \in K$.*

3. *if $(Z5)$ holds, we have that the metric S is complete and for any $x_0 \in \mathbb{R}^{2n}$ we have*
$$\lim_{|x| \to \infty} S(x_0, x) = +\infty.$$

Proof. See Proposition 3.5 and Proposition 3.6 in [16] and Formulas 3.2, 3.3 in [17]. □

Then, the equivalent of Theorem 3.1 is given.

Theorem 3.3. *For every $x_0 \in \mathbb{R}^{2n}$ the function $u(x) := S(x_0, x)$ is a supersolution of $H(x, Du(x)) = 0$ in $\mathbb{R}^{2n} \setminus \{x_0\}$ and a subsolution in $\mathbb{R}^{2n}$.*

Proof. See Theorem 4.1 and Proposition 4.2 in [16]. □

Now, one may wonder if there is some relation between the metric S and L in the case the set Z is convex or, in a more general setting, between the metric S and the metric L defined by $\mathrm{co}(Z(x))$. However, in general S and L are different (as shown in [16, Example 4.1]). Note in particular that

Proposition 3.14. *Let S and L defined, respectively, by Def. 3.12 and Def. 3.4 but to consider $\mathrm{co}(Z(x))$. L is a path metric in the sense of Definition 5.15 while, in general, S is not a path metric.*

Proof. See [16, Proposition 1.4]. □

However, some relations occurr and, in some special cases, the metrics coincide.

Proposition 3.15. *Let S and L defined as in the previous Proposition. Then, there holds*

1. *$S \leq L$;*
2. *if Z does not depend on the state $S = L$;*
3. *if $Z(x)$ is convex then $S = L$.*

Proof.

1. See [16, Theorem 3.2].
2. By Jensen's inequality, for each curve such that $y = x + \int_0^1 \zeta(t)\, dt$ we have:
$$\int_0^1 \delta_Z(\zeta(t)) \geq \delta_Z(y - x).$$
Let $p_0 \in Z$ such that $\langle p_0, y - x\rangle = \delta_Z(y - x)$ so that $L(x, y) = \langle p_0, y - x\rangle$ and $p_0 \in \partial Z$. Then, letting the definition of S $\eta(t) \equiv p_0$ leads to
$$S(x, y) \geq \inf_{\gamma \in \Gamma_{y,x}} \left\{ \int_0^1 \langle -\gamma[p_0](t), p_0\rangle\, dt \right\} = \langle p_0, y - x\rangle = L(x, y).$$
3. In the case of a convex $Z(x)$ the fact that $S = L$ is a direct consequence of the uniqueness result of Theorem 3.1 and Theorem 3.3.

□

Now, we would like to conclude this Section with some applications of the previous results to plasticity theory. More precisely, we are going to characterize geodesics in the set of admissible stresses, in the case when this set does not depend on the state (so that in particular $S = L$).

Proposition 3.16. *Let the Hamiltonian H such that Z (defined by Cond. 3.2) does not depend on the state. Then straight lines are geodesics. Moreover, if H is strictly convex, the unique geodesic between two given states is the line segment joining them.*

Proof. By Prop. 3.15 (cf. (2)) $S = L$ since H. Hence, one can prove that for any $x, y \in \mathbb{R}^{2n}$ there exists $p_0 \in Z$ with $L(x, y) = \langle p_0, y - x \rangle$. In the strictly convex case, the (unique) minimum in the definition of L is assumed on the segment by Jensen's inequality. □

Example 3.17. In the engineering literature, classical example of yield functions are

1. $f = |\boldsymbol{\sigma}| - \sigma_0$
2. $f = |\mathbf{s}| - c$, where $\mathbf{s}$ is the deviatoric part of $\boldsymbol{\sigma}$, i.e. $\mathbf{s} = \boldsymbol{\sigma} - \frac{\mathrm{tr}\,\boldsymbol{\sigma}}{3}\mathbf{1}$.
3. $f = \mathrm{tr}^{1/2}(\mathbf{s}^2)\left(1 + \alpha \frac{\mathrm{tr}^2(\mathbf{s}^3)}{\mathrm{tr}^3(\mathbf{s}^2)}\right)$
4. $f = \mathbf{sAs}$,

A being a positive definite matrix, σ_0 and α constants. As all these functions do not depend on the state variables, they satisfy the assumptions of Proposition 3.16.

