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On the geometrically exact beam model: A consistent, effective and simple derivation from three-dimensional finite-elasticity

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ABSTRACT

The goal of the present work is to emphasize some intrinsic features of the three-dimensional beam model originally proposed by Simo [Simo, J.C., 1985. A finite-strain beam formulation. The three-dimensional dynamic problem. Part I. Comput. Methods Appl. Mech. Eng. 49, 55–70] and to derive the model equations for the general finite-deformation case as well as for the finite-deformation small-strain case in a consistent but simpler way with respect to what it is generally done in the literature.

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1. Introduction

During the past decades several researchers focused their attention on geometrically nonlinear beam models. In this context, one of the major contributions is certainly awarded to the works of Simo (1985) and Simo and Vu-Quoc (1986, 1991). In fact, in 1985 Simo provided a compact and clear expression for the beam deformation map, based on the introduction of a rotation tensor as a measure of the cross-section three-dimensional finite rotation, defining a priori the stress resultants and achieving beam strain measures through the application of a three-dimensional power equation. Then, Simo and Vu-Quoc (1986) derived a weak formulation and an associated finite element formulation of the model, and, later, again Simo and Vu-Quoc (1991) extended the formulation to account for warping phenomena, introducing for the first time the denomination *geometrically exact beam* (indeed, the denomination *geometrically exact* was used for the first time two years earlier by Simo and Fox (1989), but with reference to shells).

In the same period, Cardona and Géradin (1988) contributed to better pinpoint the beam formulation as a particular case of the three-dimensional nonlinear continuum theory, clarifying also the concept of co-rotational derivative, already suggested by Simo. Moreover, they were the first who classified the beam model finite element formulations according to the different linearizations and parameterizations of the rotation tensor, introducing the now common classification in *Eulerian*, *Total Lagrangian* and *Updated Lagrangian*.

From the Nineties up to now, researchers have taken a great effort to develop new finite element formulations of Simo's model, with the main purpose of handling efficiently rotations. In particular, Ibrahimbegović et al. (1995a) developed a Total

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Lagrangian formulation, which has been recently reformulated in a very clear and systematic way by Ritto-Corrêa and Camotim (2002). Then, Ibrahimbegović and Taylor (2002) provided sharp finite element implementations for the case of both Eulerian and Updated Lagrangian. Jelenić and Crisfield (1999) proposed their own finite element formulation specifically designed to avoid non-objectivity of the discretized strain measures and presented interesting performance comparisons between several finite element schemes. Many other finite element formulations have been proposed, as for instance a formulation based only on rotational degrees of freedom (Jelenić and Saje, 1995), a formulation based on the interpolation of director vectors (Betsch and Steinmann, 2002) or others designed for initially curved beam elements (Ibrahimbegović, 1995b; Kapania and Li, 2003). Also a mixed finite element formulation has been recently developed (Nukala and White, 2004). Furthermore, in an interesting work of Mata et al. (2007), the geometrically exact model has been extended to account for nonlinear constitutive behavior.

In the already wide realm of finite-deformation beam models, the present paper goes back to Simo's original idea for the definition of the beam deformation map function. Starting from this deformation map, we rewrite the beam kinematics in the context of the finite-elasticity three-dimensional theory and, thus, we calculate the three-dimensional kinematic measures as the deformation gradient, the left and right Cauchy–Green tensors, the Euler–Almansi and the Green–Lagrange strain tensors.

Moreover, taking inspiration from recent works in the theory of finite elasticity, as for instance Boulanger and Hayes (2001) or Jarić et al. (2006), we propose an *extended polar decomposition* of the beam deformation gradient, based on the composition of a rotation tensor, here corresponding to the cross-section rotation tensor, with a positive-definite non-symmetric pure stretch tensor. This decomposition enables the clear individuation of beam strain measures, both from a physical and from a mathematical point of view.

The deformation gradient and the Green–Lagrange strain tensor are then used within a three-dimensional internal virtual work expression written either in term of the first or of the second Piola–Kirchhoff stress tensors. Integrating the three-dimensional work, we naturally attain the beam traction and moment resultants as well as the 1D internal virtual work, both consistent with our kinematics. Moreover, we highlight some interesting features of the traction and moment resultants, specific for this finite-deformation finite-strain model.

The theory described above and detailed in the first part of the paper is then *exact*, being fully consistent with the adopted beam deformation map and with a finite-deformation finite-strain elasticity theory. On the other hand, we may be interested also in the development of a beam model consistent with a finite-deformation but small-strain elasticity theory. We focus on the latter problem since we found that in the literature it is still missing a clear explanation on how to join the finite-strain kinematics with a small-strain linear elastic beam constitutive relation. In fact, Simo and Vu-Quoc (1986) postulated such a kind of relation for the model proposed in the earlier work by Simo (1985). Later on, Simo and Vu-Quoc (1991) justified that relation, as well as Mäkinen (2007) recently did, but both in a quite complex way. In this paper, we attain the same goal through a rather simple and intuitive procedure. In fact, we first approximate the Green–Lagrange tensor neglecting its quadratic pure strain term; consequently, we evaluate the corresponding approximated small-strain traction and moment resultants within the internal virtual work; finally, we introduce a linear elastic isotropic relation between the second Piola–Kirchhoff stress tensor and the small-strain Green–Lagrange tensor and we naturally attain Simo's constitutive relation. We highlight that this procedure is small-strain consistent, since we neglect the high-order strain terms in a pure strain measure; moreover, it is based on a well-known three-dimensional constitutive equation, fully suited for the approximated kinematics. In the light of our approach, it is clear why Simo's model (1985) can be defined finite-deformation finite-strain if considering kinematics and equilibrium only, while, if considering also the linear elastic beam stress–strain relations, it should be defined finite-deformation small-strain.

The paper is constituted by three main sections. The first one is devoted to the description of the model kinematics; the second one deals with the internal virtual work for the exact model; the last one concerns the approximation of the exact model in the finite-deformation small-strain regime and the consequent definition of the beam linear elastic constitutive relations.

2. Kinematics

The aim of this section is to describe in detail the model kinematics. In particular, in the following, we present the beam reference and current configurations and we introduce the deformation map and its gradient. Then, we show a decomposition of the deformation gradient which allows a clear identification of the beam strain measures. We finally provide the virtual variations of some beam kinematic measures which will be useful for subsequent developments.

2.1. Reference and current configurations

As usual in finite-elasticity, we describe body kinematics considering a reference and a current configuration, defining both configurations with respect to a global reference system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and an associate set of coordinates $\{X_1, X_2, X_3\}$.

In particular, we assume that in the reference configuration the beam has a straight axis and uniform cross-sections and, in order to describe this configuration, we also introduce a right-handed orthonormal frame $\{O, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, called *reference frame*, with O located on the axis and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ oriented such that $\mathbf{E}_1$ and $\mathbf{E}_2$ lay parallel to a generic cross-section and $\mathbf{E}_3$ is parallel to the beam axis. Assuming for simplicity that the reference frame coincides with the global one, as shown in Fig. 1, the *beam reference configuration* is described by the reference position vector field $\mathbf{X} \in \mathscr{R}^3$

$$\mathbf{X} = \mathbf{X}_0(X_3) + \mathbf{r}^r(X_\alpha), \tag{1}$$

where

$$\mathbf{X}_0(X_3) = X_3\mathbf{E}_3, \quad \mathbf{r}^r = X_\alpha \mathbf{E}_\alpha. \tag{2}$$

Henceforth, we use the summation convention with Latin indices ranging from 1 to 3 and with Greek indices ranging from 1 to 2. Being X_3 the reference axis coordinate, $\mathbf{X}_0$ represents the position of an axis point (Fig. 1), while being X_1 and X_2 the reference cross-section coordinates, $\mathbf{r}^r$ represents the position of a point within a cross-section, reason why we call it *reference cross-section position vector*.

To describe now the beam current configuration, we introduce another right-handed orthonormal frame, $\{o, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, called *moving* or *current frame*, with o located on the current axis and $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ oriented such that $\mathbf{t}_1$ and $\mathbf{t}_2$ lay parallel to a generic cross-section in the current configuration and $\mathbf{t}_3$ is normal to each cross-section in the current configuration.

Pointing out that the moving frame is function only of the reference axis coordinate X_3, i.e. $\mathbf{t}_I = \mathbf{t}_I(X_3)$, and observing that the moving frame and the reference frame are both orthonormal, we may introduce a one-parameter rotation tensor $\mathbf{\Lambda}(X_3) \in \mathscr{G}^{orth+}$, with $\mathscr{G}^{orth+}$ the proper orthogonal group, relating the moving and the reference frame as

$$\mathbf{t}_I(X_3) = \mathbf{\Lambda}(X_3)\mathbf{E}_I, \tag{3}$$

i.e. we may define the moving frame as the rotated reference frame.

With the help of the introduced quantities, following Simo (1985), we assume to describe the *beam current configuration* by the position vector field $\boldsymbol{x} \in \mathscr{R}^3$

$$\boldsymbol{x} = \boldsymbol{x}_0(X_3) + \mathbf{r}(X_\alpha, X_3), \tag{4}$$

where

$$\mathbf{r} = X_\alpha \mathbf{t}_\alpha, \tag{5}$$

such that

$$\boldsymbol{x} = \boldsymbol{x}_0(X_3) + X_\alpha \mathbf{t}_\alpha(X_3). \tag{6}$$

In Eq. (4), $\boldsymbol{x}_0 = \boldsymbol{x}_0(X_3)$ represents the position of an axis point in the current configuration, that is the position of the moving frame's origin o for a cross-section (Fig. 1). On the other side, $\mathbf{r}$ represents the position of a point within a cross-section in the current configuration, reason why we call it *current cross-section position vector*.

According to Eq. (3), we have

$$\mathbf{r} = \mathbf{\Lambda}(X_3)\mathbf{r}^r(X_\alpha), \tag{7}$$

that is, $\mathbf{r}$ is obtained by a rotation of the point's reference cross-section position vector and, since any $\mathbf{r}^r$ of a same cross-section rotates of the same quantity $\mathbf{\Lambda}(X_3)$, it follows that the cross-section rigidly moves from the reference to the current configuration and that $\mathbf{\Lambda}$ represents the *cross-section rigid rotation*.

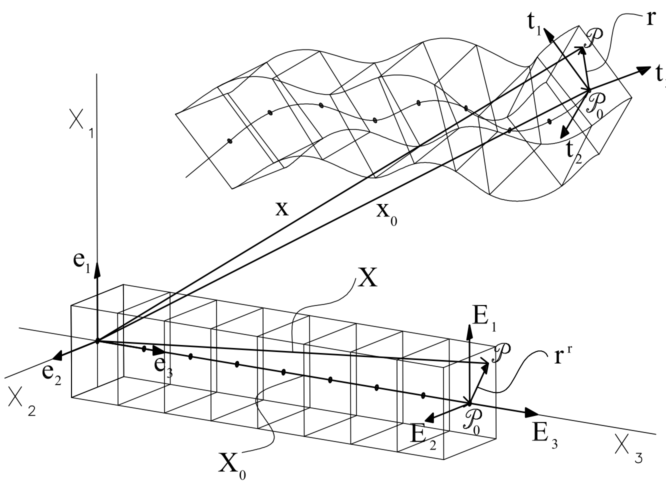

Fig. 1. Three-dimensional representation of the coordinate system, beam reference configuration and beam current configuration: global reference system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and set of coordinates $\{X_1, X_2, X_3\}$; reference frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, reference axis position vector $\mathbf{X}_0$, reference cross-section position vector $\mathbf{r}^r$ and reference position vector $\mathbf{X}$; moving frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, current axis position vector $\boldsymbol{x}_0$, current cross-section position vector $\mathbf{r}$ and current position vector $\boldsymbol{x}$.

