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ISOGEOMETRIC METHODS IN STRUCTURAL DYNAMICS AND WAVE PROPAGATION

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Abstract. *We review the discretization properties of classical finite element and NURBS-based isogeometric approximations on problems of structural vibrations and wave propagation. We find that, on the basis of equal numbers of degrees-of-freedom and bandwidth, NURBS have superior approximation properties. In fact, we observe that the high-mode behavior of classical finite elements is divergent with the order of approximation, a surprisingly negative result. On the other hand, NURBS offer almost spectral approximation properties, and all modes converge with increasing order of approximation.*

We also initiate the study of collocation methods for NURBS-based isogeometric analysis. The goal is to combine the accuracy of isogeometric analysis with the low computational cost of collocation to develop accurate and efficient procedures for large-scale structural dynamics and wave propagation problems. To this end, we present results for some simple one-dimensional model problems. We consider the cases of boundary-value and eigenvalue problems on periodic and finite domains, employing the so-called Greville abscissae as collocation points. The numerical results obtained are encouraging and motivate more extensive evaluation.

1 INTRODUCTION

Isogeometric analysis is a computational mechanics technology based on functions used to represent geometry (see [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]). The idea is to build a geometric model, e.g. through a computer aided design (CAD) system, and to directly use in the analysis the functions describing the geometry, rather than approximating it through a finite element mesh. In CAD systems, non-uniform rational B-splines (NURBS) are the dominant technology. When a NURBS model is constructed, the basis functions used to define the geometry can be systematically enriched by h-, p-, or k-refinement (i.e., smooth order elevation; see [**?**]) *without* altering the geometry or its parameterization. This means that mesh refinement techniques can be utilized without a link to the CAD database, in contrast with finite element methods. This appears to be a distinct advantage of isogeometric analysis over finite element analysis. In addition, on a per degree-of-freedom basis, isogeometric analysis has exhibited superior accuracy and robustness compared with finite element analysis (see [**?**, **?**, **?**]). This is particularly true when a k-refined basis is adopted. In this case, the upper part of the discrete spectrum shows a much better behavior (see [**?**]), resulting in better conditioned discrete systems. It appears that isogeometric analysis offers several important advantages over classical finite element analysis.

However, the use of high-degree basis function, as in k-refinement, poses the delicate issue of an efficient implementation. Until now, Isogeometric Analysis has been implemented by Galerkin formulations. In this case, the efficiency issue is related to the numerical quadrature rules which are adopted when assembling the system of equations. While element-wise Gauss quadrature is optimal in the Finite Element context, it is sub-optimal when k-refined Isogeometric discretizations of Galerkin type are considered. More efficient special rules have been derived for Isogeometric Analysis in a recent work [**?**]. Taking inspiration by [**?**], we initiate here the study of a cheaper collocation-based isogeometric approach. We propose very simple one-dimensional second order coercive model problems to test the accuracy versus computational costs. Both source and eigenvalue problems are considered and the obtained results are very interesting and indeed deserve further investigation.

2 STRUCTURAL VIBRATIONS AND WAVE PROPAGATION

In this section we briefly recall the main equations of structural vibrations and of wave propagation; for elaboration, the interested reader may refer to [**?**, **?**, **?**] for structural vibrations; and to [**?**, **?**] for wave propagation (note that in these reviews particular emphasis is on acoustics).

2.1 Structural vibrations: natural frequencies and modes

Given a linear (∞-dimensional) structural system, the undamped, unforced equations of motion, which govern free vibrations, are

$$\mathcal{M}\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} + \mathcal{K}\mathbf{u} = \mathbf{0}, \tag{1}$$

where $\mathcal{M}$ and $\mathcal{K}$ are, respectively, the mass and stiffness operators, and $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the displacement.

The n^{th} normal mode $\boldsymbol{\phi}_n$ and its frequency ω_n are obtained from the following eigenvalue problem

$$\mathcal{K}\boldsymbol{\phi}_n = \omega_n^2 \mathcal{M}\boldsymbol{\phi}_n.$$

We remark that the normal modes form a basis in space.

Then, we can separate variables as

$$\mathbf{u}(t,\mathbf{x}) = \sum_n \hat{u}_n(t)\boldsymbol{\phi}_n(\mathbf{x}),$$

and, using equation (1), obtain

$$\frac{\mathrm{d}^2\hat{u}_n(t)}{\mathrm{d}t^2} + \omega_n^2\hat{u}_n(t) = 0.$$

Each mode coefficient $\hat{u}_n$ oscillates at a frequency ω_n, and we can write

$$\hat{u}_n = C_- e^{-\imath\omega_n t} + C_+ e^{\imath\omega_n t}.$$

After discretization, the following discrete equations of motion are obtained

$$\mathbf{M}\frac{\mathrm{d}^2\mathbf{u}^h}{\mathrm{d}t^2} + \mathbf{K}\mathbf{u}^h = \mathbf{0}, \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K}$ are, respectively, the finite-dimensional consistent mass and stiffness matrices, and $\mathbf{u}^h = \mathbf{u}^h(t,\mathbf{x})$ is the discrete displacement vector.

