Generalized midpoint integration algorithms for $J_2$ plasticity with linear hardening

E. Artioli$^{1,2,*}$, F. Auricchio$^{1,2}$ and L. Beirão da Veiga$^3$

$^1$Dipartimento di Meccanica Strutturale, Università di Pavia, Via Ferrata 1, Pavia 27100, Italy
$^2$Istituto di Matematica Applicata e Tecnologie Informatiche del CNR, Via Ferrata 1, Pavia 27100, Italy
$^3$Dipartimento di Matematica ‘F. Enriques’, Università di Milano, Via Saldini 50, Milano 20133, Italy

SUMMARY

We consider four schemes based on generalized midpoint rule and return map algorithm for the integration of the classical $J_2$ plasticity model with linear hardening. The comparison, aiming to establish which is the preferable scheme among the four considered, is both theoretical and numerical. On one side, extending and completing the existing results in the literature, we investigate the four schemes from the theoretical viewpoint, addressing in particular the existence of solution, long-term behaviour, accuracy and stability. On the other hand, we develop an extensive set of numerical tests, based on pointwise stress–strain loading histories, iso-error maps and initial boundary-value problems. Copyright © 2007 John Wiley & Sons, Ltd.

Received 15 November 2006; Revised 15 January 2007; Accepted 15 January 2007

KEY WORDS: plasticity; generalized midpoint rule; return map; second order method; linear hardening

1. INTRODUCTION

In the present paper we address the analysis and the comparison of four schemes based on generalized midpoint return mapping and herein applied to the classical $J_2$ plasticity model with linear isotropic and kinematic hardening. Generalized midpoint algorithms are among the most cited and successful second order methods in plasticity and a total of four different schemes of this type can be found in the literature [1–3]. Nevertheless, it seems that an exhaustive theoretical and numerical comparison between the four methods is missing.

Other kinds of methods exhibiting second order accuracy may be found as well. Runge–Kutta methods [4], generalized Runge–Kutta methods [5] and multi-step methods [6], for instance, share
this property, even if they are based on a higher level complexity from the mathematical point of view.

The four integration schemes considered in this work are labelled, respectively, as SMPT1 \([1]\), SMPT2 \([2]\), DMPT1 \([3]\) and DMPT2 \([3]\). The first two schemes are based on a traditional generalized midpoint integration rule, combined with a return mapping procedure for the enforcement of the yield consistency condition. The last two methods are instead based on a two-stage algorithm, dividing each time step into two subintervals, in which equations are solved sequentially.

The aim of this paper is to assess, through a complete numerical and theoretical study, which of the aforementioned midpoint methods performs best in terms of reliability and robustness of the discrete solution. From the numerical viewpoint, we develop an extensive comparison based on three different kinds of tests: pointwise mixed stress–strain loading histories, iso-error maps and initial boundary-value problems on a perforated strip. From the theoretical standpoint, we investigate in particular the existence of solution, accuracy, stability and the algorithm behaviour for long time steps. The theoretical study is carried out completing and extending the results already present in the literature.

The paper is organized as follows. In Section 2 we resume briefly the \(J_2\) plasticity model under consideration. In Section 3 we present the four methods, detailing in particular the underlying mathematical structure, the algorithm formulation in terms of integration scheme and solution procedure. In Section 4 we address the mathematical analysis of the methods. In Section 5, we develop and comment an extensive set of numerical tests; we consider also the well-established radial return map method based on backward Euler implicit rule (labelled BE method) \([2,5,7]\) and the recent ESC\(^2\) exponential-based algorithm \([8]\), for comparison’s reasons. Finally, in Appendix A we complete the investigation of various theoretical issues, while Appendix B reports the concise form of the tangent operators of the examined methods.

Remark 1.1

In the present paper we purposefully consider a simple, although quite popular in applications, plasticity model. The reason of this choice is double. First of all, such a choice allows a solid and complete theoretical comparison of the methods. Moreover, the interpretation of the algorithm behaviour is better understood on such a model. The reader should be anyway aware that, in general, the extension of midpoint methods to models with more complicated yield surfaces can encounter additional difficulties, such as computing the return map algorithm solution. A study on the application of a midpoint method to more complex models is presented for instance in \([9]\).

2. TIME CONTINUOUS MODEL

We consider a von-Mises plasticity model with linear isotropic and kinematic hardening within a small deformation regime \([5,7,10]\). Splitting the stress and the strain tensor, \(\mathbf{\sigma}\) and \(\mathbf{\varepsilon}\), in deviatoric and volumetric part we have

\[
\mathbf{\sigma} = \mathbf{s} + p \mathbf{I} \quad \text{with} \quad p = \frac{1}{3} \text{tr}(\mathbf{\sigma})
\]

\[
\mathbf{\varepsilon} = \mathbf{e} + \frac{1}{3} \theta \mathbf{I} \quad \text{with} \quad \theta = \text{tr}(\mathbf{\varepsilon})
\]

where \text{tr} indicates the trace operator, while \(\mathbf{I}, \mathbf{s}, p, \mathbf{e}, \theta\) are, respectively, the second order identity tensor, the deviatoric and volumetric stress, the deviatoric and volumetric strain.
The equations for the model are

\[ \begin{align*}
    p &= K \theta \\
    \mathbf{s} &= 2G(e - e^p) \\
    \Sigma &= \mathbf{s} - \mathbf{\alpha} \\
    F &= \|\Sigma\| - \sigma_y \\
    \dot{e}^p &= \dot{\gamma} \mathbf{n} \\
    \sigma_y &= \sigma_{y,0} + H_{\text{iso}} \dot{\gamma} \\
    \dot{\mathbf{\alpha}} &= H_{\text{kin}} \dot{\gamma} \mathbf{n} \\
    \dot{\gamma} &\geq 0, \quad F \leq 0, \quad \dot{\gamma} F = 0
\end{align*} \]