We may now rewrite the deformation map (4) as

$$\boldsymbol{x} = \boldsymbol{x}_0(X_3) + X_\alpha \boldsymbol{\Lambda}(X_3)\mathbf{E}_\alpha, \tag{8}$$

which clearly shows how the current configuration is uniquely defined by $\boldsymbol{x}_0(X_3)$ and $\boldsymbol{\Lambda}(X_3)$, and the three-dimensional kinematics is reduced to a one-dimensional kinematics.

Remark 2.1.1. According to Eq. (8), in the current configuration the beam can be physically seen as a line, i.e. the axis individuated by $\boldsymbol{x}_0$, and a set of attached cross-sections obtained by a rigid rotation of the cross-sections in the reference configuration. Moreover, we observe that Eq. (6) does not impose any constraint on the orientation of $\mathbf{t}_3$, and in particular it does not impose the parallelism of $\mathbf{t}_3$ with the axis tangent, that is, the introduced kinematics allows for shear strains.

Remark 2.1.2. From Eq. (3), we can obtain the following expression for $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda}(X_3) = \mathbf{t}_I(X_3) \otimes \mathbf{E}_I, \tag{9}$$

which explicitly shows that $\boldsymbol{\Lambda}$ is a two-point tensor. Moreover, we recall that, as an orthogonal tensor, $\boldsymbol{\Lambda}$ is such that

$$\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T = \mathbf{I}. \tag{10}$$

2.2. Deformation gradient decompositions

From Eq. (8), we can now compute all the standard three-dimensional deformation measures, starting from the deformation gradient $\mathbf{F} = \partial\boldsymbol{x}/\partial\mathbf{X}$, where the symbol $\partial(\cdot)$ stands for a partial derivative. Expressing a component of the reference position vector field $\mathbf{X}$ as $X_I = \mathbf{X} \cdot \mathbf{E}_I$,

$$\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial X_I} \otimes \mathbf{E}_I, \tag{11}$$

which turns out to be a useful expression. In fact, introducing notation $(\cdot)_{,3}$ for derivatives with respect to X_3 and substituting Eq. (6) into Eq. (11), we obtain

$$\mathbf{F} = \mathbf{t}_\alpha \otimes \mathbf{E}_\alpha + (\boldsymbol{x}_{0,3} + X_\alpha \mathbf{t}_{\alpha,3}) \otimes \mathbf{E}_3. \tag{12}$$

Adding and subtracting the tensor $\mathbf{t}_3 \otimes \mathbf{E}_3$ to the right-hand-side, and recognizing that $\mathbf{t}_I \otimes \mathbf{E}_I = \boldsymbol{\Lambda}$, we may compactly write Eq. (12) as

$$\mathbf{F} = \boldsymbol{\Lambda} + (\boldsymbol{x}_{0,3} - \mathbf{t}_3 + X_\alpha \mathbf{t}_{\alpha,3}) \otimes \mathbf{E}_3 = \boldsymbol{\Lambda} + \mathbf{a} \otimes \mathbf{E}_3, \tag{13}$$

where we set

$$\mathbf{a} = \boldsymbol{\gamma} + X_\alpha \boldsymbol{\kappa}_\alpha, \tag{14}$$

with

$$\boldsymbol{\gamma} = \boldsymbol{x}_{0,3} - \mathbf{t}_3, \quad \boldsymbol{\kappa}_\alpha = \mathbf{t}_{\alpha,3}. \tag{15}$$

Recalling Eq. (10), we can collect the rotation tensor $\boldsymbol{\Lambda}$ in Eq. (13), writing the deformation gradient as

$$\mathbf{F} = [\mathbf{I} + (\mathbf{a} \otimes \mathbf{E}_3)\boldsymbol{\Lambda}^T]\boldsymbol{\Lambda} = \mathbf{A}\boldsymbol{\Lambda}, \tag{16}$$

where

$$\mathbf{A} = \mathbf{I} + \mathbf{a} \otimes \mathbf{t}_3. \tag{17}$$

Since the deformation gradient determinant must be positive and the rotation tensor determinant is 1, from Eq. (16) we may conclude that the determinant of $\mathbf{A}$ is positive and then from Eq. (17), using Jacobi's criterion, it follows that $\mathbf{A}$ is a positive-definite tensor. Hence, Eq. (16) is a decomposition of the deformation gradient into a rotation tensor $\boldsymbol{\Lambda}$, on the right, and a positive-definite tensor $\mathbf{A}$, in general non-symmetric, on the left.

In our opinion, expression (16) is a fundamental relation since it allows a clear physical interpretation of the beam deformation. This interpretation is first of all based on the observation that, as shown in Appendix A, for a beam rigid motion we have $\boldsymbol{\gamma} = \mathbf{0}$ and $\boldsymbol{\kappa}_\alpha = \mathbf{0}$ and, consequently, $\mathbf{a} = \mathbf{0}$ and $\mathbf{A} = \mathbf{I}$. Therefore, expression (16) can be interpreted as a decomposition of the deformation gradient into the cross-section physical rotation, $\boldsymbol{\Lambda}$, followed by $\mathbf{A}$, which is a pure stretch within a section point neighborhood in the current configuration. Taking inspiration from the three-dimensional theory of elasticity (see, e.g., Boulanger and Hayes (2001, 2006) and Jarić et al. (2006)), we may call decomposition (16) a *left extended polar decomposition*, which is clearly a local decomposition, i.e. a decomposition defined point by point within the beam, since the deformation gradient is function of all the coordinates X_I.

Accordingly, $\mathbf{A}$ is called *current local stretch tensor*, where the term "local" highlights the fact that $\mathbf{A}$ varies within the cross-section.

Being $\mathbf{A}$ a local stretch measure in the current configuration, we can naturally introduce a *current local strain tensor* $\mathbf{L}$ defined as

$$\mathbf{L} = \mathbf{A} - \mathbf{I}. \tag{18}$$

Using Eq. (17), we obtain the simple expression

$$\mathbf{L} = \mathbf{a} \otimes \mathbf{t}_3, \tag{19}$$

which explicitly shows that $\mathbf{L}$ is uniquely defined by vector $\mathbf{a}$ within the current frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$. Accordingly, we deduce that in the current configuration $\mathbf{a}$ contains all the information about the local strain and this is the reason why we call it *current local strain vector*.

It is interesting to observe that the strain vector $\mathbf{a}$ is composed by a term uniform within the cross-section, $\boldsymbol{\gamma} = \boldsymbol{x}_{0,3} - \mathbf{t}_3$, and by a term linear within the cross-section, $X_\alpha \boldsymbol{\kappa}_\alpha$. This structure is identical to the one obtained in the standard geometrically linear beam theory (Hjelmstad (1997)), where the uniform term accounts for the shear-axial strain while the linear term accounts for the bending-torsional strain. This analogy leads us to analyze the physical meaning of $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}_\alpha$ in more details. Since $\mathbf{A}$, as well as $\mathbf{L}$, belongs to a left decomposition, they are defined in a rotated configuration, hence we are induced to read $\mathbf{a}$ within the current frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ in order to catch its physical meaning.

Considering $\boldsymbol{\gamma} = \boldsymbol{x}_{0,3} - \mathbf{t}_3$, we recognize that $\boldsymbol{x}_{0,3}$ is the axis tangent vector in the current configuration, being the derivative of the current axis position vector $\boldsymbol{x}_0(X_3)$ with respect to the axis parameter X_3; thus, $\boldsymbol{\gamma}$ represents the difference between the axis tangent vector and the normal cross-section vector $\mathbf{t}_3$ in the current configuration. Computing the components $\gamma_i = \boldsymbol{\gamma} \cdot \mathbf{t}_i$ we have

$$\gamma_\alpha = (\boldsymbol{x}_{0,3})_\alpha, \quad \gamma_3 = (\boldsymbol{x}_{0,3})_3 - 1, \tag{20}$$

where $(\boldsymbol{x}_{0,3})_i = \boldsymbol{x}_{0,3} \cdot \mathbf{t}_i$. With the help of Fig. 2, we see that $(\boldsymbol{x}_{0,3})_\alpha$ represents the relative displacement along $\mathbf{t}_\alpha$ of an axis point with respect to another point infinitesimally close along X_3. It follows that $(\boldsymbol{x}_{0,3})_\alpha$ accounts for the shear strain in the current configuration. Adding that $(\boldsymbol{x}_{0,3})_\alpha$ is constant over the cross-section, we may conclude that γ_1 and γ_2 are the cross-section shear strains in the current configuration. On the other side, since γ_3 is the component along $\mathbf{t}_3$ of the tangent vector reduced by one, we deduce that γ_3 accounts for the axial strain in the current configuration. Hence, as in the geometrically linear beam theory we have that $\boldsymbol{\gamma}$ is the *current cross-section shear-axial strain vector*.

A clear interpretation of $\boldsymbol{\kappa}_\alpha$ follows from the introduction of the rotation tensor derivative. Given a rotation tensor, $\boldsymbol{\Lambda}(X_3)$, its derivative with respect to the parameter X_3 may be expressed in general as

$$\boldsymbol{\Lambda}_{,3} = \boldsymbol{\Omega}\boldsymbol{\Lambda}, \tag{21}$$

where $\boldsymbol{\Omega} = \boldsymbol{\Omega}(X_3)$ is a skew-symmetric tensor. Moreover, given a skew-symmetric tensor $\boldsymbol{\Omega}$, there always exists a vector $\boldsymbol{\omega}$ such that

$$\boldsymbol{\Omega}\mathbf{b} = \boldsymbol{\omega} \times \mathbf{b} \quad \forall \mathbf{b} \in \mathscr{R}^3, \tag{22}$$

where

$$\{\boldsymbol{\omega}\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}, \quad [\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{23}$$

With this notation in hand, we can write the derivative of $\mathbf{t}_\alpha = \boldsymbol{\Lambda}\mathbf{E}_\alpha$ as

$$\mathbf{t}_{\alpha,3} = (\boldsymbol{\Lambda}\mathbf{E}_\alpha)_{,3} = \boldsymbol{\Lambda}_{,3}\mathbf{E}_\alpha = \boldsymbol{\Omega}\boldsymbol{\Lambda}\mathbf{E}_\alpha = \boldsymbol{\Omega}\mathbf{t}_\alpha = \boldsymbol{\omega} \times \mathbf{t}_\alpha, \tag{24}$$

and, thus, since $\boldsymbol{\kappa}_\alpha = \mathbf{t}_{\alpha,3}$, we have that

$$\boldsymbol{\kappa}_\alpha = \boldsymbol{\omega} \times \mathbf{t}_\alpha. \tag{25}$$

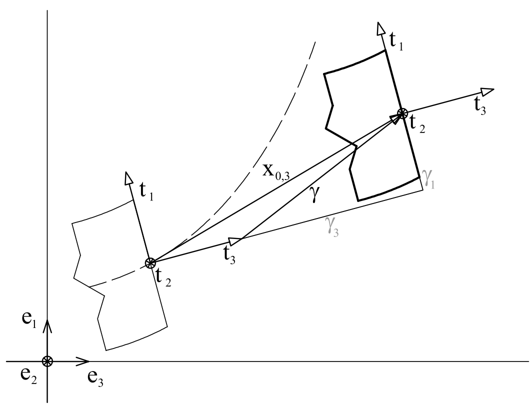

Fig. 2. Bi-dimensional representation of the current shear-axial strain vector $\boldsymbol{\gamma}$: $\boldsymbol{x}_{0,3}$ axis tangent vector, γ_1 shear component of $\boldsymbol{\gamma}$, γ_3 axial component of $\boldsymbol{\gamma}$.