3.4 Admissible directions and admissible dissipations

Now, we are in the position of writing constitutive relations for dissipative forces $\mathbf{F}^d$ in accordance with (3.1). It is clear that, once $\mathbf{F}^d$ is fixed, we recover by some "duality" arguments the thermodynamically admissible directions for the evolution. In the classical literature this problem was tackle assuming that *along admissible directions for the evolution dissipation increases*. This fact will find some correspondance with our theory, as we are going to discuss in a moment.

We give constitutive relations in terms of Clarke's generalized gradient of the metric S, as we aim to face the more general setting in which it is not a priori required any convexity for the sublevels of the yield function, which in addition possibly depends on the state. Indeed, as S is Lipschitz continuous,

we cannot involve just super- or subdifferentials which may be empty (note that S is not necessarily convex).

Let us first introduce two notions of "tangent spaces".

Definition 3.18. Let $S : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to [0, +\infty[$ be the dissipative metric induced by H. Let $x_0 \in \mathbb{R}^{2n}$ be fixed. Let $S_{x_0} : \mathbb{R}^{2n} \to [0, +\infty[$ be the Lipschitz continuous function defined by $S_{x_0}(y) = S(x_0, y)$. For every $x \in \mathbb{R}^{2n}$ define the following sets:

$$\begin{aligned} T_{x_0}(x) &:= \{v \in \mathbb{R}^{2n} : (-S_{x_0})^\circ(x; v) \leq 0\} = \{v \in \mathbb{R}^{2n} : S^\circ_{x_0}(x; -v) \leq 0\} \\ &= \{-v \in \mathbb{R}^{2n} : S^\circ_{x_0}(x; v) \leq 0\} = \{v \in \mathbb{R}^{2n} : \langle p, v\rangle \geq 0 \text{ for all } p \in \partial S_{x_0}(x)\}. \\ G_{x_0}(x) &:= \{v \in \mathbb{R}^{2n} : S^\circ_{x_0}(x; v) \geq 0\} = \{v \in \mathbb{R}^{2n} : \exists p \in \partial S_{x_0}(x) \text{ s.t. } \langle p, v\rangle \geq 0\}. \end{aligned}$$

In particular, there holds $0 \in T_{x_0}(x) \subseteq G_{x_0}(x)$.

Remark 3.19. Let us comment on the fact that, in general, the inclusion $T_{x_0}(x) \subseteq G_{x_0}(x)$ is strict. We consider, e.g., the case when $Z(x)$ is convex $\forall x \in \mathbb{R}^{2n}$. Consequently we have $S = L$ and $\partial S_{x_0}(x_0) = Z(x_0)$ (see Proposition 3.11). Then, by hypothesis $Z(x_0) \supseteq B(0, \rho)$, $\rho > 0$. Due to the definition of T_{x_0} it follows

$$0 \in T_{x_0}(x_0) \subseteq \{v \in \mathbb{R}^n : \langle v, p\rangle \geq 0 \text{ for all } p \in B(0, \rho)\} = \{0\},$$

leading to $T_{x_0}(x_0) = 0$. On the other hand, as $x \mapsto S_{x_0}(x)$ has a (global) minimum at x_0 we can also deduce $0 \in \partial S_{x_0}(x_0)$ and (recall the definition of $G_{x_0}(x_0)$ choosing $p = 0$) $G_{x_0}(x_0) = \mathbb{R}^{2n}$.

However, it is straightforward to observe that if S_{x_0} is differentiable at x it is $T_{x_0}(x) = G_{x_0}(x) = \{v : \langle v, \nabla S_{x_0}(x)\rangle \geq 0\}$.