where \( K \) is the material bulk modulus, \( G \) is the material shear modulus, \( e^p \) is the traceless plastic strain, \( \Sigma \) is the relative stress in terms of the backstress \( \mathbf{\alpha} \), introduced to describe the shifting of the yield surface in deviatoric stress space due to kinematic hardening. Moreover, \( F \) is the von-Mises yield function, \( \mathbf{n} \) is the normal to the yield surface, \( \sigma_y \) is the yield surface radius, \( \sigma_{y,0} \) the initial yield stress, \( H_{\text{kin}} \) and \( H_{\text{iso}} \) are the kinematic and isotropic linear hardening moduli. Finally, Equations (10) are the loading conditions, expressed in the so-called Kuhn–Tucker form, i.e. as constrained optimality conditions [11]. In particular, the second equation limits the relative stress within the admissible convex set bounded by the yield surface \( F = 0 \), while the other two determine the type of loading phase the material is experiencing.

**Remark 2.1**

Due to the linearity of the constitutive equation relating the volumetric part of stress and strain, the numerical schemes treated in the following deal only with the deviatoric part of the model.

3. RETURN MAPPING MIDPOINT INTEGRATION SCHEMES

In this section we present the four methods object of this study, which, for convenience, are grouped distinguishing into single-step and double-step methods. Such a distinction follows the main logical difference between the two sets of schemes. Namely, the single-step algorithms perform one return map procedure within each time-integration interval, while the double-step algorithms subdivide each time-integration interval into two subintervals and perform the return map twice, i.e. once per each subinterval.

The acronyms for the four methods are listed as follows:

- **SMPT1** single-step midpoint method no. 1 (Section 3.1).
- **SMPT2** single-step midpoint method no. 2 (Section 3.1).
- **DMPT2** double-step midpoint method no. 1 (Section 3.2).
- **DMPT2** double-step midpoint method no. 2 (Section 3.2).
In what follows the structure of each algorithm is presented specifying, for each step, the time-integration rule adopted and the solution algorithm utilized to solve the ensuing non-linear algebraic problem. An exception to this layout is made in the case of the DMPT2 method whose second step does not operate through a standard integration rule (see Section 3.2 for further details) but still makes use of a return-map type solution. The consistent tangent operators for the four methods object of discussion may be found in Appendix B.

We consider a loading history assigned over the time interval \([0, T]\), divided into \(N\) subintervals defined by the points \(0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = T\). The generic time-integration interval amplitude is indicated by \(\Delta t = t_{n+1} - t_n\). For simplicity, \(\Delta t\) is assumed constant for each loading history.

Fixing the scalar \(\alpha \in [0, 1]\) it is possible to determine a midpoint instant \(t_{n+\alpha}\), for each subinterval \([t_n, t_{n+1}]\), such that \(t_n \leq t_{n+\alpha} \leq t_{n+1}\) and

\[
\alpha = \frac{t_{n+\alpha} - t_n}{\Delta t}
\]

Given the variables \(\{s_n, e_n, \gamma_n, e^p_n, \alpha_n\}\) at time \(t_n\) and the deviatoric strain \(e_{n+1}\) at time \(t_{n+1}\), the problem consists of constructing a numerical scheme able to update the variables at time \(t_{n+1}\).

**Remark 3.1**

For the sake of generality, in the following theoretical developments the notation for \(\alpha\) is kept as indicated above. Nevertheless, the reader should be aware that throughout the numerical tests (see Section 5), it is assumed \(\alpha = \frac{1}{2}\), which represents the true midpoint method.

### 3.1. Single-step midpoint methods

Both schemes detailed in this section apply a generalized midpoint rule for the integration of the differential equations of the system. The integration is performed once per time interval (single-step methods). The difference between the two single-step methods lays in the enforcement of the yield consistency condition, which enables to compute the value of the plastic consistency parameter and to solve the related algebraic system through the return map algorithm. In the SMPT1 method, the yield consistency condition is applied at the end of the time-integration interval; in the SMPT2 method, the condition is instead enforced at the midpoint instant.

#### 3.1.1. Time-integration scheme over the interval \([t_n, t_{n+1}]\).

Using a generalized midpoint integration rule for the evolutive equations (7) and (9), one may write

\[
\begin{align*}
e^p_{n+1} &= e^p_n + \lambda n_{n+\alpha} \\
\alpha_{n+1} &= \alpha_n + \lambda H_{\text{kin}} n_{n+\alpha}
\end{align*}
\]

which permits to update the remaining variables as

\[
\begin{align*}
s_{n+1} &= 2G(e_{n+1} - e^p_{n+1}) \\
\Sigma_{n+1} &= s_{n+1} - \alpha_{n+1} \\
\gamma_{n+1} &= \gamma_n + \dot{\lambda}
\end{align*}
\]
The scalar $\lambda$ represents the incremental plastic multiplier

$$\lambda = \int_{t_n}^{t_{n+1}} \dot{\gamma} \, dt$$

to be determined enforcing the discrete plastic consistency condition $F(\Sigma_{n+1}) = 0$. To compute $n_{n+2}$ we note that the quantities evaluated at the midpoint instant $t_{n+2}$ are related to the corresponding values at $t_n$ and $t_{n+1}$, respectively, by a linear interpolation of the following kind:

$$e_{n+2} = e_{n+1} + (1 - \lambda) e_n$$
$$e^p_{n+2} = e^p_{n+1} + (1 - \lambda) e^p_n$$
$$\alpha_{n+2} = \alpha_{n+1} + (1 - \lambda) \alpha_n$$
$$s_{n+2} = s_{n+1} + (1 - \lambda) s_n$$
$$\Sigma_{n+2} = \Sigma_{n+1} + (1 - \lambda) \Sigma_n$$