From the equation above, it is immediately obtained that $\boldsymbol{\kappa}_\alpha \cdot \mathbf{t}_\alpha = 0$ (no sum on repeated indices). So, we have that of the six components of $\boldsymbol{\kappa}_\alpha$ only four are different from zero and they can be simply computed in terms of ω_i (being $\omega_i = \boldsymbol{\omega} \cdot \mathbf{t}_i$ the components of $\boldsymbol{\omega}$ in the current frame) as

$$(\boldsymbol{\kappa}_1)_2 = -(\boldsymbol{\kappa}_2)_1 = \omega_3, \quad (\boldsymbol{\kappa}_1)_3 = -\omega_2, \quad (\boldsymbol{\kappa}_2)_3 = \omega_1, \tag{26}$$

where $(\boldsymbol{\kappa}_\alpha)_i = \boldsymbol{\kappa}_\alpha \cdot \mathbf{t}_i$. Since in our beam theory $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(X_3)$ represents the cross-section rotation, $\boldsymbol{\Omega} = \boldsymbol{\Omega}(X_3)$ represents the rate of change for the cross-section rotation, and accordingly it can be interpreted as a torsional-bending curvature measure. In particular, we observe that $(\boldsymbol{\kappa}_1)_2$ and $(\boldsymbol{\kappa}_2)_1$ can be interpreted as components of the *torsion curvature*, while $(\boldsymbol{\kappa}_1)_3$ and $(\boldsymbol{\kappa}_2)_3$ can be interpreted as *bending curvatures*.

To better understand the effect of $\boldsymbol{\kappa}_\alpha$ in terms of strain (i.e. its contribution to $\mathbf{a}$), we have to explicitly compute $X_\alpha\boldsymbol{\kappa}_\alpha$ using expression (25), obtaining

$$X_\alpha\boldsymbol{\kappa}_\alpha = \boldsymbol{\omega} \times X_\alpha\mathbf{t}_\alpha = -X_2\omega_3\mathbf{t}_1 + X_1\omega_3\mathbf{t}_2 + (X_2\omega_1 - X_1\omega_2)\mathbf{t}_3. \tag{27}$$

Using again the notation $(X_\alpha\boldsymbol{\kappa}_\alpha)_i = X_\alpha\boldsymbol{\kappa}_\alpha \cdot \mathbf{t}_i$, we observe that $(X_\alpha\boldsymbol{\kappa}_\alpha)_1$ and $(X_\alpha\boldsymbol{\kappa}_\alpha)_2$ are defined through $\omega_3(X_3)$; with the help of Fig. 3, we may deduce that, as expected, ω_3 accounts for the cross-section torsional strain in the current configuration, hence $(X_\alpha\boldsymbol{\kappa}_\alpha)_1$ and $(X_\alpha\boldsymbol{\kappa}_\alpha)_2$ are local torsional strain components. On the other side, $(X_\alpha\boldsymbol{\kappa}_\alpha)_3$ is defined through ω_1 and ω_2; looking again at Fig. 3, we see that ω_1 accounts for the cross-section bending strain around $\mathbf{t}_1$ and ω_2 for the cross-section bending strain around $\mathbf{t}_2$, hence $(X_\alpha\boldsymbol{\kappa}_\alpha)_3$ is a local bending strain component.

For successive developments, it is also useful to rewrite expression (14) for $\mathbf{a}$, making use of the new expression (25) for $\boldsymbol{\kappa}_\alpha$, as follows

$$\mathbf{a} = \boldsymbol{\gamma} + \boldsymbol{\omega} \times X_\alpha\mathbf{t}_\alpha. \tag{28}$$

We remark that all the measures introduced up to now are defined in the current configuration, i.e. in a configuration rotated with respect to the reference configuration. Our next goal is to find a representation of the same quantities in the reference configuration.

In order to do that, we start recalling the relations between a vector, or a tensor, and their rotated counterparts. Given two orthogonal bases $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ related through Eq. (3), i.e. $\mathbf{t}_I = \boldsymbol{\Lambda}\mathbf{E}_I$, we consider two vectors, $\mathbf{b}$ and $\mathbf{b}^r$, and two tensors, $\mathbf{V}$ and $\mathbf{V}^r$, such that

$$\mathbf{b} = \boldsymbol{\Lambda}\mathbf{b}^r, \quad \mathbf{V} = \boldsymbol{\Lambda}\mathbf{V}^r\boldsymbol{\Lambda}^T. \tag{29}$$

Expressing their components in the form

$$b_i = \mathbf{b} \cdot \mathbf{t}_i, \quad V_{ij} = \mathbf{t}_i \cdot \mathbf{V}\mathbf{t}_j, \tag{30}$$

$$b^r_i = \mathbf{b}^r \cdot \mathbf{E}_i, \quad V^r_{ij} = \mathbf{E}_i \cdot \mathbf{V}^r\mathbf{E}_j, \tag{31}$$

it is obvious that the components of $\mathbf{b}$ and $\mathbf{V}$ in the frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ are equal to the components of $\mathbf{b}^r$ and $\mathbf{V}^r$ in the frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, that is

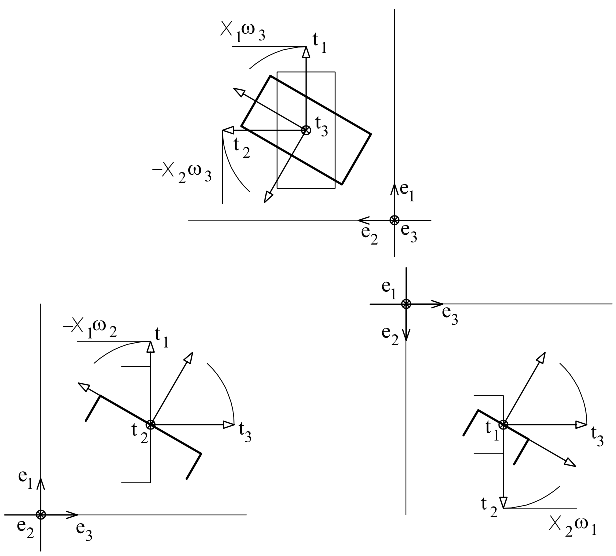

Fig. 3. Bi-dimensional representation of the local torsional-bending strain components of $X_\alpha\boldsymbol{\kappa}_\alpha$ via $\boldsymbol{\omega} : X_1\omega_3$ and $-X_2\omega_3$ torsional components (on the top, frontal beam view), $-X_1\omega_2$ and $X_2\omega_1$ bending components (on the left bottom, lateral beam view; on the right bottom, from above beam view).

$$b_i = b_i^r, \quad V_{ij} = V_{ij}^r. \tag{32}$$

We may call $\mathbf{b}$ the *rotate-forward* form of $\mathbf{b}^r$ or, viceversa, $\mathbf{b}^r$ the *rotate-back* form of $\mathbf{b}$, and the same for $\mathbf{V}$ and $\mathbf{V}^r$. Therefore, considering the current strain vector $\mathbf{a}$, we may define the vector

$$\mathbf{a}^r = \mathbf{\Lambda}^T \mathbf{a}, \tag{33}$$

such that $\mathbf{a}^r$ measured in the reference frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ has the same components of $\mathbf{a}$ measured in the current frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, i.e. $a_i^r = a_i$ with $a_i^r = \mathbf{a}^r \cdot \mathbf{E}_i$ and $a_i = \mathbf{a} \cdot \mathbf{t}_i$ (Fig. 4). Accordingly, we may say that $\mathbf{a}^r$ is the rotate-back form of $\mathbf{a}$, i.e. it is the vector strain measure in the reference configuration, and then we may call $\mathbf{a}^r$ the *reference local strain vector*. Equivalently, we may define the *reference cross-section shear-axial strain vector* γ^r and the *reference cross-section torsional-bending strain vector* $\boldsymbol{\omega}^r$ as

$$\gamma^r = \mathbf{\Lambda}^T \gamma, \quad \boldsymbol{\omega}^r = \mathbf{\Lambda}^T \boldsymbol{\omega}, \tag{34}$$

where $\gamma^r = \mathbf{\Lambda}^T \boldsymbol{x}_{0,3} - \mathbf{E}_3$. Moreover, recalling Eq. (28), from Eq. (33) we can derive that

$$\mathbf{a}^r = \gamma^r + \boldsymbol{\omega}^r \times X_\alpha \mathbf{E}_\alpha. \tag{35}$$

Similarly, we may define the *reference local strain tensor* $\mathbf{L}^r$ as the rotate-back form of $\mathbf{L}$, i.e. $\mathbf{L}^r = \mathbf{\Lambda}^T \mathbf{L} \mathbf{\Lambda}$, and the *reference local stretch tensor* $\mathbf{A}^r$ as the rotate-back form of $\mathbf{A}$, i.e. $\mathbf{A}^r = \mathbf{\Lambda}^T \mathbf{A} \mathbf{\Lambda}$, which are, respectively, the local strain and stretch measures in the reference frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Recalling that $\mathbf{L} = \mathbf{a} \otimes \mathbf{t}_3, \mathbf{L}^r$ can be expressed as

$$\mathbf{L}^r = \mathbf{\Lambda}^T (\mathbf{a} \otimes \mathbf{t}_3) \mathbf{\Lambda} = \mathbf{a}^r \otimes \mathbf{E}_3, \tag{36}$$

while, recalling that $\mathbf{A} = \mathbf{I} + \mathbf{a} \otimes \mathbf{t}_3$, $\mathbf{A}^r$ takes the form

$$\mathbf{A}^r = \mathbf{\Lambda}^T (\mathbf{I} + \mathbf{a} \otimes \mathbf{t}_3) \mathbf{\Lambda} = \mathbf{I} + \mathbf{a}^r \otimes \mathbf{E}_3. \tag{37}$$

It is important to observe that, besides obtaining all the reference strain and stretch measures through a consistent rotate-back operation as done above, they can also be naturally obtained moving from a *right extended polar decomposition* for the deformation gradient. In fact, recalling from Eq. (13) that $\mathbf{F} = \mathbf{\Lambda} + \mathbf{a} \otimes \mathbf{E}_3$, we can collect $\mathbf{\Lambda}$ on the left, rather than on the right as in Eq. (16), and obtain

$$\mathbf{F} = \mathbf{\Lambda}[\mathbf{I} + \mathbf{\Lambda}^T (\mathbf{a} \otimes \mathbf{E}_3)] = \mathbf{\Lambda} \mathbf{A}^r, \tag{38}$$

where $\mathbf{A}^r$ is indeed given by expression (37).

2.3. Three-dimensional stretch and strain tensors

Besides clearly individuating the beam strain measures both in the current and in the reference configurations, the left and right extended polar decompositions (16) and (38) are also useful to express in very simple forms the standard finite-elasticity stretch and strain tensors in term of the beam stretch and strain measures.