Analogously to the continuum case, the discrete normal modes $\boldsymbol{\phi}_n^h$ and the frequencies ω_n^h are obtained from the eigenproblem

$$\mathbf{K}\boldsymbol{\phi}_n^h = (\omega_n^h)^2\mathbf{M}\boldsymbol{\phi}_n^h, \tag{3}$$

and separating variables we get

$$\mathbf{u}^h(t,\mathbf{x}) = \sum_n \hat{u}_n^h(t)\boldsymbol{\phi}_n^h(\mathbf{x}),$$

with $\hat{u}_n^h$ oscillating at a frequency ω_n^h, that is,

$$\hat{u}_n^h = C_- e^{-\imath\omega_n^h t} + C_+ e^{\imath\omega_n^h t}.$$

The n^{th} discrete normal mode $\boldsymbol{\phi}_n^h$ is an approximation of the n^{th} exact normal mode $\boldsymbol{\phi}_n$, for $n = 1,\ldots,N$, being N the total number of degrees-of-freedom.

The corresponding discrete and exact frequencies are of course different (see, e.g., Figure **??**).

A fundamental question is, how close are the discrete frequencies to the continuous ones? In other words, how well does the discrete spectrum approximate the exact spectrum?

2.2 Wave propagation: the Helmholtz equation

The classical equation governing wave propagation is

$$\nabla^2 u - \frac{1}{c^2}\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = 0, \tag{4}$$

where c is the wave propagation speed. Particular solutions of **(??)** are plane waves of frequency ω traveling in the direction $\mathbf{n}$ at a speed c, which can be expressed as the time-harmonic wave train

$$u(\mathbf{x},t) = Re(Ae^{\imath(k\mathbf{n}\cdot\mathbf{x}-\omega t)}), \tag{5}$$

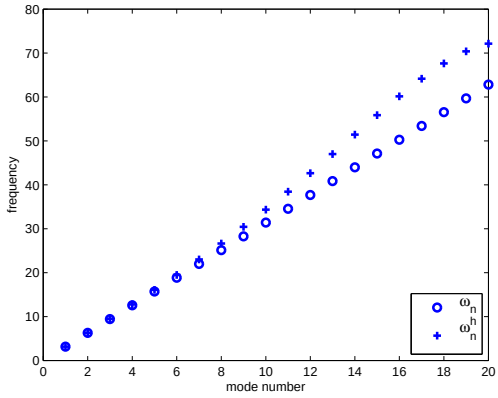

Figure 1: Exact and discrete natural frequencies for the one-dimensional model problem of free vibration of an elastic rod with homogeneous Dirichlet boundary conditions. The discrete method is based on linear finite elements.

where $k = \omega/c$ is the wave-number, ω is the angular frequency, and A is a complex number. The wavelength (with units of length) is defined by $\lambda = 2\pi/k$, while the dual measure of period (with units of time) is defined by $T = 2\pi/\omega$.

Assuming time-harmonic solutions, that is, with abuse of notation, $u(t, \mathbf{x}) = e^{\imath\omega t}u(\mathbf{x})$, the linear wave equation (**??**) reduces to the Helmholtz equation

$$\nabla^2 u + k^2 u = 0, \tag{6}$$

whose solutions in $\mathcal{R}^n$ are linear combinations of plane waves in space $u(\mathbf{x}) = e^{\imath k\mathbf{n}\cdot\mathbf{x}}$. After discretization, equation (**??**) gives rise to

$$(\mathbf{K} - k^2\mathbf{M})\mathbf{u}^h = \mathbf{0}. \tag{7}$$

The numerical solution of the above equation is a linear combination of plane waves having numerical wave-number k^h, where, in general, $k^h \neq k$.

Thus, discrete and exact waves have different wavelengths, $2\pi/k^h$ and $2\pi/k$ (see Figure **??**).

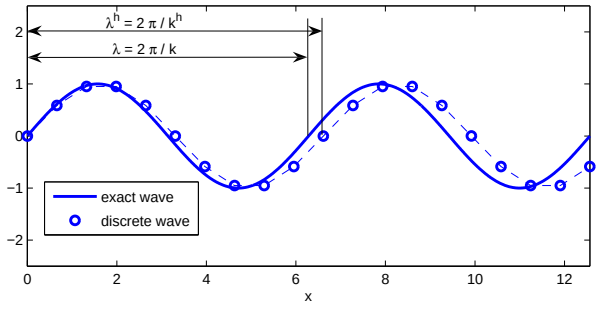

Figure 2: Different exact and numerical wave-numbers produce waves with different wavelengths.

The fundamental issue, which is addressed by dispersion analysis, is to determine the dispersion of a numerical method, that is, how close the discrete wave-number k^h is to its continuous counterpart k.

3 NURBS-BASED ISOGEOMETRIC ANALYSIS

Non-Uniform Rational B-Splines (NURBS) are a standard tool for describing and modeling curves and surfaces in computer aided design and computer graphics (see Piegl and Tiller [**?**] and Rogers [**?**] for an extensive description of these functions and of their properties). In this work, we use NURBS as an analysis tool, referred to as "isogeometric analysis", recently introduced by Hughes *et al.* [**?**]. The aim of this section is to present a brief overview of features and properties of this new approach, that we will apply to the problems introduced in the previous sections.

3.1 Isogeometric analysis

In the following we present a summary of the main features of isogeometric analysis. The interested reader may find more details and applications in [**?**, **?**, **?**, **?**].

- A mesh for a NURBS patch is defined by the product of knot vectors.
- Knot spans subdivide the domain into "elements".
- The support of each basis function consists of a small number of elements.
- The control points associated with the basis functions define the geometry.
- The isoparametric concept is invoked, that is, the unknown variables are represented in terms of the basis functions which define the geometry. The coefficients of the basis functions are the degrees-of-freedom, or *control variables*.
- Three different mesh refinement strategies are possible: analogues of classical *h*-refinement (by knot insertion) and *p*-refinement (by order elevation of the basis functions), and a new possibility referred to as *k*-refinement, which increases smoothness in addition to order.
- The element arrays constructed from isoparametric NURBS can be assembled into global arrays in the same way as finite elements (see Hughes [**?**], chapter 2).
- Dirichlet boundary conditions are applied to the control variables, in the same way as in finite elements. Neumann boundary conditions are satisfied naturally as in standard finite element formulations (see Hughes [**?**], chapters 1 and 2).