(13)

3.1.2. Solution algorithm. Given the endpoint value $e_{n+1}$, the evolution of the history variables over $[t_n, t_{n+1}]$ is initially supposed to be elastic. This leads to the trial values at the final instant $t_{n+1}$

$$e^p_{n+1}^{TR} = e^p_n$$
$$s_{n+1}^{TR} = 2G(e_{n+1} - e^p_n)$$
$$\alpha_{n+1}^{TR} = \alpha_n$$
$$\Sigma_{n+1}^{TR} = s_{n+1}^{TR} - \alpha_{n+1}^{TR}$$
$$\gamma_{n+1}^{TR} = \gamma_n$$

(14)

If the resulting trial relative stress is admissible, i.e.

$$\|\Sigma_{n+1}^{TR}\| \leq \sigma_{y,0} + H_{iso} \gamma_{n+1}^{TR}$$

(15)

the whole step is assumed to be elastic and the variables are updated with the trial ones. On the other hand, if $\Sigma_{n+1}^{TR}$ violates the yield limit, a plastic correction is introduced.

Due to the construction (13), at $t_{n+2}$, it holds

$$\alpha_{n+2} = \alpha_{n+1}^{TR} + \lambda H_{kin} \lambda n_{n+2}$$
$$s_{n+2} = s_{n+1}^{TR} - 2G \alpha \lambda n_{n+2}$$
$$\Sigma_{n+2} = \Sigma_{n+1}^{TR} - \lambda Y \lambda n_{n+2}$$

(16)
where
\[ s^\text{TR}_{n+1} = 2G(e_{n+1} - e^p_n) \]
\[ \alpha^\text{TR}_{n+1} = \alpha_n \]
\[ \Sigma^\text{TR}_{n+1} = s^\text{TR}_{n+1} - \sigma^\text{TR}_{n+1} \]
\[ Y^\dot{\lambda} = (2G + H_{\text{kin}})\dot{\lambda} \]

The updated values at \( t_{n+1} \) are computed as
\[ e^p_{n+1} = e^p_{n+1} + \dot{\gamma}n_{n+1} \]
\[ \alpha_{n+1} = \alpha^\text{TR}_{n+1} + H_{\text{kin}}\dot{\gamma}n_{n+1} \]
\[ s_{n+1} = s^\text{TR}_{n+1} - 2G\dot{\gamma}n_{n+1} \]
\[ \Sigma_{n+1} = \Sigma^\text{TR}_{n+1} - Y^\dot{\lambda}n_{n+1} \]
\[ \dot{\gamma}_{n+1} = \gamma^\text{TR}_{n+1} + \dot{\lambda} \]

where the normal tensor \( n_{n+1} = \Sigma_{n+1}/||\Sigma_{n+1}|| \) at the midpoint instant can be computed observing that Equation (16) induces the following co-alignment relation:
\[ n_{n+1} = \Sigma_{n+1}/||\Sigma_{n+1}|| = \Sigma^\text{TR}_{n+1}/||\Sigma^\text{TR}_{n+1}|| \]

(18)

The update procedure (17) is completed by deriving the value of the plastic multiplier increment \( \dot{\lambda} \). This task is achieved enforcing the plastic consistency condition to the updated solution.

While the SMPT1 scheme enforces the plastic consistency condition to the final instant solution \( \Sigma_{n+1} \), the SMPT2 scheme enforces the plastic consistency condition to the midpoint solution \( \Sigma_{n+1} \). This is actually the stage where the two single-step schemes differ.

**SMPT1 Scheme** (consistency at \( t_{n+1} \)): Due to the fact that \( \Sigma_{n+1} \) and \( \Sigma^\text{TR}_{n+1} \) are parallel (cf. Equation (16)), it holds
\[ ||\Sigma_{n+1}|| = ||\Sigma^\text{TR}_{n+1}|| - \alpha Y^\dot{\lambda} \]

(19)

Hence, the discrete consistency condition at the final instant \( t_{n+1} \)
\[ ||\Sigma_{n+1}||^2 = (\sigma_{y,n} + \lambda H_{\text{iso}})^2 \]

(20)

can be rewritten in terms of the norm of the midpoint relative stress \( ||\Sigma_{n+1}|| \). Recalling the midpoint rule (13),
\[ \Sigma_{n+1} = \frac{1}{2} \Sigma_{n+1} - \left( \frac{1 - \alpha}{\alpha^2} \right) \Sigma_n \]

(21)

Equation (20) becomes
\[ \frac{1}{\alpha^2} ||\Sigma_{n+1}||^2 - 2 \left( \frac{1 - \alpha}{\alpha^2} \right) (\Sigma_{n+1} : \Sigma_n) + \left( \frac{1 - \alpha}{\alpha} \right)^2 ||\Sigma_n||^2 - (\sigma_{y,n} + \lambda H_{\text{iso}})^2 = 0 \]

(22)
which, with some algebra, results in the following second order equation in \( \lambda \) to be solved for the minimum positive root
\[
a\lambda^2 + b\lambda + c = 0
\] (23)

where
\[
a = (2G + H_{\text{kin}})^2 - H_{\text{iso}}^2
\]
\[
b = 2 \left( \frac{1 - \alpha}{\alpha} \right) \frac{2G + H_{\text{kin}}}{\|\Sigma_{n+2}\|} \Sigma_{n+2} : \Sigma_n - 2\sigma_{y,n} H_{\text{iso}}
\]
\[
c = \|\Sigma_{n+2} - \Sigma_n\|^2 - \frac{1 - 2\alpha}{\alpha^2} \|\Sigma_n\|^2 - \frac{2}{\alpha^2} \Sigma_{n+2} : \Sigma_n - \sigma_{y,n}^2
\]

Figures 1 and 2 illustrate the SMPT1 updating procedure in stress space, respectively, in the case of zero hardening moduli and non-zero hardening moduli.