As an example, exploiting the left decomposition $\mathbf{F} = \mathbf{A}\mathbf{\Lambda}$, we can express the current stretch tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^T$, known as the left Cauchy–Green tensor, in term of the current beam measures as

$$\mathbf{b} = (\mathbf{A}\mathbf{\Lambda})(\mathbf{A}\mathbf{\Lambda})^T = \mathbf{A}\mathbf{A}^T = \mathbf{I} + 2(\mathbf{a} \otimes \mathbf{t}_3)^s + \mathbf{a} \otimes \mathbf{a}, \tag{39}$$

where $(\cdot)^s$ denotes the symmetric part of a tensor. Wishing to evaluate the current strain Euler–Almansi tensor $\mathbf{e} = [\mathbf{I} - \mathbf{b}^{-1}]/2$, we compute the inverse of $\mathbf{b}$ first evaluating the inverse of $\mathbf{A}$ and its transpose. Being $J = 1 + a_3$ the third invariant of $\mathbf{A}$, as shown in Appendix C, it is easy to prove that $\mathbf{A}^{-1}$ and $\mathbf{A}^{-T}$ take the form

$$\mathbf{A}^{-1} = \mathbf{I} - \frac{1}{J}(\mathbf{a} \otimes \mathbf{t}_3), \quad \mathbf{A}^{-T} = \mathbf{I} - \frac{1}{J}(\mathbf{t}_3 \otimes \mathbf{a}). \tag{40}$$

Accordingly, $\mathbf{b}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1}$ follows as

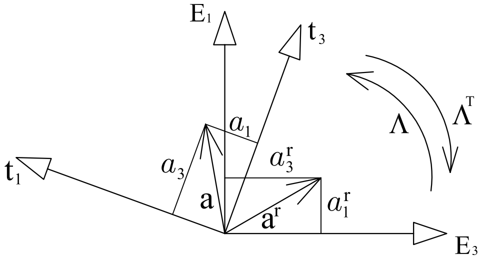

Fig. 4. Bi-dimensional representation of a frame and vector rotation.

$$\mathbf{b}^{-1} = (\mathbf{A}\mathbf{\Lambda})^{-T}(\mathbf{A}\mathbf{\Lambda})^{-1} = \mathbf{A}^{-T}\mathbf{A}^{-1} = \mathbf{I} - 2\frac{1}{J}(\mathbf{a}\otimes\mathbf{t}_3)^s + \frac{1}{J^2}(\mathbf{a}\cdot\mathbf{a})\mathbf{t}_3\otimes\mathbf{t}_3, \tag{41}$$

and, therefore, the Euler–Almansi tensor takes the form

$$\mathbf{e} = \frac{1}{J}(\mathbf{a}\otimes\mathbf{t}_3)^s - \frac{1}{2J^2}(\mathbf{a}\cdot\mathbf{a})\mathbf{t}_3\otimes\mathbf{t}_3. \tag{42}$$

Similarly, exploiting the right decomposition $\mathbf{F} = \mathbf{\Lambda}\mathbf{A}^r$, we can compactly write the reference finite-elasticity tensors in term of the reference beam measures. For instance, we can compute the stretch tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, known as the right Cauchy–Green tensor, in the form

$$\mathbf{C} = (\mathbf{\Lambda}\mathbf{A}^r)^T(\mathbf{\Lambda}\mathbf{A}^r) = (\mathbf{A}^r)^T\mathbf{A}^r = \mathbf{I} + 2(\mathbf{a}^r\otimes\mathbf{E}_3)^s + (\mathbf{a}^r\cdot\mathbf{a}^r)\mathbf{E}_3\otimes\mathbf{E}_3. \tag{43}$$

Accordingly, the reference strain Green–Lagrange tensor $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$, simply follows from Eq. (43) as

$$\mathbf{E} = (\mathbf{a}^r\otimes\mathbf{E}_3)^s + \frac{1}{2}(\mathbf{a}^r\cdot\mathbf{a}^r)\mathbf{E}_3\otimes\mathbf{E}_3. \tag{44}$$

We observe that $\mathbf{b}^{-1}$ and $\mathbf{C}$ hold the same algebraic structure, as well as $\mathbf{e}$ and $\mathbf{E}$. We moreover remark that matrix expressions for the beam stretch and strain tensors are provided in Appendix B, along with some interesting observations.

We now wish to briefly comment expression (44), having a crucial role in the forthcoming calculations. First of all, we observe that $\mathbf{E}$ is defined only in term of $\mathbf{a}^r$ and of the fixed vector $\mathbf{E}_3$, stressing once more that $\mathbf{a}^r$ contains the whole information about the strain in the reference configuration. Furthermore, expression (44) clearly shows that $\mathbf{E}$ is obtained by an additive composition of a term linear in $\mathbf{a}^r$ and a term quadratic in $\mathbf{a}^r$, i.e. of a *linear pure strain* term and a *quadratic pure strain* term. In our opinion, this is an outstanding feature; in fact, in the case we are interested in developing an effective beam theory within a finite-deformation regime but with limitation to small strains, we should just neglect the quadratic part of $\mathbf{E}$ and work only with the linear part, i.e.

$$\mathbf{E} \approx \mathbf{E}^{\text{lin}} = (\mathbf{a}^r\otimes\mathbf{E}_3)^s = (\mathbf{L}^r)^s. \tag{45}$$

Remark 2.3.1. Recalling that $\mathbf{a}\otimes\mathbf{t}_3 = \mathbf{L}$ and $\mathbf{a}^r\otimes\mathbf{E}_3 = \mathbf{L}^r$, we can rearrange even more compactly the current tensors $\mathbf{b}, \mathbf{b}^{-1}$ and $\mathbf{e}$ as

$$\mathbf{b} = \mathbf{I} + 2\mathbf{L}^s + \mathbf{L}\mathbf{L}^T, \tag{46}$$

$$\mathbf{b}^{-1} = \mathbf{I} - 2\frac{1}{J}\mathbf{L}^s + \frac{1}{J^2}\mathbf{L}^T\mathbf{L}, \tag{47}$$

$$\mathbf{e} = \frac{1}{J}\mathbf{L}^s - \frac{1}{2J^2}\mathbf{L}^T\mathbf{L}, \tag{48}$$

as well as the reference tensors $\mathbf{C}$ and $\mathbf{E}$ as

$$\mathbf{C} = \mathbf{I} + 2(\mathbf{L}^r)^s + (\mathbf{L}^r)^T\mathbf{L}^r, \tag{49}$$

$$\mathbf{E} = (\mathbf{L}^r)^s + \frac{1}{2}(\mathbf{L}^r)^T\mathbf{L}^r. \tag{50}$$

Remark 2.3.2. It is interesting to observe that the expression of $\mathbf{E}$ as a composition of a linear and of a quadratic pure strain terms is not immediate or trivial when we deal with finite rotations (see also Gerstmayr and Schöberl (2006) and Sharf (1999)). In fact, $\mathbf{E}$ could be computed also exploiting the ordinary finite-elasticity expression

$$\mathbf{E} = \boldsymbol{\varepsilon}^s + \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}, \tag{51}$$

where $\boldsymbol{\varepsilon} = \nabla_{\mathbf{X}}\mathbf{u}$ is the material gradient of the displacement field $\mathbf{u}$. Even if expressions (50) and (51) look alike, the latter is not a composition of a linear pure strain term with a quadratic pure strain term, since $\boldsymbol{\varepsilon}^s$ as well as $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}$ are not pure strain tensors within our finite-deformation beam model. To prove this, we may first compute the displacement gradient $\boldsymbol{\varepsilon} = \mathbf{F} - \mathbf{I}$ through the right extended polar decomposition $\mathbf{F} = \mathbf{\Lambda}\mathbf{A}^r$ as

$$\boldsymbol{\varepsilon} = \mathbf{\Lambda}\mathbf{A}^r - \mathbf{I} = \mathbf{\Lambda}(\mathbf{I} + \mathbf{L}^r) - \mathbf{I} = \mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{L}^r - \mathbf{I}, \tag{52}$$

and, thus, evaluate $\boldsymbol{\varepsilon}^s$

$$\boldsymbol{\varepsilon}^s = \mathbf{\Lambda}^s + (\mathbf{\Lambda}\mathbf{L}^r)^s - \mathbf{I}, \tag{53}$$

and $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}/2$

$$\frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon}^s + (\mathbf{L}^r)^s + \frac{1}{2}(\mathbf{L}^r)^T\mathbf{L}^r. \tag{54}$$

Noting that for a beam rigid motion $\mathbf{L}^r = \mathbf{0}$ (see Appendix A), we can exert this condition in Eq. (53) and, observing that $\boldsymbol{\varepsilon}^s$ does not vanish, we can conclude that $\boldsymbol{\varepsilon}^s$ is not a measure of pure strain. Consequently, from Eq. (54) we deduce that also $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}/2$ is not a measure of pure strain. Thereby, expression (51) is not suitable to develop a finite-deformation small-strain beam theory, on the contrary of expressions (50) or (44).

2.4. Virtual variations of the kinematic measures

We are now interested in providing the virtual variations of the beam kinematic measures. Introducing the notation $\delta(\cdot)$ for a virtual variation, it is straightforward to compute the virtual variation of the reference local strain vector $\mathbf{a}^r$, that is

$$\delta\mathbf{a}^r = \delta(\boldsymbol{\gamma}^r + \boldsymbol{\omega}^r \times X_\alpha \mathbf{E}_\alpha) = \delta\boldsymbol{\gamma}^r + \delta\boldsymbol{\omega}^r \times X_\alpha \mathbf{E}_\alpha, \tag{55}$$

where $\delta\boldsymbol{\gamma}^r$ is the virtual variation of the shear-axial strain vector $\boldsymbol{\gamma}^r$ and $\delta\boldsymbol{\omega}^r$ is the virtual variation of the torsional-bending strain vector $\boldsymbol{\omega}^r$. Then, depending $\mathbf{A}^r$ and $\mathbf{E}$ only on $\mathbf{a}^r$, their virtual variations can be simply written as

$$\delta\mathbf{A}^r = \delta(\mathbf{I} + \mathbf{a}^r \otimes \mathbf{E}_3) = \delta\mathbf{a}^r \otimes \mathbf{E}_3, \tag{56}$$

$$\delta\mathbf{E} = \delta((\mathbf{a}^r \otimes \mathbf{E}_3)^s + \frac{1}{2}(\mathbf{a}^r \cdot \mathbf{a}^r)\mathbf{E}_3 \otimes \mathbf{E}_3) = (\delta\mathbf{a}^r \otimes \mathbf{E}_3)^s + (\delta\mathbf{a}^r \cdot \mathbf{a}^r)\mathbf{E}_3 \otimes \mathbf{E}_3. \tag{57}$$

In order to compute the deformation gradient virtual variation, $\delta\mathbf{F}$, we first recall that the virtual variation of the rotation tensor, $\delta\boldsymbol{\Lambda}$, can be expressed through the product composition of the rotation $\boldsymbol{\Lambda}$ followed by a skew-symmetric tensor $\mathbf{W}$, i.e.

$$\delta\boldsymbol{\Lambda} = \mathbf{W}\boldsymbol{\Lambda}. \tag{58}$$

Moreover, we introduce the co-rotational or Lie variation for a current beam measure, defined as

$$\delta_{\boldsymbol{\Lambda}}(\cdot) = \boldsymbol{\Lambda}\delta[\boldsymbol{\Lambda}^T(\cdot)], \tag{59}$$

i.e. as the quantity obtained first by a back-rotation of the current beam measure from the current to the reference configuration, followed then by the virtual variation and finally by a forward-rotation to the current configuration (see Cardona and Géradin (1988) for more details). For instance, the co-rotational variation of the current local strain vector $\mathbf{a}$ takes the form

$$\delta_{\boldsymbol{\Lambda}}\mathbf{a} = \boldsymbol{\Lambda}\delta\mathbf{a}^r, \tag{60}$$

since $\mathbf{a}^r = \boldsymbol{\Lambda}^T\mathbf{a}$. Moreover, recalling Eq. (55) and multiplying both sides by $\boldsymbol{\Lambda}$, we can arrange $\delta_{\boldsymbol{\Lambda}}\mathbf{a}$ as

$$\delta_{\boldsymbol{\Lambda}}\mathbf{a} = \delta_{\boldsymbol{\Lambda}}\boldsymbol{\gamma} + \delta_{\boldsymbol{\Lambda}}\boldsymbol{\omega} \times X_\alpha \mathbf{t}_\alpha, \tag{61}$$

where $\delta_{\boldsymbol{\Lambda}}\boldsymbol{\gamma} = \boldsymbol{\Lambda}\delta\boldsymbol{\gamma}^r = \boldsymbol{\Lambda}\delta(\boldsymbol{\Lambda}^T\boldsymbol{\gamma})$ and $\delta_{\boldsymbol{\Lambda}}\boldsymbol{\omega} = \boldsymbol{\Lambda}\delta\boldsymbol{\omega}^r = \boldsymbol{\Lambda}\delta(\boldsymbol{\Lambda}^T\boldsymbol{\omega})$. Finally, with this notation in hand, the virtual variation of the deformation gradient $\mathbf{F} = \boldsymbol{\Lambda}\mathbf{A}^r$ takes the form

$$\delta\mathbf{F} = \delta(\boldsymbol{\Lambda}\mathbf{A}^r) = \delta\boldsymbol{\Lambda}\mathbf{A}^r + \boldsymbol{\Lambda}\delta\mathbf{A}^r = \mathbf{W}\boldsymbol{\Lambda}\mathbf{A}^r + \boldsymbol{\Lambda}(\delta\mathbf{a}^r \otimes \mathbf{E}_3) = \mathbf{W}\mathbf{F} + (\delta_{\boldsymbol{\Lambda}}\mathbf{a}) \otimes \mathbf{E}_3. \tag{62}$$