Remark 3.20. Now, we would like to give a physical interpretation of these objects. From a physical point of view a direction v belongs to $G_{x_0}(x)$ if there exist $y_n \to x$ and $t_n \to 0^+$ such that

$$\lim_{n \to +\infty} \frac{S_{x_0}(y_n + t_n v) - S_{x_0}(y_n)}{t_n} \geq 0.$$

On the other hand, a direction v belongs to $T_{x_0}(x)$ iff we have:

$$\limsup_{\substack{y \to x \\ t \to 0^+}} \frac{-S_{x_0}(y + tv) + S(y)}{t} = -\liminf_{\substack{y \to x \\ t \to 0^+}} \frac{S_{x_0}(y + tv) - S(y)}{t} \leq 0,$$

hence

$$\liminf_{\substack{y \to x \\ t \to 0^+}} \frac{S_{x_0}(y + tv) - S(y)}{t} \geq 0.$$

This implies that for *every* sequence of points $y_n \to x$ and times $t_n \to 0^+$ we have:

$$\lim_{n\to+\infty} \frac{S_{x_0}(y_n + t_n v) - S(y_n)}{t_n} \geq 0,$$

providing that the limit exists.
Comparing the two definitions, we have that $T_{x_0}(x)$, which is also the *polar* to the Clarke's generalized gradient of S_{x_0}, is the sets of directions along which a *stronger* nondecresing property of S_{x_0} is satisfied.

Now, we give the constitutive relation for dissipative forces.

Definition 3.21. Given $x_0, x \in \mathbb{R}^{2n}$, $p \in \partial S_{x_0}(x)$, let $v \in N_{\partial S_{x_0}(x)}(p) \cap G_{x_0}(x)$, then $\langle p, v\rangle = \delta_{\partial S_{x_0}(x)}(v) \geq 0$ and v will be called the *maximal dissipation direction associated to p relative to x_0*.
If we give the constitutive relation for the admissible dissipations:

$$\mathbf{F}^d \in \partial S_{x_0}(x), \tag{3.4}$$

the duality relation (3.1) is satisfied for every

$$v := (\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}^p{}_t) \in N_{\partial S_{x_0}(x)}(\mathbf{F}^d) \cap G_{x_0}(x).$$

Remark 3.22. Recalling that $0 \in \partial S_{x_0}(x_0)$, for $x = x_0$ it follows that $G_{x_0}(x_0) = \mathbb{R}^{2n}$. In particular, in the convex case, $Z(x_0) = \partial S_{x_0}(x_0)$ (see Proposition 3.11). Hence, for $p \in Z(x_0)$, the maximal dissipation directions can be given in terms of the following inclusion

$$v \in N_{Z(x_0)}(p).$$

In the case when the Hamiltonian H does not depend on the state (so $L = S$), we have $\mathrm{co}(Z(x_0)) = \partial S_{x_0}(x_0)$ (also if $Z(x_0) \equiv Z$ is not convex). However, this cannot be extended to the general nonconvex case, as it is not yet available a relation between $\partial S_{x_0}(x_0)$ and the elastic region $Z(x_0)$.

4 Conclusions and open questions

Let us now point out some conclusions of our analysis, based on the Hamiltonian formulation of the yield function presented in the previous sections. First, we have introduced a new constitutive relations for dissipative forces (3.4) holding also in a non-convex framework, in the sense that the new yield function (and consequently the set of admissible dissipations) is not still necessarily convex (see Def. 3.21).

Then, we show the possibility to define a dissipative metric, in the set of states, which is the solution of a suitable Hamilton-Jacobi equation. This yields exactly the results known in the literature for the convex case (see Remark 3.1) and extends the theory to some nonconvex cases (see Theorem 3.1 and 3.3). In particular, it is shown that, under suitable assumptions on the yield function, we are still able to reduce to the convex case, e.g. in the case when the yield function is quasiconvex or it does not depend on the state variable (see also Prop. 3.15 and Remarks 3.8). Note that in this last case, we can characterize geodesics as straight line segments (see Prop. 3.16). This fact justifies some *convexification* procedures that are used to recover the yield function starting from experimental data.

Now, let us briefly discuss some still open questions. Indeed, in the case when the set of admissible dissipations is not convex (and depending on the state), we can consider two different metrics: the dissipation metric S (defined as viscosity solution) or the path metric L associated to it. In the convex case and also in other relevant cases, these two metric coincides. However in the general nonconvex case the dissipation metric is not a path metric (see Prop. 3.14). As a consequence, we are not able to give neither a proper definition of geodesic, nor to recover the dissipation between two states as infimum of dissipations along all the possible transformation from the starting state to the final. This fact seems to be related to the possibility for the material to develop fracture, hence drastically different behavior among different processes. On the other hand, the path metric associated to the dissipative metric has no longer the property to be a solution of the Hamilton-Jacobi equation.