Figure 1. Return map update in relative stress space for the SMPT1 method: no hardening case.

Figure 2. Return map update in relative stress space for the SMPT1 method: combined linear isotropic-kinematic hardening.
SMPT2 Scheme (consistency at $t_{n+2}$): Following this scheme, the consistency condition is enforced at the midpoint instant $t_{n+2}$. Due to identity (19), the midpoint plastic consistency condition

$$\|\Sigma_{n+2}\| = \sigma_{y,n} + \alpha \lambda H_{iso}$$  \hspace{1cm} (24)

is equivalent to

$$\|\Sigma_{n+2}^{TR}\| = \sigma_{y,n} + \alpha \lambda H_{iso} + Y^G$$  \hspace{1cm} (25)

which returns the following solution for $\lambda$

$$\lambda = \frac{\|\Sigma_{n+2}^{TR}\| - \sigma_{y,n}}{\alpha(2G + H_{kin} + H_{iso})}$$

Figures 3 and 4 illustrate the SMPT2 updating procedure in stress space, respectively, in the case of zero hardening moduli and non-zero hardening moduli. For the sake of clearness the SMPT1 and SMPT2 schemes are reported compactly in Table I.

Remark 3.2
It is noted that, choosing $\alpha = 1$, both the SMPT1 and SMPT2 procedures become the well-established backward Euler integration rule coupled with the return map projection. This efficient first order method is utilized in the following sections as a reference algorithm for comparison purposes. The aforementioned scheme is indicated as BE method in the sequel.

Remark 3.3
The SMPT2 scheme is not endpoint yield consistent as it can be checked in Figures 3 and 4.

3.2. Double-step midpoint methods

Double-step midpoint methods apply the integration scheme and the return map solution algorithm twice per time interval. The solution in terms of variables is calculated first at $t_{n+2}$ and then at $t_{n+1}$. As a consequence, there is a gain in the simplicity of the scalar non-linear equation for the incremental plastic multiplier, but the operational steps are doubled.

In both methods, the first step consists in a classical backward Euler integration rule over the subinterval $[t_n, t_{n+2}]$ coupled with a return map update. As a result, all the variables at $t_{n+2}$ are computed. The second step of the scheme is instead where the two methods differ.
Figure 4. Return map update in relative stress space for the SMPT2 method: combined linear isotropic-kinematic hardening.

Table I. SMPT1 and SMPT2 schemes.

<table>
<thead>
<tr>
<th>SMPT1</th>
<th>SMPT2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen. MPT integration rule over ([t_n, t_{n+1}])</td>
<td>Gen. MPT integration rule over ([t_n, t_{n+1}])</td>
</tr>
<tr>
<td>Consistency at (t_{n+1})</td>
<td>Consistency at (t_{n+2})</td>
</tr>
<tr>
<td>Return map projection at (t_{n+1})</td>
<td>Return map projection at (t_{n+1})</td>
</tr>
</tbody>
</table>

The DMPT1 scheme applies a midpoint integration rule over the time interval \([t_n, t_{n+1}]\) and adopts a return map update, imposing the plastic correction at the final instant. In this case, the return map makes use of the normal tensor \(n_{n+2}\) calculated in the first step.

On the other hand, the easiest way to interpret the DMPT2 second step is as an extrapolation of non-standard (elastoplastic) trial state at final instant \(t_{n+1}\) combined with a return map update of the ensuing algebraic problem. The peculiarity of this procedure is twofold. First, no specific rule is adopted for the integration of the evolution equations. Second, the return map applies to a non-standard (elastoplastic) trial state which is calculated interpolating the values of the history variables obtained at \(t_{n+2}\) with the corresponding values at \(t_n\), assuming a linear evolution in time over \([t_n, t_{n+1}]\). The DMPT2 scheme considered here constitutes a generalization of a scheme labelled ALGO 3 proposed by Simo for the case of perfect \(J_2\) plasticity [3]. It is noted that the DMPT2 is still based on the use of a generalized midpoint integration concept and on the application of a return map type algorithm, i.e. a predictor–corrector solution strategy. Such a method is referred in the literature also as generalized projected midpoint return map method [3].

In the following we start considering the STEP 1 over \([t_n, t_{n+2}]\) which, as mentioned above, is the same for both methods.

3.2.1. **STEP 1: time-integration scheme over the subinterval \([t_n, t_{n+2}]\).** The evolutionary equations (7) and (9) are discretized over the first subinterval \([t_n, t_{n+2}]\) by means of the backward
Euler rule
\begin{align}
\dot{\mathbf{e}}_{p,n+z} &= \mathbf{e}_{p,n} + \lambda_1 \mathbf{n}_{n+z} \\
\mathbf{\alpha}_{n+z} &= \mathbf{\alpha}_n + \lambda_1 \mathbf{H}_{\text{kin}} \mathbf{n}_{n+z} 
\end{align}
(26)

where \( \lambda_1 \) represents the plastic rate parameter increment over \([t_n, t_{n+z}]\)
\[
\lambda_1 = \int_{t_n}^{t_{n+z}} \dot{\gamma} \, dt
\]
to be computed requiring the subinterval endpoint plastic consistency, i.e. \( F(\mathbf{\Sigma}_{n+z}) = 0 \). The updates at \( t_{n+z} \) hence become
\begin{align}
\mathbf{s}_{n+z} &= 2G(\mathbf{e}_{n+z} - \mathbf{e}_{n}^p) \\
\mathbf{\Sigma}_{n+z} &= \mathbf{s}_{n+z} - \mathbf{\alpha}_{n+z} \\
\dot{\gamma}_{n+z} &= \dot{\gamma}_n + \lambda_1 
\end{align}
(27)