3. Beam internal virtual works

The goal of this section is to derive possible beam internal virtual work expressions starting from the three-dimensional finite-elasticity theory. We recall that for a three-dimensional body the internal virtual work can be expressed in the finite-elasticity context with two different relations (see for example Hjelmstad (1997) and Holzapfel (2000)) as

$$\delta W^P_{\text{int}} = \int_{\Omega_0} \mathbf{P} : \delta\mathbf{F} \, \mathrm{d}v_0, \tag{63}$$

$$\delta W^S_{\text{int}} = \int_{\Omega_0} \mathbf{S} : \delta\mathbf{E} \, \mathrm{d}v_0, \tag{64}$$

that is, as the integral of the first Piola–Kirchhoff stress tensor $\mathbf{P}$ double contracted with the deformation gradient virtual variation $\delta\mathbf{F}$, or alternatively as the integral of the second Piola–Kirchhoff stress tensor $\mathbf{S}$ double contracted with the Green–Lagrange strain tensor virtual variation $\delta\mathbf{E}$. We call δW^P_{int} the current internal virtual work and δW^S_{int} the reference internal virtual work, noting that in both cases the integrals are evaluated over the reference body volume Ω_0.

Considering first δW^P_{int}, we can substitute Eq. (62) for $\delta\mathbf{F}$, obtaining

$$\begin{aligned}\delta W^P_{\text{int}} &= \int_{\Omega_0} [\mathbf{P} : (\mathbf{W}\mathbf{F} + \delta_{\boldsymbol{\Lambda}}\mathbf{a} \otimes \mathbf{E}_3)] \, \mathrm{d}v_0 = \int_{\Omega_0} [\mathbf{P} : (\mathbf{W}\mathbf{F})] \, \mathrm{d}v_0 + \int_{\Omega_0} [\mathbf{P} : (\delta_{\boldsymbol{\Lambda}}\mathbf{a} \otimes \mathbf{E}_3)] \, \mathrm{d}v_0 \\ &= \int_{\Omega_0} [(\mathbf{P}\mathbf{F}^T) : \mathbf{W}] \, \mathrm{d}v_0 + \int_{\Omega_0} [(\mathbf{P}\mathbf{E}_3) : \delta_{\boldsymbol{\Lambda}}\mathbf{a}] \, \mathrm{d}v_0.\end{aligned} \tag{65}$$

At this level we observe that $(\mathbf{PF}^T):\mathbf{W}=0$ being, from a mathematical point of view, the double contraction of a symmetric tensor with a skew-symmetric one. The disappearance of such a term finds also a simple and clear physical justification. In fact, using Eq. (62), the vanishing term can be expressed as $(\mathbf{PF}^T):\mathbf{W}=\mathbf{P}:(\delta\mathbf{\Lambda}\mathbf{A}^r)$ and, thus, it can be interpreted as the internal virtual work produced by $\mathbf{P}$ through the virtual variation of the cross-section rotation, $\delta\mathbf{\Lambda}\mathbf{A}^r$. Being the cross-section rotation a rigid rotation, its virtual variation cannot produce any virtual internal work. Using this observation, Eq. (65) can be simplified as

$$\delta W^P_{\mathrm{int}}=\int_{\Omega_0}[(\mathbf{PE}_3):\delta_\Lambda\mathbf{a}]\,\mathrm{d}v_0=\int_{\Omega_0}[\mathbf{p}_3\cdot\delta_\Lambda\mathbf{a}]\,\mathrm{d}v_0, \tag{66}$$

where $\mathbf{p}_3=\mathbf{PE}_3$ is called *current local traction vector*, since it is the stress vector locally acting in the current configuration on a cross-section of normal $\mathbf{E}_3$ in the reference configuration. Recalling expression (61) for $\delta_\Lambda\mathbf{a}$, the previous equation becomes

$$\delta W^P_{\mathrm{int}}=\int_{\Omega_0}[\mathbf{p}_3\cdot(\delta_\Lambda\gamma+\delta_\Lambda\boldsymbol{\omega}\times X_\alpha\mathbf{t}_\alpha)]\,\mathrm{d}v_0=\int_{\Omega_0}[\mathbf{p}_3\cdot\delta_\Lambda\gamma+(X_\alpha\mathbf{t}_\alpha\times\mathbf{p}_3)\cdot\delta_\Lambda\boldsymbol{\omega}]\,\mathrm{d}v_0, \tag{67}$$

where $X_\alpha\mathbf{t}_\alpha\times\mathbf{p}_3$ is the *current local moment vector*, i.e. the cross product of the current local traction vector $\mathbf{p}_3$ with the current cross-section position vector $X_\alpha\mathbf{t}_\alpha$. Noting that $\delta_\Lambda\gamma$ and $\delta_\Lambda\boldsymbol{\omega}$ do not depend on the cross-section coordinates X_α, we can split the volume integral as follows

$$\delta W^P_{\mathrm{int}}=\int_{L_0}[\delta_\Lambda\gamma\cdot\int_{A_0}\mathbf{p}_3\,\mathrm{d}a_0+\delta_\Lambda\boldsymbol{\omega}\cdot\int_{A_0}(X_\alpha\mathbf{t}_\alpha\times\mathbf{p}_3)\,\mathrm{d}a_0]\,\mathrm{d}l_0, \tag{68}$$

where L_0 is the reference axis length and A_0 is the reference cross-section area. We immediately recognize the surface integral of $\mathbf{p}_3$ as the *current traction resultant* and the surface integral of $X_\alpha\mathbf{t}_\alpha\times\mathbf{p}_3$ as the *current moment resultant*; accordingly, introducing the notation

$$\mathbf{f}=\int_{A_0}\mathbf{p}_3\,\mathrm{d}a_0,\quad \mathbf{m}=\int_{A_0}X_\alpha\mathbf{t}_\alpha\times\mathbf{p}_3\,\mathrm{d}a_0, \tag{69}$$

we get the *beam current internal virtual work* as

$$\delta W^P_{\mathrm{int}}=\int_{L_0}[\mathbf{f}\cdot\delta_\Lambda\gamma+\mathbf{m}\cdot\delta_\Lambda\boldsymbol{\omega}]\,\mathrm{d}l_0. \tag{70}$$

Considering now $\delta W^S_{\mathrm{int}}$, using Eq. (57) for $\delta\mathbf{E}$ and exploiting the symmetry of $\mathbf{S}$, we obtain

$$\begin{aligned}\delta W^S_{\mathrm{int}}&=\int_{\Omega_0}[\mathbf{S}:((\delta\mathbf{a}^r\otimes\mathbf{E}_3)^s+(\delta\mathbf{a}^r\cdot\mathbf{a}^r)\mathbf{E}_3\otimes\mathbf{E}_3)]\,\mathrm{d}v_0=\int_{\Omega_0}[\mathbf{S}:(\delta\mathbf{a}^r\otimes\mathbf{E}_3)^s+\mathbf{S}:(\delta\mathbf{a}^r\cdot\mathbf{a}^r)\mathbf{E}_3\otimes\mathbf{E}_3]\,\mathrm{d}v_0\\&=\int_{\Omega_0}[\mathbf{s}_3\cdot\delta\mathbf{a}^r+\mathbf{s}_3\cdot(\delta\mathbf{a}^r\cdot\mathbf{a}^r)\mathbf{E}_3]\,\mathrm{d}v_0,\end{aligned} \tag{71}$$

where $\mathbf{s}_3=\mathbf{SE}_3$ is called *reference local traction vector*, since it is the stress vector locally acting in the reference configuration on a cross-section of normal $\mathbf{E}_3$ in the reference configuration. Collecting $\delta\mathbf{a}^r$, we can rearrange the previous equation in the form

$$\delta W^S_{\mathrm{int}}=\int_{\Omega_0}[((\mathbf{I}+\mathbf{a}^r\otimes\mathbf{E}_3)\mathbf{s}_3)\cdot\delta\mathbf{a}^r]\,\mathrm{d}v_0=\int_{\Omega_0}[(\mathbf{A}^r\mathbf{s}_3)\cdot\delta\mathbf{a}^r]\,\mathrm{d}v_0, \tag{72}$$

where we identify $\mathbf{I}+\mathbf{a}^r\otimes\mathbf{E}_3=\mathbf{A}^r$. Then, we can substitute expression (55) for $\delta\mathbf{a}^r$, that is

$$\delta W^S_{\mathrm{int}}=\int_{\Omega_0}[(\mathbf{A}^r\mathbf{s}_3)\cdot(\delta\gamma^r+\delta\omega^r\times X_\alpha\mathbf{E}_\alpha)]\,\mathrm{d}v_0=\int_{\Omega_0}[(\mathbf{A}^r\mathbf{s}_3)\cdot\delta\gamma^r+(X_\alpha\mathbf{E}_\alpha\times(\mathbf{A}^r\mathbf{s}_3))\cdot\delta\omega^r]\,\mathrm{d}v_0, \tag{73}$$

and, observing that $\delta\gamma^r$ and $\delta\omega^r$ do not depend on the cross-section coordinates X_α, we can split the volume integral as

$$\delta W^S_{\mathrm{int}}=\int_{L_0}[\delta\gamma^r\cdot\int_{A_0}(\mathbf{A}^r\mathbf{s}_3)da_0+\delta\omega^r\cdot\int_{A_0}(X_\alpha\mathbf{E}_\alpha\times(\mathbf{A}^r\mathbf{s}_3))\,\mathrm{d}a_0]\,\mathrm{d}l_0. \tag{74}$$

It is interesting to observe that it is not immediate to give a physical meaning to the surface integrals

$$\mathbf{f}^r=\int_{A_0}\mathbf{A}^r\mathbf{s}_3da_0,\quad \mathbf{m}^r=\int_{A_0}X_\alpha\mathbf{E}_\alpha\times(\mathbf{A}^r\mathbf{s}_3)\,\mathrm{d}a_0, \tag{75}$$

as previously done for $\mathbf{f}$ and $\mathbf{m}$ in Eq. (69). However, we note that $\mathbf{s}_3$ can be related to $\mathbf{p}_3$ through the equation

$$\mathbf{p}_3=\mathbf{Fs}_3=\mathbf{\Lambda}\mathbf{A}^r\mathbf{s}_3, \tag{76}$$

which follows from $\mathbf{P}=\mathbf{FS}$. Accordingly, we have that $\mathbf{A}^r\mathbf{s}_3=\mathbf{\Lambda}^T\mathbf{p}_3$ and, thus, $\mathbf{f}^r$ and $\mathbf{m}^r$ can be given as

$$\mathbf{f}^r = \int_{A_0} \mathbf{\Lambda}^T \mathbf{p}_3 = \mathbf{\Lambda}^T \int_{A_0} \mathbf{p}_3 = \mathbf{\Lambda}^T \mathbf{f}, \tag{77}$$

$$\mathbf{m}^r = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times (\mathbf{\Lambda}^T \mathbf{p}_3)\ \mathrm{d}a_0 = \int_{A_0} \mathbf{\Lambda}^T (X_\alpha \mathbf{t}_\alpha \times \mathbf{p}_3)\ \mathrm{d}a_0 = \mathbf{\Lambda}^T \int_{A_0} (X_\alpha \mathbf{t}_\alpha \times \mathbf{p}_3)\ \mathrm{d}a_0 = \mathbf{\Lambda}^T \mathbf{m}. \tag{78}$$

Hence, $\mathbf{f}^r$ can be interpreted as the rotate-back form of the current traction resultant $\mathbf{f}$, and thus it is called *reference traction resultant*, while $\mathbf{m}^r$ can be interpreted as the rotate-back form of the current moment resultant $\mathbf{m}$, and thus it is called *reference moment resultant*.