5 Notation, definitions, and basic results on Hamilton-Jacobi equations

In this section we fix notations and recall some basic facts about convexity and viscosity solutions of Hamilton-Jacobi equations. Our environment will be the Euclidean n-dimensional space $\mathbb{R}^n$. Our main reference for the properties of convex sets in $\mathbb{R}^n$ is [15]. A detailed introduction to the theory of viscosity solutions and its applications is given in [1].

Remark 5.1. Throughout this paper we have been concerned with *Hamilton–Jacobi* partial differential equations, i.e. the equation

$$H(x, Du(x)) = 0$$

in the unknown $u : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n$, and $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

The terminology is related to classical mechanics, indeed assume that $H(\mathbf{q},\mathbf{p})$ is the Hamiltonian of an autonomous mechanical system. If we are looking for a function $u = u(\mathbf{q})$ generating a canonical transformation sending $(\mathbf{q},\mathbf{p})$ to $(\mathbf{Q},\mathbf{P})$ in such a way that the new Hamiltonian reduces to $K(\mathbf{P}) = \text{const} = K$, then u solves the equation

$$H(\mathbf{q}, Du(\mathbf{q})) - K = 0.$$

In the new system of coordinates given by u, the system of Hamilton equations:

$$\begin{cases} \dot{Q}_j &= -\dfrac{\partial K(\mathbf{P})}{\partial P_j} \\ \\ \dot{P}_j &= \dfrac{\partial K(\mathbf{P})}{\partial Q_j} = 0, \end{cases}$$

can be easily solved. Since $p_j = \partial_j u(\mathbf{q})$ and $P_j = -\partial_j u(\mathbf{Q})$ by definition of canonical transformation, if we can invert these relations we can pass from the easier solution in the system $(\mathbf{Q},\mathbf{P})$ to the solution expressed in the original system of coordinates.

Definition 5.2. Let $x, y \in \mathbb{R}^n$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, K be a closed subset of $\mathbb{R}^n$, $r > 0$. We denote by:

$$\begin{aligned}
\text{int}(K),\ \partial K &\qquad \text{(resp. the } \textit{interior} \text{ and the } \textit{closure} \text{ of } K \text{ in } \mathbb{R}^n) \\
\bar{K} &\qquad \text{(the } \textit{topological boundary} \text{ of } K \text{ in } \mathbb{R}^n) \\
\langle y, x\rangle = \sum_{i=1}^{n} x_i y_i &\qquad \text{(the usual scalar product in } \mathbb{R}^n), \\
d_E(x,y) = |y-x| = \sqrt{\langle y-x, y-x\rangle} &\qquad \text{(the } \textit{euclidean} \text{ distance in } \mathbb{R}^n), \\
d_E(x,K) = \min_{z\in K} |z-x| &\qquad \text{(the } \textit{euclidean} \text{ distance from } x \text{ to } K), \\
d_E^\sharp(x,K) = 2d_E(x,K) - d_E(x,\partial K) &\qquad \text{(the } \textit{signed} \text{ distance from } x \text{ to } K), \\
B_E(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\} &\qquad \text{(the } \textit{euclidean} \text{ ball centered at } x \text{ of radius } r).
\end{aligned}$$

Definition 5.3. Let K_1, K_2 be compact subsets of $\mathbb{R}^n$. The *Hausdorff distance* between K_1 and K_2 is given by:

$$d_H(K_1,K_2) := \max\left\{ \sup_{x\in K_1}\{d_E(x,K_2)\}, \sup_{y\in K_2}\{d_E(y,K_1)\} \right\}.$$

Definition 5.4. Let $K \subset \mathbb{R}^n$, $a \in K$. We say that K is *star-shaped with respect to* a iff for all $\lambda \in [0,1]$ and $x \in K$ we have $\lambda a + (1-\lambda)x \in K$;

in particular K is *convex* iff it is star-shaped with respect to y for every $y \in K$. Given $K \subseteq \mathbb{R}^n$, we will denote by $\mathrm{co}(K)$ its convex hull, namely the intersection of all convex subsets of $\mathbb{R}^n$ containing K.