Finally, it is important to remark that in structural analysis NURBS elements represent all rigid body motions and constant strain states exactly (see Hughes [**?**]). Consequently, structures assembled from compatible NURBS elements pass standard "patch tests" (see Hughes [**?**], chapters 3 and 4, for a description of patch tests).

3.2 Linear and nonlinear parameterizations

When dealing with NURBS, an important issue is the choice of the parameterization to be used. Take as an example a 1D domain: the simplest (and more natural) option is to employ a linear parameterization, but in some situations a nonlinear choice can be more suitable.

The isogeometric procedure originally proposed by Hughes *et al.* [**?**] is based on a distribution of control points which leads to a linear parameterization (i.e., constant Jacobian determinant), but in Cottrell *et al.* [**?**] it has been shown that when studying structural vibrations

a nonlinear parameterization such that the control points are uniformly spaced gives better results. In Figure **??**, we show the 1D distribution of 21 control points obtained for the two cases using cubic NURBS (top), along with plots of the corresponding parameterization $x = x(\xi)$ and Jacobian $J(\xi) = \dfrac{\mathrm{d}\, x(\xi)}{\mathrm{d}\, \xi}$ (bottom).

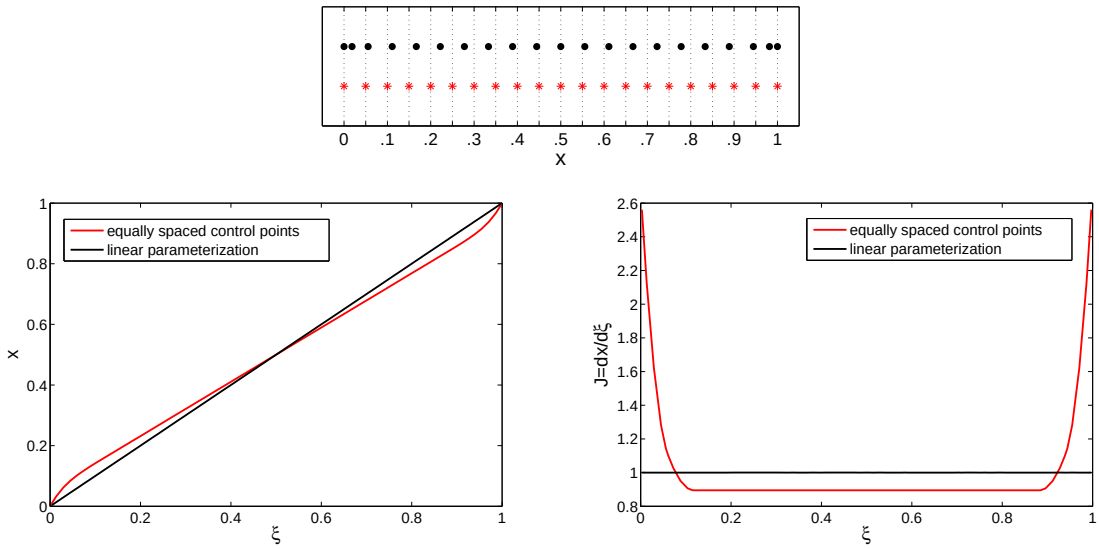

Figure 3: Linear parameterization versus a nonlinear one with uniformly-spaced control points (cubic NURBS, 21 control points). Top: distribution of control points; dots correspond to linear parameterization control points and asterisks to uniformly-spaced control points. Bottom: plot of the parameterization (left) and of its Jacobian (right) for the two cases.

3.3 *k*-method and *p*-method

We conclude this section on isogeometric analysis by briefly pointing out what we mean in this paper with the terms "*k*-method" and "*p*-method". Referring to the already cited *k*- and *p*-refinement strategies, we call the *k*-method an analysis method exploiting the full continuity across the elements allowed by NURBS basis functions (i.e., C^{p-1} for a degree p NURBS). In the following we will simply label this method as "NURBS". Instead, we call the *p*-method an analysis method where only C^0-continuity is enforced across elements (this can be obtained with isogeometric analysis by repeating the knots of a degree p NURBS $p-1$ times). This approach, used in combination with a linear parameterization, is equivalent to classical h-p finite element methods, and in the following we will simply label it as "FEM".

4 NUMERICAL TESTS

We now report on the results of some numerical tests we performed using the k-method (i.e., NURBS) and the p-method (i.e., FEM) on a 1D vibration problem (i.e., the problem of the longitudinal structural vibrations of a rod) to obtain the numerical spectra. In [**?**], this has been pointed out to be equivalent to the classical time-harmonic dispersion analysis for 1D wave propagation ("duality principle").

Before comparing NURBS and FEM results, we briefly remark that the following plots regarding NURBS are obtained using a nonlinear parameterization (as described in section **??**) in order to avoid the so-called "outlier frequencies". These are spurious frequencies (or discrete

optical branches) that show up when a linear parameterization is employed. For a more detailed discussion on the appearance of such outliers and on the way to eliminate them, the reader is referred to [**?**].