It is also interesting to observe that $\mathbf{f}^r$ and $\mathbf{m}^r$ are composed of two parts, one depending only on the traction vector $\mathbf{s}_3$ and one depending also on the strain vector $\mathbf{a}^r$, as it can be seen making the expression of $\mathbf{A}^r$ explicit in Eqs. $(75)_1$ and $(75)_2$, i.e.

$$\mathbf{f}^r = \int_{A_0} (\mathbf{I} + \mathbf{a}^r \otimes \mathbf{E}_3)\mathbf{s}_3\ \mathrm{d}a_0 = \int_{A_0} \mathbf{s}_3\ \mathrm{d}a_0 + \int_{A_0} (\mathbf{a}^r \otimes \mathbf{E}_3)\mathbf{s}_3\ \mathrm{d}a_0, \tag{79}$$

$$\mathbf{m}^r = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times ((\mathbf{I} + \mathbf{a}^r \otimes \mathbf{E}_3)\mathbf{s}_3)\ \mathrm{d}a_0 = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{s}_3\ \mathrm{d}a_0 + \int_{A_0} X_\alpha \mathbf{E}_\alpha \times ((\mathbf{a}^r \otimes \mathbf{E}_3)\mathbf{s}_3)\ \mathrm{d}a_0. \tag{80}$$

The primary reason for such a rather unusual feature is the quadratic dependence of $\mathbf{E}$ on the strain vector $\mathbf{a}^r$. Consequently, $\delta\mathbf{E}$ and $\delta\mathbf{W}^S_{\mathrm{int}}$ depend not only on the linearization of the strain vector, but also on the strain vector itself and, therefore, $\mathbf{a}^r$ appears in the definition of the reference resultants. We may observe that, on the other hand, the current resultants $\mathbf{f}$ and $\mathbf{m}$ are functions only of $\mathbf{p}_3$, being $\mathbf{F}$ linearly dependent on the strain vector $\mathbf{a}$ and then being $\delta\mathbf{F}$ and $\delta\mathbf{W}^P_{\mathrm{int}}$ dependent only on its linearization.

According to position (75), from Eq. (74) we finally obtain the *beam reference internal virtual work* as

$$\delta W^S_{\mathrm{int}} = \int_{L_0} [\mathbf{f}^r \cdot \delta\gamma^r + \mathbf{m}^r \cdot \delta\omega^r]\ \mathrm{d}l_0. \tag{81}$$

We highlight that the derived beam internal virtual work expressions (70) and (81) are fully consistent with the beam kinematic hypotheses and the finite-elasticity theory. Hence, they can be called *finite-deformation finite-strain* beam internal virtual work expressions. In the literature, expression (70) is usually called *spatial* beam internal virtual work, while expression (81) is called *material* beam internal virtual work.

Remark 3.1. As it naturally should be, the reference and current beam internal virtual works are equivalent. In fact, using the invariance of the scalar product under a rotation, we have

$$\delta W^S_{\mathrm{int}} = \int_{L_0} [\mathbf{f}^r \cdot \delta\gamma^r + \mathbf{m}^r \cdot \delta\omega^r]\ \mathrm{d}l_0 = \int_{L_0} [(\mathbf{\Lambda}\mathbf{f}^r) \cdot (\mathbf{\Lambda}\delta\gamma^r) + (\mathbf{\Lambda}\mathbf{m}^r) \cdot (\mathbf{\Lambda}\delta\omega^r)]\ \mathrm{d}l_0 = \int_{L_0} [\mathbf{f} \cdot \delta_{\mathbf{\Lambda}}\gamma + \mathbf{m} \cdot \delta_{\mathbf{\Lambda}}\omega] dl_0 = \delta W^P_{\mathrm{int}}, \tag{82}$$

where we recognize the current resultants, $\mathbf{f}$ and $\mathbf{m}$, and the strain co-rotational variations, $\delta_{\mathbf{\Lambda}}\gamma$ and $\delta_{\mathbf{\Lambda}}\omega$. This result is consistent with the three-dimensional finite-elasticity relation $\mathbf{P} : \delta\mathbf{F} = \mathbf{S} : \delta\mathbf{E}$.

Remark 3.2. We note that the reference beam internal virtual work (81) is, here, obtained as the natural result of the reference three-dimensional internal virtual work (64); relations between reference and current resultants are picked out just a posteriori. Instead, other authors, as for instance Simo and Vu-Quoc (1986), Cardona and Géradin (1988) and Ibrahimbegović et al. (1995a), define a priori the reference resultants in term of the current ones and then obtain the reference internal virtual work (81) from the current one and not from a three-dimensional finite-elasticity expression.

Remark 3.3. We observe that the stress components $P_{\alpha\beta} = \mathbf{t}_\alpha \cdot \mathbf{P}\mathbf{E}_\beta$ and $S_{\alpha\beta} = \mathbf{E}_\alpha \cdot \mathbf{S}\mathbf{E}_\beta$ do not appear in the beam internal virtual works. This fact is not directly imposed, but it naturally derives from the tensor structure of $\mathbf{A}$ and $\mathbf{E}$, i.e. from the kinematic hypotheses.

4. Beam internal virtual work for the small-strain linear elastic case

In this section, we focus on a *finite-deformation small-strain* regime and we derive the form of the beam internal virtual work as well as an elastic constitutive relation consistent with such an assumption. Hence, as mentioned in Section 2.3, we approximate the Green–Lagrange tensor $\mathbf{E}$ neglecting its quadratic strain part, i.e.

$$\mathbf{E} = (\mathbf{a}^r \otimes \mathbf{E}_3)^s + \frac{1}{2}(\mathbf{a}^r \cdot \mathbf{a}^r)\mathbf{E}_3 \otimes \mathbf{E}_3 \approx (\mathbf{a}^r \otimes \mathbf{E}_3)^s. \tag{83}$$

It is important to observe that we focus on $\mathbf{E}$, and hence on $\delta W^S_{\mathrm{int}}$, because $\mathbf{E}$ is a quadratic strain measure and, moreover, we are able to clearly individuate its linear and quadratic part. Accordingly, we can approximate $\mathbf{E}$ consistently with a small-strain regime.

If we introduce the kinematic approximation within the three-dimensional reference internal virtual work $\delta W^S_{\mathrm{int}}$ and repeat all the derivations, we obtain the beam internal virtual work as

$$\delta W^{S,\text{lin}}_{\text{int}} = \int_{L_0} [\mathbf{f}^{r,\text{lin}} \cdot \delta\gamma^r + \mathbf{m}^{r,\text{lin}} \cdot \delta\boldsymbol{\omega}^r]\,\mathrm{d}l_0, \tag{84}$$

where

$$\mathbf{f}^{r,\text{lin}} = \int_{A_0} \mathbf{s}_3\ \mathrm{d}a_0, \quad \mathbf{m}^{r,\text{lin}} = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{s}_3\ \mathrm{d}a_0. \tag{85}$$

The traction and moment resultants $\mathbf{f}^{r,\text{lin}}$ and $\mathbf{m}^{r,\text{lin}}$ are the components of $\mathbf{f}^r$ and $\mathbf{m}^r$ that do not depend on the strain, as it can be seen comparing expressions $(85)_1$ and $(85)_2$ with expressions (79) and (80). This definition for $\mathbf{f}^{r,\text{lin}}$ and $\mathbf{m}^{r,\text{lin}}$ follows directly from the small-strain approximation and, in our opinion, it is the distinguishing key between the small-strain and the finite-strain beam cases. In fact, $\mathbf{f}^{r,\text{lin}}$ and $\mathbf{m}^{r,\text{lin}}$ have exactly the same mathematical structure of the resultants of the geometrically linear beam theory, as it can be seen for example in Hjelmstad (1997). This is the reason why we call $\mathbf{f}^{r,\text{lin}}$ *linear traction resultant* and $\mathbf{m}^{r,\text{lin}}$ *linear moment resultant*.

Focusing our attention now on a plausible elastic isotropic constitutive relation for this small-strain model, in analogy with the linear beam theory we consider a linear relation between the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the approximated Green–Lagrange tensor $\mathbf{E}^{\text{lin}} = (\mathbf{a}^r \otimes \mathbf{E}_3)^s$, as

$$\mathbf{S} = \lambda \text{tr}(\mathbf{E}^{\text{lin}})\mathbf{I} + 2\mu\mathbf{E}^{\text{lin}}, \tag{86}$$

where λ and μ are the Lamé constants and $\text{tr}(\cdot)$ is the trace operator. Using this constitutive equation, we express the local reference traction vector $\mathbf{s}_3 = \mathbf{S}\mathbf{E}_3$ in the form

$$\mathbf{s}_3 = (\lambda\text{tr}[(\mathbf{a}^r \otimes \mathbf{E}_3)^s]\mathbf{I} + 2\mu(\mathbf{a}^r \otimes \mathbf{E}_3)^s)\mathbf{E}_3 = \lambda\text{tr}[(\mathbf{a}^r \otimes \mathbf{E}_3)^s]\mathbf{E}_3 + 2\mu(\mathbf{a}^r \otimes \mathbf{E}_3)^s\mathbf{E}_3 = [(\lambda+\mu)\mathbf{E}_3 \otimes \mathbf{E}_3 + \mu\mathbf{I}]\mathbf{a}^r = \mathbf{D}\mathbf{a}^r, \tag{87}$$

where $\mathbf{D} = (\lambda+\mu)\mathbf{E}_3 \otimes \mathbf{E}_3 + \mu\mathbf{I}$. We note that the previous equation linearly relates $\mathbf{s}_3$ to the reference local strain vector $\mathbf{a}^r$, through a second order tensor, $\mathbf{D}$, which is dependent only on material parameters. Introducing $\mathbf{a}^r = \gamma^r + \boldsymbol{\omega}^r \times \mathbf{r}^r$ in Eq. (87), we can rearrange $\mathbf{s}_3$ as

$$\mathbf{s}_3 = \mathbf{D}(\gamma^r + \boldsymbol{\omega}^r \times \mathbf{r}^r) = \mathbf{D}\gamma^r + \mathbf{D}(\boldsymbol{\omega}^r \times \mathbf{r}^r). \tag{88}$$

Considering now expression $(85)_1$ for $\mathbf{f}^{r,\text{lin}}$ and using the previous equation, we obtain

$$\mathbf{f}^{r,\text{lin}} = \int_{A_0} \mathbf{D}\gamma^r\ \mathrm{d}a_0 + \int_{A_0} \mathbf{D}(\boldsymbol{\omega}^r \times \mathbf{r}^r)\ \mathrm{d}a_0. \tag{89}$$