Definition 5.5. Let $K \subseteq \mathbb{R}^n$ be closed, $x \in K$, $v \in \mathbb{R}^n$, we say that v is a *proximal normal* to K at x if there exists $c > 0$ such that for every $y \in K$ it holds:

$$\langle v, y - x \rangle \leq c|v||y - x|^2.$$

We define the *proximal normal cone* to K at x by setting:

$$N_K^P(x) := \{v \in \mathbb{R}^n : v \text{ is a proximal normal to } K \text{ at } x\}.$$

In particular, if K is closed and convex, we can take $c = 0$ in the definition of proximal normal, hence $N_K^P(x)$ reduces to the *normal cone in the sense of convex analysis*, i.e. the closed and convex set $N_K(x)$ defined by

$$N_K(x) = \{v \in \mathbb{R}^n : \langle v, y - x \rangle \leq 0 \text{ for all } y \in K\}.$$

Definition 5.6. Let K be a closed convex subset of $\mathbb{R}^n$. We define:

1. δ_K the *support function* to K is the function $\delta_K : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_K(q) = \sup_{p \in K} \langle q, p \rangle,$$

   and we recall that $\delta_K(q) = \langle p_0, q \rangle$ for some $p_0 \in K$ iff $q \in N_K(p_0)$.

2. if $0 \in \mathrm{int}(K)$, the *gauge (Minkowski) function* of K is defined by:

$$\gamma_K(q) = \inf\{\lambda > 0 : q/\lambda \in K\}.$$

3. the set $\hat{K}$ of *extreme points* of K will be

$$\hat{K} = \{x \in K : x \neq \lambda y + (1 - \lambda z) \text{ for all } 0 < \lambda < 1,\ y, z \in K \setminus \{x\}\}.$$

We recall the following result:

Proposition 5.7. *Every compact and convex subset of $\mathbb{R}^n$ is the convex hull of its extreme points, in particular if $K = \mathrm{co}(S)$ with S compact subset of $\mathbb{R}^n$, then every extreme point of K belongs to S.*

Proof. See, e.g. [15, Corollary 18.5.1, p. 167] and [15, Corollary 18.3.1, p. 165]. □

We recall also the following notions of generalized differentials.

Definition 5.8. Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be a continuous function, we define:

$$\begin{aligned} D^+u(x) &= \left\{ p : \limsup_{|v|\to 0} \frac{u(x+v) - u(x) - \langle p, v\rangle}{|v|} \leq 0 \right\}, \\ D^-u(x) &= \left\{ p : \liminf_{|v|\to 0} \frac{u(x+v) - u(x) - \langle p, v\rangle}{|v|} \geq 0 \right\}, \end{aligned}$$

the set $D^+u(x)$ (resp. $D^-u(x)$) is called the (Fréchet or viscosity) *superdifferential* (resp. *subdifferential*) of u at x. Another (equivalent) characterization is the following one: $v \in D^+u(x)$ (resp. $v \in D^-u(x)$) iff there exists a neighborhood V of x and a C^1 function $\varphi : V \to \mathbb{R}$ such that $u - \varphi$ attains its maximum (resp. minimum) in V at x and $v = D\varphi(x)$. Such φ will be called a super- (resp. sub-)tangent test function.

Definition 5.9. Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function. We denote by $\mathrm{dom}(Du)$ the set where the (classical) differential of u exists. By Rademacher's theorem, this is a full measure subset of Ω. The *Clarke's generalized gradient* of u at $x \in \Omega$ is the convex set:

$$\partial u(x) := \mathrm{co}\left\{ v \in \mathbb{R}^n : \exists \{x_n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(Du) \text{ with } v = \lim_{n\to\infty} Du(x_n) \right\}.$$

If $u : \Omega \to \mathbb{R}$ is convex, then $\partial u(x)$ coincides with the subgradient in the sense of convex analysis, namely:

$$\partial u(x) := \{ v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y - x\rangle \text{ for all } y \in \Omega \}.$$

Definition 5.10. Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function. Let $v \in \mathbb{R}^n$ and $x \in \Omega$. We define the *generalized directional derivative* of f in direction v by setting:

$$u^\circ(x; v) := \limsup_{y\to x t\to 0^+} \frac{f(y + tv) - f(y)}{t}.$$

We recall the following properties (see [3, Chapter 2, Proposition 1.1, 1.5, pp. 69–74] for further details):

1. $u^\circ(x; -v) = (-u)^\circ(x; v)$;
2. $\zeta \in \partial u(x)$ iff $u^\circ(x; v) \geq \langle \zeta, v\rangle$ for all $v \in \mathbb{R}^n$;
3. $u^\circ(x; v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial u(x) = \delta_{\partial u(x)}(v)$.