3.2.2. STEP 1: solution algorithm over the subinterval \([t_n, t_{n+z}]\). The step is initially supposed to be elastic. This leads to the following trial state at \( t_{n+z} \): \begin{align}
\mathbf{e}_{n+z}^{p,\text{TR}} &= \mathbf{e}_n^p \\
\mathbf{s}_{n+z}^{\text{TR}} &= 2G(\mathbf{e}_{n+z} - \mathbf{e}_n^p) \\
\mathbf{\alpha}_{n+z}^{\text{TR}} &= \mathbf{\alpha}_n \\
\mathbf{\Sigma}_{n+z}^{\text{TR}} &= \mathbf{s}_{n+z}^{\text{TR}} - \mathbf{\alpha}_{n+z}^{\text{TR}} \\
\dot{\gamma}_{n+z}^{\text{TR}} &= \dot{\gamma}_n
\end{align}
(28)

If the resulting trial relative stress is admissible, i.e.
\[
\|\mathbf{\Sigma}_{n+z}^{\text{TR}}\| \leq \sigma_{y,0} + H_{\text{iso}} \dot{\gamma}_{n+z}^{\text{TR}}
\]
(29)

the subinterval is assumed to be elastic and the midpoint variables are updated with the trial ones. On the other hand, if \( \mathbf{\Sigma}_{n+z}^{\text{TR}} \) violates the yield limit, a plastic correction for the variables at \( t_{n+z} \) is introduced as
\begin{align}
\mathbf{e}_{n+z}^p &= \mathbf{e}_{n+z}^{p,\text{TR}} + \lambda_1 \mathbf{n}_{n+z} \\
\mathbf{\alpha}_{n+z} &= \mathbf{\alpha}_{n+z}^{\text{TR}} + H_{\text{kin}} \lambda_1 \mathbf{n}_{n+z} \\
\mathbf{s}_{n+z} &= \mathbf{s}_{n+z}^{\text{TR}} - 2G \lambda_1 \mathbf{n}_{n+z} \\
\mathbf{\Sigma}_{n+z} &= \mathbf{\Sigma}_{n+z}^{\text{TR}} - Y \lambda_1 \mathbf{n}_{n+z} \\
\dot{\gamma}_{n+z} &= \dot{\gamma}_{n+z}^{\text{TR}} + \lambda_1
\end{align}
(30)
where the normal tensor \( \mathbf{n}_{n+2} = \Sigma_{n+2}/\|\Sigma_{n+2}\| \) at the midpoint instant is computed observing that Equation (30) induces the following co-alignment relation:

\[
\mathbf{n}_{n+2} = \frac{\Sigma_{n+2}}{\|\Sigma_{n+2}\|} = \frac{\Sigma_{n+2}}{\|\Sigma_{n+2}\|} = \frac{\Sigma^{TR}_{n+2}}{\|\Sigma^{TR}_{n+2}\|}
\]  

With some algebraic considerations the consistency equation \( F(\Sigma_{n+2}) = 0 \) returns the following value for the plastic multiplier:

\[
\lambda_1 = \frac{\|\Sigma^{TR}_{n+2}\| - (\sigma_y, 0 + H_{iso}\gamma^{TR}_{n+2})}{2G + H_{iso} + H_{kin}}
\]  

The updating of the variables at \( t_{n+2} \) is obtained via Equations (30).

### 3.2.3. Step 2. DMPT1 Scheme: time-integration scheme over the interval \([t_n, t_{n+1}]\):

After applying a BE integration scheme over \([t_n, t_{n+2}]\) combined with a radial projection at \( t_{n+2} \), as second step the DMPT1 scheme is such that the evolutionary equations (7) and (9) are discretized over the whole step \([t_n, t_{n+1}]\) by means of a generalized midpoint rule

\[
e_{n+1}^p = e_n^p + \lambda_2 \mathbf{n}_{n+2}
\]

\[
\alpha_{n+1} = \alpha_n + \lambda_2 H_{kin}\mathbf{n}_{n+2}
\]

where the midpoint normal tensor \( \mathbf{n}_{n+2} \) is calculated at the previous step according to (31).

The scalar \( \lambda_2 \) represents the increment of the plastic multiplier over \([t_n, t_{n+1}]\), and is calculated enforcing the yield consistency condition at \( t_{n+1} \) as in the SMPT1 scheme (cf. Equation (22)). The updates at \( t_{n+1} \) are

\[
s_{n+1} = 2G(e_{n+1} - e_{n+1}^p)
\]

\[
\Sigma_{n+1} = s_{n+1} - \alpha_{n+1}
\]

\[
\gamma_{n+1} = \gamma_n + \lambda_2
\]

**Remark 3.4**

In the literature (see for example [3]) it is implicitly observed that, in the case of associative plasticity, the DMPT1 scheme is equivalent to the SMPT1 scheme. The reasons why the authors preferred to present both algorithms independently is twofold. The first reason is that the integration rule of the two methods is different, which is why the respective extensions to more complicated models are in general not equivalent. The second reason is that, even for perfect plasticity, the two schemes are not identical also from the computational standpoint. Indeed, as we prove in Proposition A.2, assuming that both algorithms have a solution, such two solutions coincide; on the other hand, the existence of a solution is always verified for the DMPT1, while the SMPT1 scheme has no solution for a certain range of trial states as investigated in Section 4.2.