Recalling that $\mathbf{r}^r = X_\alpha \mathbf{E}_\alpha$, we observe that the second term on the right-hand-side vanishes because it is linear in the cross-section coordinates X_α and it is integrated in a centroidal reference system; hence, the integration of the linear traction resultant yields the beam constitutive relation

$$\mathbf{f}^{r,\text{lin}} = \mathbf{D}_f\gamma^r, \tag{90}$$

where (recalling that A_0 is the reference cross-section area)

$$\mathbf{D}_f = A_0\mathbf{D} = A_0[(\lambda+\mu)\mathbf{E}_3 \otimes \mathbf{E}_3 + \mu\mathbf{I}], \tag{91}$$

or, in matrix form,

$$[\mathbf{D}_f] = \begin{bmatrix} \mu A_0 & 0 & 0 \\ 0 & \mu A_0 & 0 \\ 0 & 0 & (\lambda+2\mu)A_0 \end{bmatrix}. \tag{92}$$

Considering then the linear moment resultant $\mathbf{m}^{r,\text{lin}}$, it is useful to introduce the skew-symmetric tensor $\mathbf{R}^r = -(\mathbf{R}^r)^T$ defined such that

$$\mathbf{R}^r\mathbf{q} = \mathbf{r}^r \times \mathbf{q}\ \forall \mathbf{q} \in \mathscr{R}^3; \tag{93}$$

accordingly, $\mathbf{R}^r$ can be expressed in the material frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ through the matrix form

$$[\mathbf{R}^r] = \begin{bmatrix} 0 & 0 & X_2 \\ 0 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{bmatrix}. \tag{94}$$

Using $\mathbf{R}^r$, we can rearrange expression $(85)_2$ for the linear moment resultant as

$$\mathbf{m}^{r,\text{lin}} = \int_{A_0} \mathbf{R}^r\mathbf{s}_3\ \mathrm{d}a_0, \tag{95}$$

and, substituting expression (88) for $\mathbf{s}_3$ in the previous equation, we obtain

$$\mathbf{m}^{r,\text{lins}} = \int_{A_0} \mathbf{R}^r\mathbf{D}\gamma^r\ \mathrm{d}a_0 + \int_{A_0} \mathbf{R}^r\mathbf{D}(\boldsymbol{\omega}^r \times \mathbf{r}^r)\ \mathrm{d}a_0. \tag{96}$$

The first integral on the right-hand-side is linear in the cross-section coordinates X_α through $\mathbf{R}^r$ and hence vanishes in the integration; on the other side, using $\boldsymbol{\omega}^r \times \mathbf{r}^r = (\mathbf{R}^r)^T \boldsymbol{\omega}^r$ and substituting the expression for $\mathbf{D}$, the second integral can be rearranged such that

$$\mathbf{m}^{r,\text{lin}} = \int_{A_0} \mathbf{R}^r[(\lambda+\mu)\mathbf{E}_3 \otimes \mathbf{E}_3 + \mu\mathbf{I}](\mathbf{R}^r)^T\boldsymbol{\omega}^r \,\mathrm{d}a_0 = \int_{A_0}[(\lambda+\mu)\mathbf{R}^r\mathbf{E}_3 \otimes \mathbf{R}^r\mathbf{E}_3 + \mu\mathbf{R}^r(\mathbf{R}^r)^T]\boldsymbol{\omega}^r \,\mathrm{d}a_0. \tag{97}$$

Using relation (94), the tensors in the previous equation can be computed in matrix form as

$$[\mathbf{R}^r\mathbf{E}_3 \otimes \mathbf{R}^r\mathbf{E}_3] = \begin{bmatrix} X_2^2 & -X_1X_2 & 0 \\ -X_1X_2 & X_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{98}$$

and

$$[\mathbf{R}^r(\mathbf{R}^r)^T] = \begin{bmatrix} X_2^2 & -X_1X_2 & 0 \\ -X_1X_2 & X_1^2 & 0 \\ 0 & 0 & X_1^2+X_2^2 \end{bmatrix}. \tag{99}$$

Therefore, the linear moment resultant can be easily integrated as

$$\mathbf{m}^{r,\text{lin}} = \mathbf{D}_m\boldsymbol{\omega}^r, \tag{100}$$

where $\mathbf{D}_m$ is

$$[\mathbf{D}_m] = \begin{bmatrix} (\lambda+2\mu)J_{22} & -(\lambda+2\mu)J_{12} & 0 \\ -(\lambda+2\mu)J_{21} & (\lambda+2\mu)J_{11} & 0 \\ 0 & 0 & \mu J_p \end{bmatrix}, \tag{101}$$

with

$$J_{\alpha\beta} = \int_{A_0} X_\alpha X_\beta \,\mathrm{d}a_0,\; J_p = \int_{A_0} X_\alpha^2 \,\mathrm{d}a_0. \tag{102}$$

We have now to remark that, due to the rigid cross-section kinematical hypothesis and to the fact that the model is derived starting from a virtual work principle, the obtained beam constitutive equations have some limitations, exactly as pointed out, for instance, by Hjelmstad (1997) in the framework of the linear beam theory. In particular, the obtained constitutive relations are correct only in the limit $\nu \to 0$ (i.e. no lateral contraction) and shear equilibrium inconsistencies, as well as lack of torsional warping effects, do appear. Therefore, as it is done by many authors in the context of both linear and nonlinear beam theories, it is necessary to perform some ad hoc substitutions in Eqs. (92) and (101), i.e.

$$\lambda + 2\mu \rightsquigarrow E, \quad \mu A_0 \rightsquigarrow \mu k_1 A_0 (\text{or} \quad \mu k_2 A_0), \quad J_p \rightsquigarrow J_{SV},$$

where E is the Young modulus, k_1 and k_2 are the shear correction factors (along directions 1 and 2, respectively), and J_{SV} is the classical Saint-Venant torsion constant. The obtained constitutive equations are exactly those postulated by Simo and Vu-Quoc (1986) and then justified in a rather more complex way by Simo and Vu-Quoc (1991) and recently by Mäkinen (2007).

We finally obtain the final expression of the *finite-deformation small-strain beam internal virtual work* substituting constitutive Eqs. (91) and (100) into expression (84) as

$$\delta W_{\text{int}}^{S,\text{lin}} = \int_{L_0} [\mathbf{D}_f\boldsymbol{\gamma}^r \cdot \delta\boldsymbol{\gamma}^r + \mathbf{D}_m\boldsymbol{\omega}^r \cdot \delta\boldsymbol{\omega}^r] \,\mathrm{d}l_0. \tag{103}$$

Consistently with the small-strain hypothesis and with the linear elastic constitutive hypothesis, axial, shear, bending and torsional strains turn out to be all uncoupled.

Eq. (103) has been widely used in the literature to develop finite-deformation models, as for instance by Simo and Vu-Quoc (1991), Ibrahimbegović et al. (1995a), Ibrahimbegović and Taylor (2002). Anyway, we believe that our approach is valuable because it clearly shows the origin of the small-strain hypothesis, not yet clarified in the literature as far as we know.

5. Conclusions

In this paper, we first analyze the kinematics and the internal virtual work of a finite-deformation finite-strain beam model, first developed by Simo (1985). In particular, we introduce a left and a right extended polar decompositions of the deformation gradient which are useful for a clear interpretation of the beam kinematics. We then write the internal virtual work consistent with the beam kinematics exploiting some three-dimensional internal virtual work expressions; the computation is compact and direct thanks to the introduced deformation gradient decompositions, and we also highlight some interesting features about the resultants. Finally, through an intuitive and effective approximation, we reduce the model to the finite-deformation small-strain case and we individuate a linear elastic constitutive relation suitable for this regime.

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Appendix A. Beam rigid motion

The aim of this appendix is to show that $\mathbf{a} = \mathbf{0}$ and $\mathbf{A} = \mathbf{I}$ for a beam rigid motion. Indicating with $\widehat{\mathbf{\Lambda}}$ the cross-section rotation tensor, constant along the beam axis, and with $\hat{\mathbf{u}}_0$ the axis displacement field, also constant along X_3, the deformation map associated with a beam rigid motion can be written as

$$\hat{\mathbf{x}} = \widehat{\mathbf{\Lambda}}(X_3\mathbf{E}_3 + \hat{\mathbf{u}}_0 + X_\alpha \mathbf{E}_\alpha), \tag{A.1}$$

where we decompose the motion into a rigid translation followed by a rigid rotation, as it can be seen in Fig. A.1. We can rewrite the previous equation also in the form

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0 + X_\alpha \widehat{\mathbf{\Lambda}}\mathbf{E}_\alpha \quad \text{with } \hat{\mathbf{x}}_0 = \widehat{\mathbf{\Lambda}}(X_3\mathbf{E}_3 + \hat{\mathbf{u}}_0), \tag{A.2}$$

where we recognize $\hat{\mathbf{x}}_0$ as the position vector of the beam axis in the current configuration. Since the derivatives with respect to X_3 of $\widehat{\mathbf{\Lambda}}$ and $\hat{\mathbf{u}}_0$ are equal to zero, i.e.

$$\widehat{\mathbf{\Lambda}}_{,3} = \mathbf{0}, \quad (\hat{\mathbf{u}}_0)_{,3} = \mathbf{0}, \tag{A.3}$$

the derivative of $\hat{\mathbf{x}}_0$ with respect to X_3 follows as

$$\hat{\mathbf{x}}_{0,3} = (\widehat{\mathbf{\Lambda}}(X_3\mathbf{E}_3 + \hat{\mathbf{u}}_0))_{,3} = \widehat{\mathbf{\Lambda}}_{,3}(X_3\mathbf{E}_3 + \hat{\mathbf{u}}_0) + \widehat{\mathbf{\Lambda}}(\mathbf{E}_3 + (\hat{\mathbf{u}}_0)_{,3}) = \widehat{\mathbf{\Lambda}}\mathbf{E}_3. \tag{A.4}$$

Computing now $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}_\alpha$ for the rigid motion, i.e. $\widehat{\boldsymbol{\gamma}} = \hat{\mathbf{x}}_{0,3} - \hat{\mathbf{t}}_3$ and $\widehat{\boldsymbol{\kappa}}_\alpha = \hat{\mathbf{t}}_{\alpha,3}$, we obtain

$$\widehat{\boldsymbol{\gamma}} = \widehat{\mathbf{\Lambda}}\mathbf{E}_3 - \widehat{\mathbf{\Lambda}}\mathbf{E}_3 = \mathbf{0}, \tag{A.5}$$

$$\widehat{\boldsymbol{\kappa}}_\alpha = (\widehat{\mathbf{\Lambda}}\mathbf{E}_\alpha)_{,3} = \widehat{\mathbf{\Lambda}}_{,3}\mathbf{E}_\alpha = \mathbf{0}. \tag{A.6}$$

Accordingly, we conclude that, for a beam rigid motion, we have

$$\hat{\mathbf{a}} = \widehat{\boldsymbol{\gamma}} + X_\alpha \widehat{\boldsymbol{\kappa}}_\alpha = \mathbf{0}, \tag{A.7}$$

$$\hat{\mathbf{L}} = \hat{\mathbf{a}} \otimes \mathbf{t}_3 = \mathbf{0}, \tag{A.8}$$

$$\hat{\mathbf{A}} = \mathbf{I} + \hat{\mathbf{L}} = \mathbf{I}. \tag{A.9}$$

Appendix B. Matrix expressions of beam stretch and strain tensors

The aim of this appendix is to provide the matrix expression for some beam stretch and strain tensors. First of all, it is interesting to observe that $\mathbf{A}, \mathbf{L}$ and $\mathbf{e}$ have extremely simple matrix expressions in the moving frame $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$. In fact, considering Eqs. (17), (19) and (42), we can compute the components $A_{ij} = \mathbf{t}_i \cdot \mathbf{A}\mathbf{t}_j, L_{ij} = \mathbf{t}_i \cdot \mathbf{L}\mathbf{t}_j$ and $e_{ij} = \mathbf{t}_i \cdot \mathbf{e}\mathbf{t}_j$, obtaining

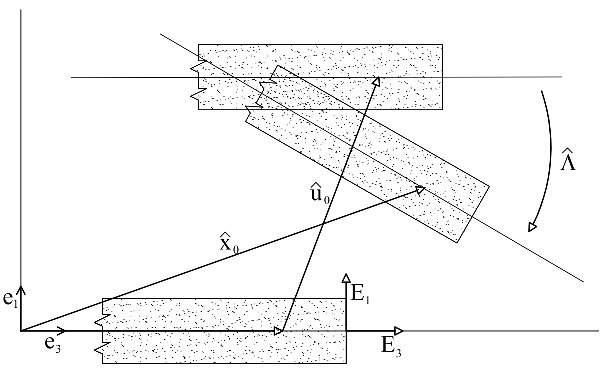

Fig. A.1. Bi-dimensional representation of a beam rigid motion.