In Figure **??**, we present on the same plot the numerical and the analytical spectra obtained with the k-method for $p = 1, ..., 3$ (all the shown numerical spectra have been computed employing 1000 control points). It is possible to observe that they are practically coincident.

In Figure **??**, we present the numerical and analytical spectra obtained with the p-method for $p = 2$ (left) and for $p = 3$ (right). In this case, the numerical spectra consist of the union of the different analytical branches arising for p-method.

Finally, Figure **??** shows a comparison of k-method and p-method numerical spectra for $p = 1, ..., 4$ (we recall that for $p = 1$ the two methods coincide). Here, the superiority of the isogeometric approach is clearly revealed, as one can see that optical branches of spectra diverge with p in the classical finite element method. This negative result shows that even higher-order finite elements have no approximability for higher modes in vibration analysis, and explains the fragility of higher-order finite element methods in nonlinear and dynamic applications in which higher modes necessarily participate.

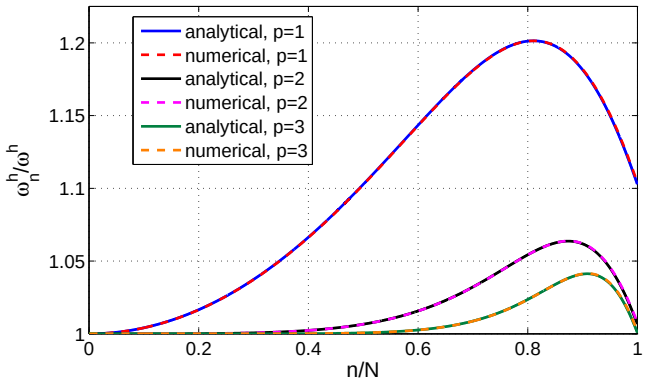

Figure 4: Numerical and analytical spectra for the k-method.

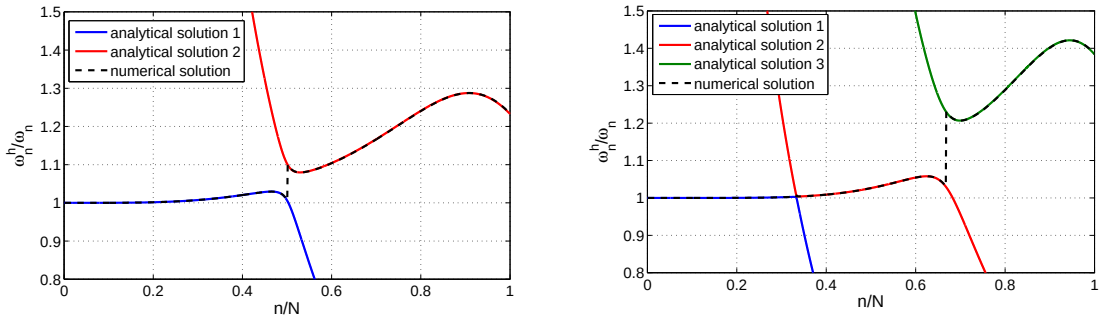

Figure 5: Numerical and analytical spectra for the p-method using quadratic (left) and cubic (right) approximations.

5 A COLLOCATION APPROACH

We present now some preliminary numerical tests based on a collocation approach, in a very simple setting, in one space dimension and with a linear parameterization of the physical domain (see [**?**]). Then, NURBS spaces reduce to classical B-spline spaces. These can be obtained with

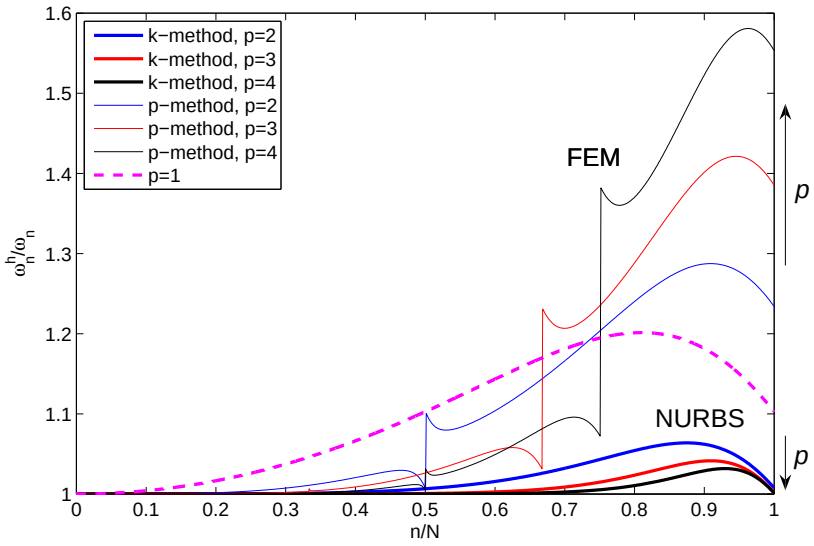

Figure 6: Comparison of k-method and p-method numerical spectra.

the construction of [**?**], for example. We will focus on the k-method, then, given a degree p, we consider the space

$$V_h = \{\text{space of } p\text{-degree piecewise polynomial of } C^{p-1} \text{ global regularity}\}.$$

Moreover, we will need the space of periodic splines on $[0,1]$, denoted by V_h^{per}. This is the subspace of V_h made of functions that have matching values and derivatives, up to order $p-1$, at the interval endpoints 0 and 1. One can also think of $[0,1]$ as a periodic domain, that is, as a circumference (gluing the endpoints 0 and 1). The periodic domain is denoted by $[0,1]^{per}$. Observe that if the dimension of V_h is $\#V_h = n$, the dimension of V_h^{per} is $\#V_h^{per} = n - p$.