**DMPT2 Scheme: extrapolation of non-standard endpoint trial state:** The DMPT2 applies a backward Euler integration rule coupled with a return map update over \([t_n, t_{n+1}]\), using a non-standard endpoint elastoplastic trial state represented by the following barred quantities:

\[
e_{n+1}^{p, TR}, \Sigma_{n+1}^{TR}, \gamma_{n+1}^{TR}
\]
The barred trial quantities, as already mentioned, are deduced from the variables calculated at the end of STEP 1, i.e. at the midpoint instant \( t_{n+\frac{1}{2}} \), assuming a linear interpolation in time according to

\[
\begin{align*}
\dot{e}^{p,TR}_{n+1} &= \frac{1}{\alpha} e^{p}_{n+\frac{1}{2}} - \left( \frac{1 - \alpha}{\alpha} \right) e^{p}_n \\
\ddot{s}_{n+1}^{TR} &= \frac{1}{\alpha} s_{n+\frac{1}{2}} - \left( \frac{1 - \alpha}{\alpha} \right) s_n \\
\dot{\alpha}^{TR}_{n+1} &= \frac{1}{\alpha} \alpha_{n+\frac{1}{2}} - \left( \frac{1 - \alpha}{\alpha} \right) \alpha_n \\
\dot{\Sigma}_{n+1}^{TR} &= \frac{1}{\alpha} \Sigma_{n+\frac{1}{2}} - \left( \frac{1 - \alpha}{\alpha} \right) \Sigma_n \\
\dot{\gamma}^{TR}_{n+1} &= \frac{1}{\alpha} \gamma_{n+\frac{1}{2}} - \left( \frac{1 - \alpha}{\alpha} \right) \gamma_n
\end{align*}
\] (35)

3.2.4. STEP 2: Solution algorithm over the interval \([t_n, t_{n+1}]\). DMPT1 Scheme: Initially the endpoint trial state is supposed to be purely elastic, namely

\[
\begin{align*}
\dot{e}^{p,TR}_{n+1} &= e^{p}_n \\
\ddot{s}_{n+1}^{TR} &= 2G(e_{n+1} - e^{p}_n) \\
\dot{\alpha}^{TR}_{n+1} &= \alpha_n \\
\dot{\Sigma}_{n+1}^{TR} &= s^{TR}_{n+1} - \alpha^{TR}_{n+1} \\
\dot{\gamma}^{TR}_{n+1} &= \gamma_n
\end{align*}
\] (36)

If the resulting relative stress is contained within the elastic domain, i.e.

\[
\|\Sigma^{TR}_{n+1}\| \leq \sigma_{y,0} + H_{iso}\gamma^{TR}_{n+1}
\] (37)

the variables at time \( t_{n+1} \) are taken as the trial ones. On the other hand, if \( \Sigma^{TR}_{n+1} \) violates the yield limit (37), a plastic correction is introduced as

\[
\begin{align*}
\dot{e}^{p}_{n+1} &= e^{p,TR}_{n+1} + \lambda_2 n_{n+\frac{1}{2}} \\
\dot{\alpha}_{n+1} &= \alpha^{TR}_{n+1} + \lambda_2 H_{kin} n_{n+\frac{1}{2}} \\
\ddot{s}_{n+1}^{TR} &= \ddot{s}_{n+1}^{TR} - 2G\lambda_2 n_{n+\frac{1}{2}} \\
\dot{\Sigma}_{n+1}^{TR} &= \Sigma^{TR}_{n+1} - \dot{Y}^{\lambda_2} n_{n+\frac{1}{2}} \\
\dot{\gamma}_{n+1}^{TR} &= \gamma_{n+\frac{1}{2}} + \lambda_2
\end{align*}
\] (38)
The solution of the algebraic system (38) is found by enforcing the discrete limit condition $F(\Sigma_{n+1}) = 0$, which provides

$$a(\lambda_2)^2 + b\lambda_2 + c = 0$$

(39)

to be solved for the minimum positive root. The coefficients of the above equation read

$$a = (2G + H_{\text{kin}})^2 - H_{\text{iso}}^2$$
$$b = -2[\Sigma_{n+1}^{\text{TR}} : (2G + H_{\text{kin}})n_{n+1}] - 2H_{\text{iso}}\sigma_{y,n}$$
$$c = \|\Sigma_{n+1}^{\text{TR}}\|^2 - \sigma_{y,n}^2$$

Figures 5 and 6 illustrate the DMPT1 updating procedure in stress space, respectively, in the case of zero hardening moduli and non-zero hardening moduli.

![Figure 5](image1.png)

Figure 5. Return map update in relative stress space for the DMPT1 method: no hardening case.

![Figure 6](image2.png)

Figure 6. Return map update in relative stress space for the DMPT1 method: combined linear isotropic-kinematic hardening.
**DMPT2 Scheme:** The return map algorithm applies directly to the non-standard trial values deduced in (??) and enables to calculate $\tilde{\lambda}_2$ which satisfies

$$\gamma_{n+1} = \tilde{\gamma}_{TR} + \tilde{\lambda}_2$$

(40)

The scheme proceeds as follows: if the barred trial relative stress $\tilde{\sigma}_{TR}$ is contained within the trial admissible domain, i.e.

$$\|\tilde{\sigma}_{TR}\| \leq \sigma_{y,0} + H_{iso}\tilde{\gamma}_{n+1}$$

(41)

the variables at the time instant $t_{n+1}$ are taken as the barred trial ones. On the other hand, if $\tilde{\sigma}_{TR}$ violates the yield limit (41), a plastic correction is introduced using a return map-type procedure

$$\begin{align*}
\mathbf{e}^{p}_{n+1} &= \mathbf{e}^{p,TR}_{n+1} + \tilde{\lambda}_2 \mathbf{n}_{n+1} \\
\mathbf{s}_{n+1} &= \mathbf{s}^{TR}_{n+1} - 2G \mathbf{\tilde{\lambda}}_2 \mathbf{n}_{n+1} \\
\mathbf{\alpha}_{n+1} &= \mathbf{\tilde{\alpha}}^{TR}_{n+1} + H_{kin}\mathbf{\tilde{\lambda}}_2 \mathbf{n}_{n+1} \\
\mathbf{\Sigma}_{n+1} &= \mathbf{\tilde{\Sigma}}^{TR}_{n+1} - \gamma^{\prime,2}_{n+1} \mathbf{n}_{n+1} \\
\gamma_{n+1} &= \tilde{\gamma}_{TR} + \tilde{\lambda}_2
\end{align*}$$