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1+a_3 \end{bmatrix}, \quad [\mathbf{L}] = \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix}, \quad [\mathbf{e}] = \frac{1}{2J}\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ a_1 & a_2 & 2a_3 - a^2/J \end{bmatrix}, \tag{B.1}$$

where $a^2 = \|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$. Similarly, $\mathbf{A}^r, \mathbf{L}^r$ and $\mathbf{E}$ have equivalent matrix expressions in the material frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. In fact, considering Eqs. (36), (37) and (44), we can evaluate $A^r_{ij} = \mathbf{E}_i \cdot \mathbf{A}^r\mathbf{E}_j$, $L^r_{ij} = \mathbf{E}_i \cdot \mathbf{L}^r\mathbf{E}_j$ and $E_{ij} = \mathbf{E}_i \cdot \mathbf{E}\mathbf{E}_j$, obtaining

$$[\mathbf{A}^r] = \begin{bmatrix} 1 & 0 & a^r_1 \\ 0 & 1 & a^r_2 \\ 0 & 0 & 1+a^r_3 \end{bmatrix}, \quad [\mathbf{L}^r] = \begin{bmatrix} 0 & 0 & a^r_1 \\ 0 & 0 & a^r_2 \\ 0 & 0 & a^r_3 \end{bmatrix}, \quad [\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & a^r_1 \\ 0 & 0 & a^r_2 \\ a^r_1 & a^r_2 & 2a^r_3 + (a^r)^2 \end{bmatrix}, \tag{B.2}$$

where $(a^r)^2 = \|\mathbf{a}^r\|^2 = \mathbf{a}^r \cdot \mathbf{a}^r$.

We note that, in the assumed bases, $\mathbf{A}$ and $\mathbf{A}^r$ are upper triangular matrices while $\mathbf{L}$ and $\mathbf{L}^r$ are rank-one matrices. Moreover, we observe that the zero values of $e_{\alpha\beta}$ and $E_{\alpha\beta}$ match with the hypothesis of rigid cross-section. In fact, we can recall from the finite-elasticity theory that, for a three-dimensional body, $\mu = \mathrm{d}\mathbf{X}\cdot\mathbf{E}\mathrm{d}\mathbf{X}$ is the change in squared length of an infinitesimal line element $\mathrm{d}\mathbf{X}$ belonging to the reference configuration, see Holzapfel (2000). In our beam case, we can consider an infinitesimal line element within a generic cross-section in the reference configuration, i.e. $\mathrm{d}\mathbf{X} = \mathrm{d}X_\alpha\mathbf{E}_\alpha$, and, exploiting expression $(\mathrm{B.2})_3$, we can compute that $\mu = 0$. Therefore, being this relation true for any line element within the cross-section, we may conclude that the cross-section remains rigid during the deformation. An equivalent observation is valid for a cross-section line element in the rotated configuration, $\mathrm{d}\boldsymbol{x} = \mathrm{d}X_\alpha\mathbf{t}_\alpha$, when mapped via $\mathbf{e}$.

Appendix C. Representation of the stretch tensors A and A^r through eigenvalues and eigenvectors

The aim of this appendix is to express $\mathbf{A}$ and $\mathbf{A}^r$ through their eigenvalues and eigenvectors.

Recalling that $\mathbf{A}$ is in general non-symmetric, we can write it by the linear combination

$$\mathbf{A} = \lambda_1\mathbf{n}_1 \otimes \mathbf{n}^1 + \lambda_2\mathbf{n}_2 \otimes \mathbf{n}^2 + \lambda_3\mathbf{n}_3 \otimes \mathbf{n}^3, \tag{C.1}$$

where

- $\mathbf{n}_i$ are the linearly independent, generally non-orthogonal, eigenvectors of $\mathbf{A}$;
- λ_i are the eigenvalues associated to $\mathbf{n}_i$;
- $\mathbf{n}^i$ are the elements of the reciprocal basis for $\mathbf{n}_i$, i.e. the basis such that $\mathbf{n}_i \cdot \mathbf{n}^j = \delta_{ij}$.

Since $[\mathbf{A}]$ is an upper triangular matrix, as shown in Eq. $(\mathrm{B.1})_1$, its eigenvalues λ_i coincide with its diagonal values, that is

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 1 + a_3. \tag{C.2}$$

The eigenvectors can then be straightforwardly evaluated solving the systems $(\mathbf{A} - \lambda_k\mathbf{I})\mathbf{n}_k = \mathbf{0}$. In fact, exploiting expression $(\mathrm{B.1})_1$, we have

$$\begin{cases} \mathbf{n}_1 = c_1\mathbf{t}_1 + c_2\mathbf{t}_2 \\ \mathbf{n}_2 = c_3\mathbf{t}_1 + c_4\mathbf{t}_2 \\ \mathbf{n}_3 = \frac{a_1}{a_3}c_5\mathbf{t}_1 + \frac{a_2}{a_3}c_5\mathbf{t}_2 + c_5\mathbf{t}_3 = \frac{c_5}{a_3}a_i\mathbf{t}_i, \end{cases} \tag{C.3}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants. The reciprocal basis is then given by the rows of the matrix $[\mathbf{G}^{-1}]$ where $[\mathbf{G}] = [\mathbf{n}_1; \mathbf{n}_2; \mathbf{n}_3]$, i.e.

$$\begin{cases} \mathbf{n}^1 = \frac{1}{c_1c_4 - c_3c_2}\left(c_4\mathbf{t}_1 - c_3\mathbf{t}_2 - \frac{a_1c_4 - a_2c_3}{a_3}\mathbf{t}_3\right) \\ \mathbf{n}^2 = \frac{1}{c_1c_4 - c_3c_2}\left(c_2\mathbf{t}_1 - c_1\mathbf{t}_2 - \frac{a_1c_2 - a_2c_1}{a_3}\mathbf{t}_3\right) \\ \mathbf{n}^3 = \frac{1}{c_5}\mathbf{t}_3. \end{cases} \tag{C.4}$$

Being c_1, c_2, c_3, c_4, c_5 arbitrary constants, we can chose $c_1 = c_4 = 1, c_2 = c_3 = 0$ and $c_5 = a_3/\|\mathbf{a}\|$, such that $\|\mathbf{n}_1\| = \|\mathbf{n}_2\| = \|\mathbf{n}_3\| = 1$; accordingly, we get

$$\begin{cases} \mathbf{n}_1 = \mathbf{t}_1 \\ \mathbf{n}_2 = \mathbf{t}_2 \\ \mathbf{n}_3 = \frac{1}{\|\mathbf{a}\|}a_i\mathbf{t}_i, \end{cases} \tag{C.5}$$

as well as

$$\begin{cases} \mathbf{n}^1 = \mathbf{t}_1 - \frac{a_1}{a_3}\mathbf{t}_3 \\ \mathbf{n}^2 = \mathbf{t}_2 - \frac{a_2}{a_3}\mathbf{t}_3 \\ \mathbf{n}^3 = \frac{\|\mathbf{a}\|}{a_3}\mathbf{t}_3. \end{cases} \tag{C.6}$$

Substituting now $\mathbf{n}_i, \mathbf{n}^i$ and λ_i into expression (C.1) for $\mathbf{A}$, we recover the definition $\mathbf{A} = \mathbf{I} + \mathbf{a} \otimes \mathbf{t}_3$.

It is interesting to observe that the eigenvalues and the eigenvectors of $\mathbf{A}$ assume a clear physical meaning in the light of the left extended polar decomposition $\mathbf{F} = \mathbf{A}\boldsymbol{\Lambda}$. In fact, specializing to our case the concepts introduced by Boulanger and Hayes (2001) and Jarić et al. (2006) for the finite-elasticity case, we can interpret the basis of eigenvectors $\mathbf{n}_i$ as a *current unsheared triad*, i.e. a triad which maintains invariant its internal angles when the beam passes from the rotated configuration to the final one. Similarly, we can interpret the eigenvalues associated with $\mathbf{n}_i$ as the stretches taking place along the triad directions. Therefore, we have that the three unit vectors $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{a}/\|\mathbf{a}\|$ remain unsheared passing from the rotated to the final configuration. Moreover, $\mathbf{t}_1$ and $\mathbf{t}_2$ remain also unstretched, being associated with unit eigenvalues, and the only beam stretch takes place along the direction $\mathbf{a}/\|\mathbf{a}\|$ of an amount $1 + a_3$. We observe that the deformations of $\mathbf{t}_1$ and $\mathbf{t}_2$ are consistent with the hypothesis of rigid cross-section.

Recalling that a_1 and a_2 account for the local shear and torsion strain in the current configuration, since $a_1 = \gamma_1 + \kappa_1$ and $a_2 = \gamma_2 + \kappa_2$, we note that if $a_1 = a_2 = 0$, i.e. if there is no shear-torsion strain, the basis of the eigenvectors and the reciprocal basis coincide with the moving frame; that is, from Eqs. (C.5) and (C.6) we have that

$$\begin{cases} \mathbf{n}_1 = \mathbf{n}^1 = \mathbf{t}_1 \\ \mathbf{n}_2 = \mathbf{n}^2 = \mathbf{t}_2 \\ \mathbf{n}_3 = \mathbf{n}^3 = \mathbf{t}_3. \end{cases} \tag{C.7}$$

Hence the moving frame remains unsheared passing from the rotated to the final configuration and the local direction of the stretch is $\mathbf{t}_3$. Moreover, in this case the tensor $\mathbf{A}$ is symmetric and, thus, the left extended polar decomposition reduces to the classical one.

Exploiting expression (C.2), it is immediate to evaluate the determinant of $\mathbf{A}$, J, as

$$J = \lambda_1\lambda_2\lambda_3 = 1 + a_3. \tag{C.8}$$

Since the deformation gradient can be expressed as $\mathbf{F} = \mathbf{A}\boldsymbol{\Lambda}$ and since the determinant of a rotation tensor is 1, it follows that the determinant of the deformation gradient is equal to J, too. Recalling that the deformation gradient determinant measures the change of the body volume when the configuration changes, it follows that, under the assumed kinematics, a_1 and a_2 provide an isochoric transformation while only a_3, the axial and bending local strain, gives a volume variation.

All these computations and observations can be performed for $\mathbf{A}^r$ as well, simply evaluating all the quantities in the material frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and using the right extended polar decomposition.

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