Definition 5.11. Let Ω be an open subset of $\mathbb{R}^n$, $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function (the *Hamiltonian*), $a \in \mathbb{R}$. Consider the following general nonlinear first order equation:

$$H(x, Du) = a \tag{5.1}$$

in the unknown $u : \Omega \to \mathbb{R}$.
Let v be a continuous function from Ω to $\mathbb{R}$. We say that v is:
- a *viscosity subsolution* of (5.1) if it satisfies $H(x, p) \leq a$ for all $x \in \Omega$, $p \in D^+v(x)$;
- a *strict viscosity subsolution* if there exists $\bar{a} < a$ such that $H(x, D^+v(x)) \leq \bar{a}$ in Ω.
- a *viscosity supersolution* of (5.1) if it satisfies $H(x, q) \geq a$ for all $x \in \Omega$, $q \in D^-v(x)$;
- a *viscosity solution* of (5.1) if it is both sub- and supersolution.
We define the *critical value* of H by setting:

$$c = \inf\{a \in \mathbb{R} : \text{ equation (5.1) admits subsolutions}\}.$$

Definition 5.12. Let $T > 0$ be fixed and $B_1, B_2 \subseteq B^T := L^\infty(]0, T[, \mathbb{R}^n)$. A nonanticipating strategy is a map $\gamma : B_1 \to B_2$ such that if $t \in]0, T[$ and $\eta_1, \eta_2 \in B_1$ are such that $\eta_1 = \eta_2$ a.e. in $]0, t[$, then $\gamma[\eta_1] = \gamma[\eta_2]$ a.e. in $]0, t[$.

Definition 5.13. For every $x, y \in \mathbb{R}^n$, we define the set of velocities of curves joining x to y:

$$B^T_{x,y} := \left\{ \zeta \in B^T : x + \int_0^T \zeta(t)\, dt = y \right\},$$

and will denote by $\Gamma^T, \Gamma^T_{x,y}$ the strategies from B^T to B^T and from B^T to $B^T_{x,y}$ respectively. In the case $T = 1$, we will write $B, B_{x,y}, \Gamma, \Gamma_{x,y}$ instead of $B^1, B^1_{x,y}, \Gamma^1, \Gamma^1_{x,y}$. Given $x, y \in \mathbb{R}^n$ and $\zeta \in B^T_{x,y}$ we define:

$$\xi^{x,\zeta}(t) := x + \int_0^t \zeta(s)\, ds.$$

Definition 5.14. Let $D : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ be a map. We say that D is a *Finsler metric* if the following conditions are satisfied for every $x, y, z \in \mathbb{R}^n$:

1. we have $D(x, y) = 0$ if and only if $x = y$;
2. $D(x, y) \leq D(x, z) + D(z, y)$. Notice that it is *not* required the symmetric condition $D(x, y) = D(y, x)$.

Definition 5.15. Let $\psi : [0,1] \to \mathbb{R}^n$ be a Lipschitz continuous curve. If D is a Finsler metric on $\mathbb{R}^n$, we define the *intrinsic lenght* of ψ by setting:

$$\ell_D(\psi) := \sup \sum_i D(\psi(t_{i-1}) - \psi(t_i)),$$

where the supremum is taken with respect to all finite increasing sequences $\{t_1, ..., t_n\}$ with $t_1 = 0$, $t_n = 1$. The intrinsic length induce a metric by setting:

$$D_\ell(x,y) = \inf\{\ell_D(\psi) : \psi \in AC([0,1], \mathbb{R}^{2n}, \psi(0) = x, \psi(1) = y)\}.$$

We will say that D_ℓ is a *path metric* iff $D = D_\ell$. If D is a path metric, a curve $\psi \in AC([0,1], \mathbb{R}^{2n})$ will be called a *D-geodesic* if

$$D(\psi(0), \psi(1)) = \ell_D(\psi),$$

i.e. ψ realizes the minimum of intrinsic length between all curves joining $\psi(0)$ and $\psi(1)$.

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