Given a differential equation $\mathcal{L}u = f$ on $[0,1]^{per}$, the collocation method looks for a discrete solution $u_h \in V_h^{per}$ such that the differential equation is exactly satisfied at the collocation points τ_j, that is

$$(\mathcal{L}u_h)(\tau_j) = f(\tau_j), \quad \forall j = 1, \ldots, \#V_h.$$

The non-periodic case is analogous, but typically there are boundary conditions that are enforced in a strong way on the discrete solution u_h. Eigenvalue/eigenvector formulations are similar as well. A relevant issue is the selection of collocation points: here, we use as collocation points the so-called Greville abscissae (see [**?**, **?**]). Since we deal with a uniform knot vector, the Greville abscissa τ_j coincides with the point of maximum of the basis function $N_{j,p}$, when $N_{j,p}$ is supported in the interior of the domain $[0,1]$.

6 NUMERICAL TESTS

In this section we propose some promising preliminary numerical results for the collocation approach outlined above, obtained solving source and eigenvalue problems on both a periodic and a finite one dimensional domain.

6.1 One dimensional source problem, on the periodic domain

The first test case we study is the following source problem defined on the periodic domain $[0,1]^{per}$:

$$-u'' + u = (1 + 4\pi^2)\sin(2\pi x), \quad x \in [0,1]^{per}, \tag{8}$$

which admits exact solution:

$$u = \sin(2\pi x). \tag{9}$$

The problem is numerically solved using the collocation method briefly outlined in the previous section. In Figures **??-??**, we report log-scale plots of the relative errors for different degrees of approximations in L^2-, H^1-, and H^2-norms, respectively. Indicating with p the degree of the approximation, the figures show that in the first two norms an order of convergence p is attained for even degrees, while an order $p-1$ is attained for odd degrees. Instead in H^2-norm, we observe the expected optimal order of convergence, i.e., p, for all approximation degrees. This error behaviour, at least for odd p, is theoretically understood in [**?**].

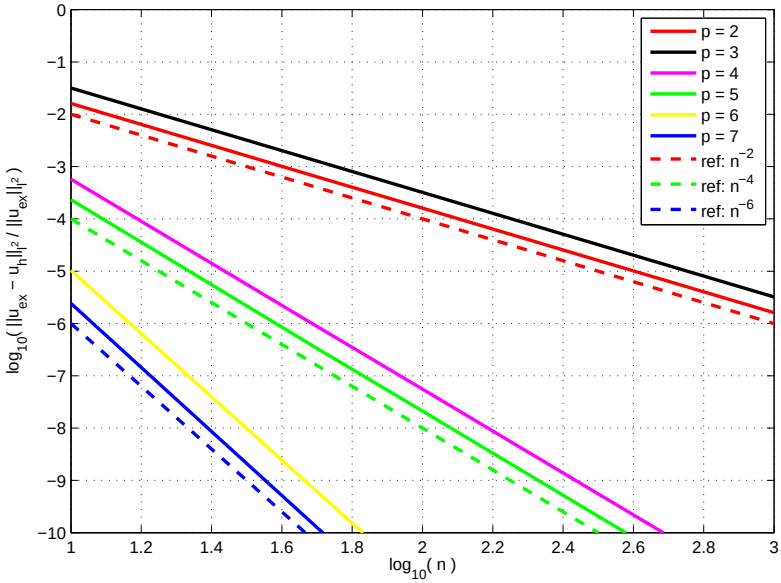

Figure 7: 1D source problem on the periodic domain. Relative error in L^2-norm for different degrees of approximation.

6.2 One dimensional eigenvalue problem, on the periodic domain

We now study the eigenvalue problem on the periodic domain, i.e.:

$$-u'' + \omega^2 u = 0, \quad x \in [0,1]^{per}, \tag{10}$$

for which the exact frequencies ω are given by:

$$\omega = 2\pi n, \quad \text{with } n = 1,1,2,2,3,3,\ldots \tag{11}$$

Also this problem can be numerically tackled using the collocation method outlined in the previous section and the results, in terms of normalized discrete spectra, is reported in Figure **??**

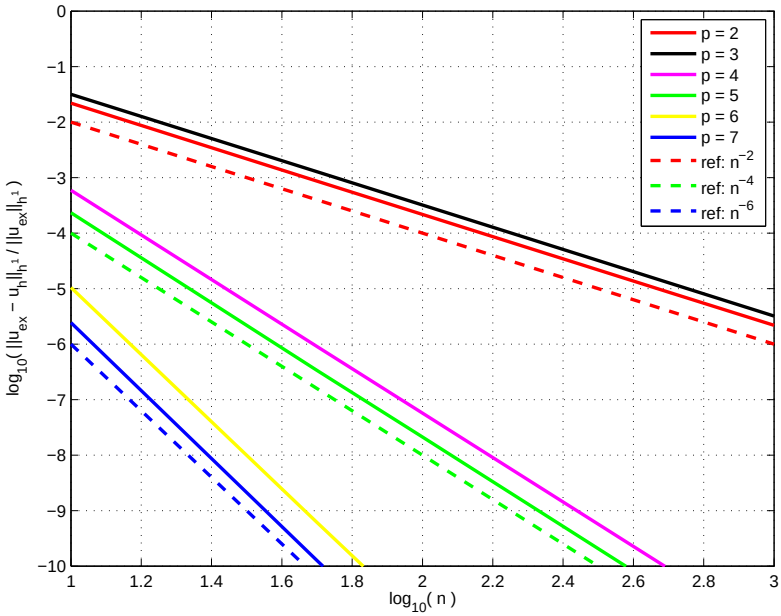

Figure 8: 1D source problem on the periodic domain. Relative error in H^1-norm for different degrees of approximation.