(42)

The endpoint normal $\mathbf{n}_{n+1}$ along which the plastic corrector is applied is defined by

$$\mathbf{n}_{n+1} = \frac{\mathbf{\Sigma}_{n+1}}{\|\mathbf{\Sigma}_{n+1}\|} = \frac{\mathbf{\tilde{\Sigma}}^{TR}_{n+1}}{\|\mathbf{\tilde{\Sigma}}^{TR}_{n+1}\|}$$

The solution of the algebraic system (42) is found by enforcing the discrete limit condition $F(\mathbf{\Sigma}_{n+1}) = 0$, which gives

$$\tilde{\lambda}_2 = \frac{\|\mathbf{\tilde{\Sigma}}^{TR}_{n+1}\| - (\sigma_{y,0} + H_{iso}\tilde{\gamma}_{n+1})}{2G + H_{iso} + H_{kin}}$$

(43)

Once the scalar $\tilde{\lambda}_2$ is known, it is possible to update the variables at the final state according to Equations (42).

Figures 7 and 8 illustrate the DMPT2 updating procedure in stress space, respectively, in the case of zero hardening moduli and non-zero hardening moduli. For the sake of clearness the DMPT1 and DMPT2 schemes are reported compactly in Table II.

**Remark 3.5**

The non-standard trial state used in the DMPT2 scheme corresponds exactly to the solution of the SMPT2 method. The DMPT2 method may be therefore interpreted as an improvement of the SMPT2 algorithm on a yield consistency basis. In fact the SMPT2 method is characterized by a non-consistent solution at $t_{n+1}$, while the DMPT2 can be interpreted as a projection of the SMPT2 final state onto the yield surface [3].
Figure 7. Return map update in relative stress space for the DMPT2 method: no hardening case.

Figure 8. Return map update in relative stress space for the DMPT2 method: combined linear isotropic-kinematic hardening.

Table II. DMPT1 and DMPT2 schemes.

<table>
<thead>
<tr>
<th></th>
<th>DMPT1</th>
<th>DMPT2</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 1</td>
<td>BE integration rule over $[t_n, t_{n+1}]$</td>
<td>BE integration rule over $[t_n, t_{n+2}]$</td>
</tr>
<tr>
<td></td>
<td>Consistency at $t_{n+2}$</td>
<td>Consistency at $t_{n+2}$</td>
</tr>
<tr>
<td></td>
<td>Return map projection at $t_{n+2}$</td>
<td>Return map projection at $t_{n+2}$</td>
</tr>
<tr>
<td></td>
<td>Gen. MPT integration rule over $[t_n, t_{n+1}]$</td>
<td>Gen. BE algorithm</td>
</tr>
<tr>
<td></td>
<td>Consistency at $t_{n+1}$</td>
<td>Consistency at $t_{n+1}$</td>
</tr>
<tr>
<td></td>
<td>Return map projection at $t_{n+1}$</td>
<td>Return map projection at $t_{n+1}$</td>
</tr>
</tbody>
</table>

Remark 3.6
At the end of plastic steps, the SMPT1 and DMPT1 methods satisfy

$$\| \Sigma_{n+1} \| = \sigma_{y,n+1}$$  \hspace{1cm} (44)
which is a stronger condition than $\|\Sigma_{n+1}\| \leq \sigma_{y,n+1}$. Condition (44) is correct, because during plastic phases the relative stress lays on the yield surface and not inside it. Inspecting the DMPT2 method, instead, one notes that condition (44) is not directly guaranteed by the algorithm construction. The reason is that the trial radius $\tilde{\sigma}_{y,n+1}^{TR}$ may be, in principle, larger than the norm of $\Sigma_{n+1}^{TR}$; in such case the scheme stops and no additional correction is introduced. Proposition A.1 in Appendix A addresses this point proving that (44) holds at the end of all plastic steps also for the DMPT2 scheme, for the specific model under investigation.

4. THEORETICAL ANALYSIS

In this section, we gather and complete, on the basis of the existing literature, the theoretical analysis of the four midpoint methods presented in Sections 3.1 and 3.2. In particular, we take into consideration the geometrical interpretation of the algorithms in stress space, together with the issues of existence of solution, long-term behaviour, accuracy and stability.

4.1. Geometrical interpretation

In this section, we show the stress space geometrical interpretation of the four algorithms for an explicit visualization of their updating schemes. In Figures 1–8, the plastic step geometrical idealization, respectively, for the SMPT1, SMPT2, DMPT1 and DMPT2, both for the case of perfect and hardening plasticity, is presented. Note that SMPT1 and DMPT1 produce the same stress output at the end of the time step (as proved in Proposition A.2), although such observation is subjected to Remark 3.4. Using basic geometrical arguments and triangle similarities, the following result is easy to check (see Figure 9).

Observation 4.1

The solution $\Sigma_{n+1}$ of the DMPT1 scheme lays on the intersection between the updated yield surface and the line connecting $\Sigma_{n+1}^{TR}$ and $-\Sigma_n$. If more than one intersection exists, the point nearer to $\Sigma_{n+1}^{TR}$ has to be chosen.