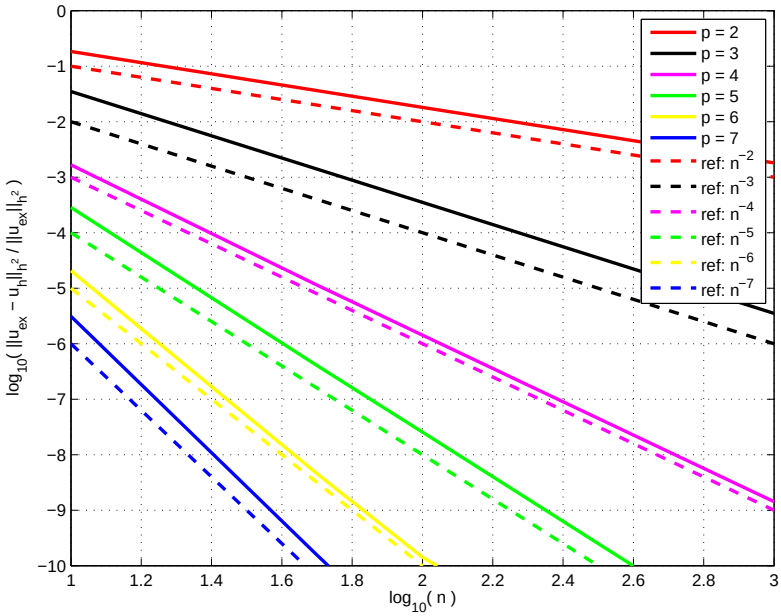

Figure 9: 1D source problem on the periodic domain. Relative error in H^2-norm for different degrees of approximation.

for different degrees of approximation (1000 d.o.f.'s have been used to produce each spectrum). The behavior of the numerical spectra can be better appreciated separating the plots with an even degree from those with an odd degree, as done in Figures **??** and **??**, respectively. In those figures it is possible to appreciate the behavior of the numerical spectra, clearly converging with p.

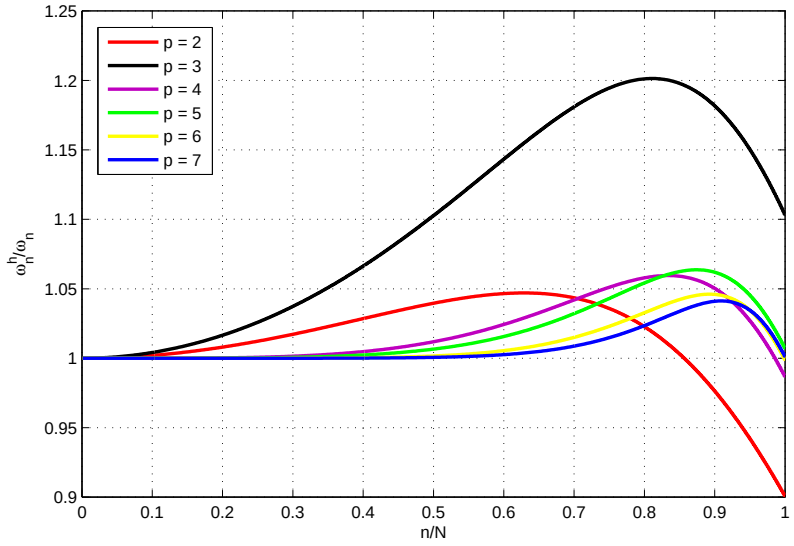

Figure 10: 1D eigenvalue problem on the periodic domain. Normalized spectra for different degrees of approximation.

6.3 One dimensional source problem, with Dirichlet boundary conditions

We now move to the domain $[0, 1]$ and we study the same problem as before, but with Dirichlet boundary conditions, i.e.:

$$\begin{aligned} -u'' + u &= (1 + 4\pi^2)\sin(2\pi x), \quad x \in [0, 1], \\ u(0) = u(1) &= 0. \end{aligned} \tag{12}$$

also in this case the exact solution has the expression:

$$u = \sin(2\pi x). \tag{13}$$

At the discrete level, the boundary conditions are strongly imposed on the first ($\tau_1 = 0$) and last ($\tau_n = 1$) Greville abscisse, and collocation is implemented at $\tau_2, \ldots, \tau_{n-1}$. We obtain the convergence plots of In Figures **??-??**, which show the same results and call for the same comments as in the periodic case.

6.4 One dimensional eigenvalue problem, with Dirichlet boundary conditions

We finally propose also the eigenvalue problem on the finite domain, i.e.:

$$\begin{aligned} -u'' + \omega^2 u &= 0, \quad x \in [0, 1], \\ u(0) = u(1) &= 0. \end{aligned} \tag{14}$$

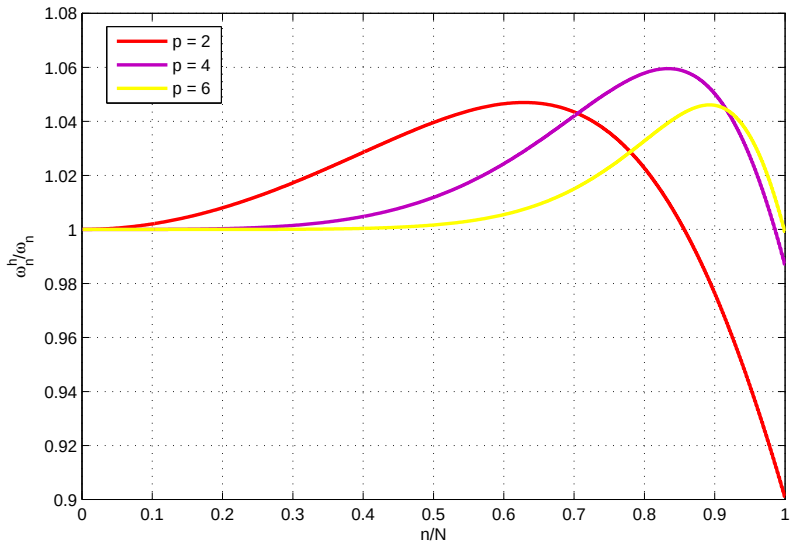

Figure 11: 1D eigenvalue problem on the periodic domain. Normalized spectra for different even degrees of approximation.

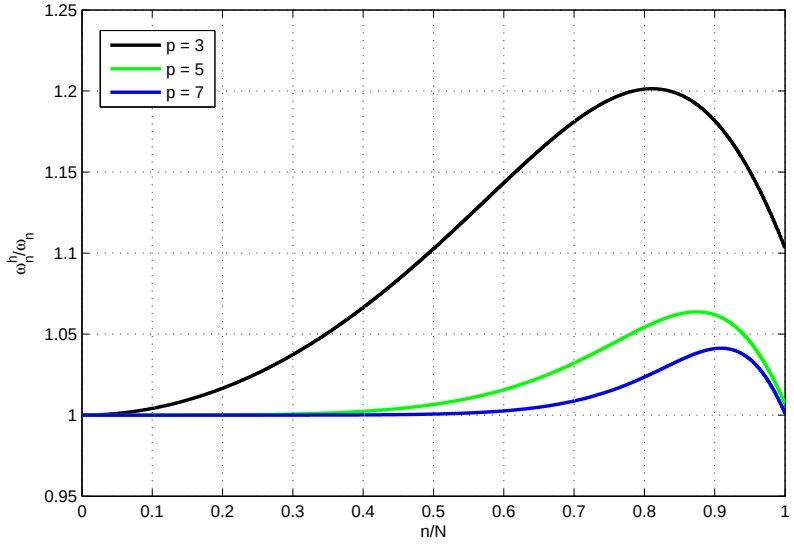

Figure 12: 1D eigenvalue problem on the periodic domain. Normalized spectra for different odd degrees of approximation.

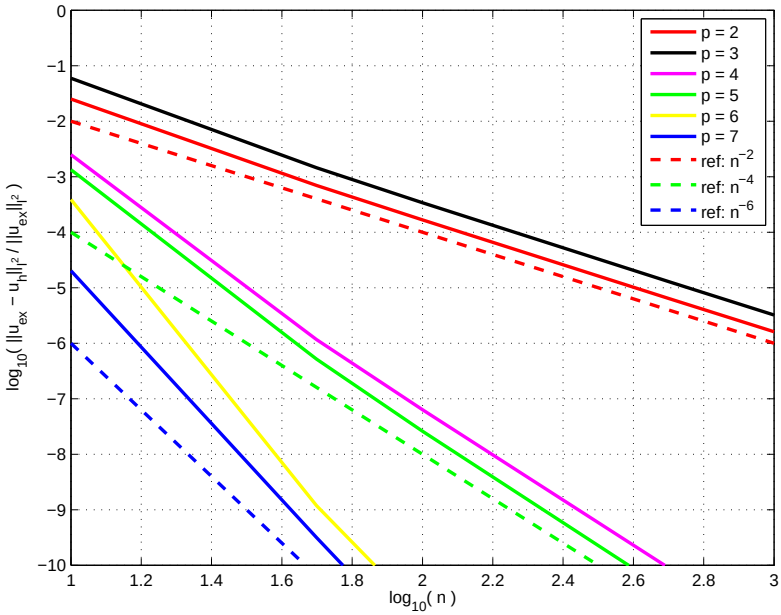

Figure 13: 1D source problem with Dirichlet boundary conditions. Relative error in L^2-norm for different degrees of approximation.

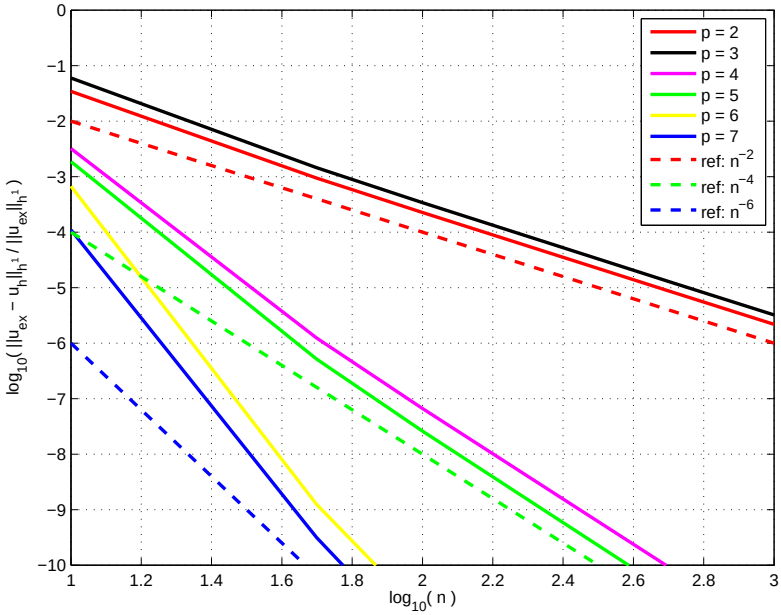

Figure 14: 1D source problem with Dirichlet boundary conditions. Relative error in H^1-norm for different degrees of approximation.

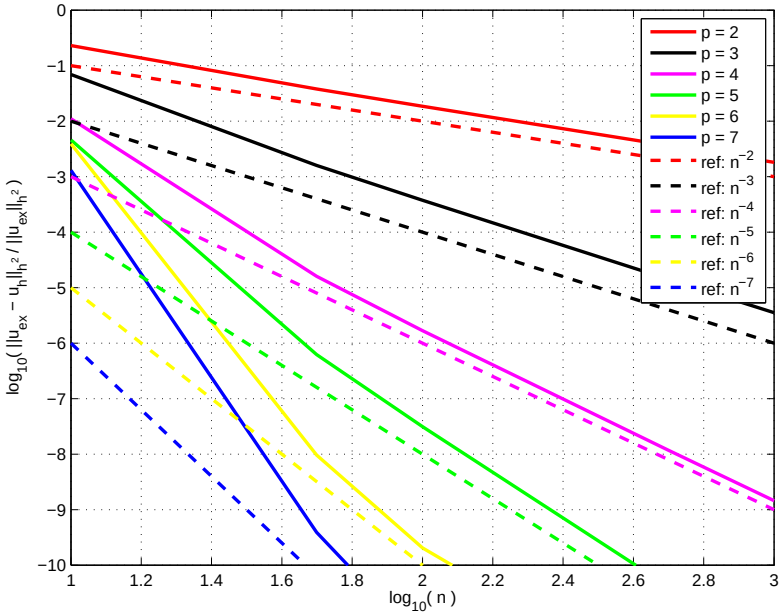

Figure 15: 1D source problem with Dirichlet boundary conditions. Relative error in H^2-norm for different degrees of approximation.

for which the exact frequencies ω are given by:

$$\omega = 2\pi n, \quad \text{with } n = 1, 2, 3, \dots \tag{15}$$

Also in this case, we obtain basically the same numerical spectra error plot as for the case of the periodic domain and the results are shown in Figure **??**. We have to highlight that in this case outlier frequencies appear at the very end of the spectrum. This behaviour is analogous to the one of Galerkin-based Isogeometric discretization and can be circumvented resorting to a suitable nonlinear parametrization of the domain $[0, 1]$, obtained by a uniform control mesh (see [**?**] for more details). This issue deserves further study for collocation-based discretizations.

7 CONCLUSIONS

In this work, we first reviewed the properties of NURBS-based discretizations for problems of structural vibration and wave propagation, taking advantage of the duality principle.

We then proposed a preliminary study on collocation methods for NURBS-based isogeometric analysis. The idea consists of trying to connect the superior isogeometric accuracy properties with the low computational costs typical of collocation approaches, aiming at being able in the future to tackle complicate dynamics problems. We focused on very simple one-dimensional model problems, considering the cases of periodic and finite domains and studying both source and eigenvalue problems. We decided to choose as collocation points the so-called Greville abscissae and the obtained numerical results are particularly interesting and encouraging. In fact, for the studied source problems, we observed an order of convergence of p, where p is the approximation degree. This is a very good (and optimal) result for a collocation method. We also studied eigenvalue problems, highlighting good accuracy and convergence properties for an increasing degree of the approximation.

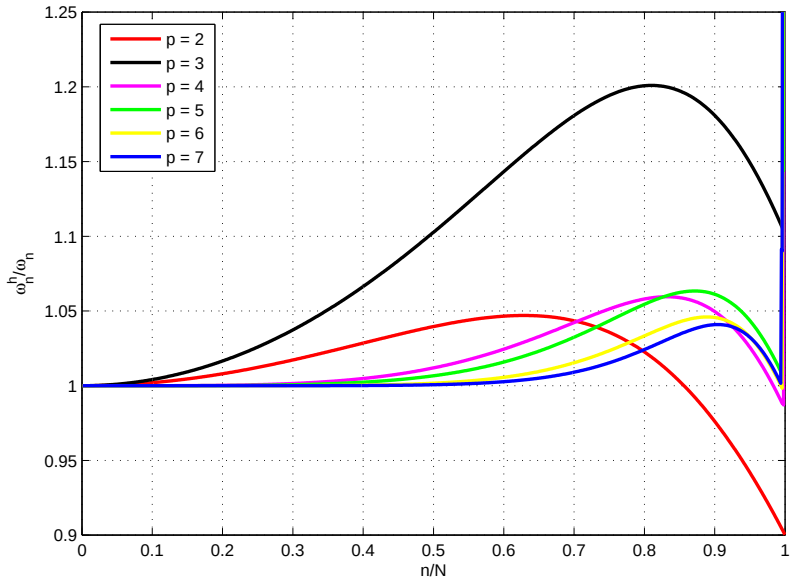

Figure 16: 1D eigenvalue problem with Dirichlet boundary conditions. Normalized spectra for different degrees of approximation.

We may conclude that this work shows an interesting research way that, given these promising preliminary results, deserves to be deeply explored, in particular for potential applications in structural and fluid dynamics, and in general in all those problems where computational costs are highly relevant.

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