Finally, observe that the trial state $\tilde{s}_{n+1}^{TR}$ of the DMPT2 corresponds to the solution of the SMPT2, as noted in Remark 3.5.

4.2. Existence of solution

As discussed for example in [3], for the present plasticity model the BE scheme has always a solution for any possible set of data $\{s_n, e_n, e_{ip}, \gamma_n, \alpha_n, e_{n+1}\}$. As a consequence, the same result holds for the SMPT2, DMPT1 and DMPT2 methods, because such schemes are based on the assembly of BE methods. In other words, given any initial step data $\{s_n, e_n, e_{ip}, \gamma_n, \alpha_n, \}$, and a final step strain $e_{n+1}$, the above schemes are always able to compute a final step state $\{s_{n+1}, e_{n+1}, e_{ip+1}, \gamma_{n+1}, \alpha_{n+1}\}$.

We now analyse the existence of solution for the SMPT1 scheme. We start from the following result, proven in Appendix A.

Proposition 4.1

The SMPT1 and DMPT1 methods are equivalent, provided both schemes have a solution.
Figure 9. DMPT1 updating procedure in relative stress space.

**Proof**

Due to Proposition 4.1, the only candidate solution for the SMPT1 scheme corresponds to the solution of the DMPT1 method. Moreover, in order for the SMPT1 scheme to be well defined in plastic steps, the midpoint normal

\[ \mathbf{n}_{n+1/2} = \frac{\Sigma_{n+1} + \Sigma_n}{\|\Sigma_{n+1} + \Sigma_n\|} \]  

(45)

needs also to be computable. Therefore, given \( \Sigma_n \) as the starting relative stress and \( \Sigma_{n+1} \) as the solution of the DMPT1 scheme, the SMPT1 scheme holds a solution if and only if (45) is well defined; in other words if and only if

\[ \Sigma_{n+1} + \Sigma_n \neq \mathbf{0} \]  

(46)

where \( \mathbf{0} \) is the null second order tensor. It is immediate to check that (46) is true for all plastic steps if \( H_{iso} \neq 0 \). Indeed, if \( H_{iso} \neq 0 \), there exists no plastic step for which it holds \( \Sigma_{n+1} = -\Sigma_n \); such claim easily follows observing that

\[ \| -\Sigma_n \| \leq \sigma_{y,n} < \sigma_{y,n+1} = \| \Sigma_{n+1} \| \]  

(47)

Therefore, if isotropic hardening is included in the model, also the SMPT1 scheme has a solution for all possible sets of data \( \{ s_n, e_n, e^p_n, \gamma_n, \mathbf{x}_n \} \).

Differently, if \( H_{iso} = 0 \), it may happen that the solution \( \Sigma_{n+1} \) of the DMPT1 scheme corresponds to \(-\Sigma_n\). In such cases condition (46) holds and the SMPT1 method has no solution. Recalling Observation 4.1, such situation is verified if and only if

\[ \| \Sigma_n \| = \sigma_{y,n} \quad \text{and} \quad -\Sigma_{n}^{TR} : \Sigma_n \geq \Sigma_n : \Sigma_n \]  

(48)

in other words if \( \Sigma_n \) lays on the yield surface and the trial relative stress \( \Sigma_n^{TR} \) is in the demispace shown in Figure 10.

From the computational standpoint it happens that, whenever the trial stress falls in the range depicted in Figure 10, the second order equation for \( \lambda \) shown in (23) has no real solutions.
Differently, for the same data the analogous equation (39) for the DMPT1 scheme has one or more real roots.

4.3. Long-term behaviour of the schemes

In this section we study the behaviour at infinity of the four methods in the case of perfect plasticity, i.e. \( H_{iso} = H_{kin} = 0 \). Such asymptotic analysis is considered in the literature to be a good general indication on the method behaviour for long time steps, see, for example, [5].

Given a stress–strain couple \((s_0, e_0)\) at the initial instant \(t_0\), let \(s(t)\) represent the solution at time \(t\) corresponding to the strain history \(e(t) = e_0 + t \Delta e\), \(\Delta e\) assigned, \(||\Delta e|| = 1\) (49)

It is known that

\[
\lim_{t \to +\infty} s(t) = s_{\text{exact}}^*, \quad s_{\text{exact}}^* = \sigma_{y,0} \Delta e
\] (50)

Given a general numerical method for the stress updating and the same initial stress–strain couple \((s_0, e_0)\), let \(s_1\) be the solution obtained applying a single step of the method with \(e_1 = e_0 + t \Delta e\). For BE method it holds

\[
\lim_{t \to +\infty} s_1^{\text{BE}} = s_{\text{exact}}^*
\] (51)

in other words the scheme is able to mimic behaviour of the exact solution for large time steps [5].

As noted in the literature, in general, this property is not shared by midpoint methods. In what follows, the limit values for the four algorithms under investigation are presented; such results can be obtained also from the geometrical representation of the schemes.

\[
\lim_{t \to +\infty} s_1^{\text{SMPT1}} = \lim_{t \to +\infty} s_1^{\text{DMPT1}} = - s_0 + t_* \Delta e \quad \text{with}
\]

\[
t_* = s_0 : \Delta e + \sqrt{(s_0 : \Delta e)^2 + \sigma_{y,0}^2 - ||s_0||^2}
\]
\[
\lim_{t \to +\infty} s_1^{\text{SMPT2}} = 2s_{\text{exact}}^n - s_0
\]
\[
\lim_{t \to +\infty} s_1^{\text{DMPT2}} = \frac{2s_{\text{exact}}^n - s_0}{\|2s_{\text{exact}}^n - s_0\|}
\]

Note that for the SMPT1 the result holds only if \(\|s_n\|<\sigma_{y,0}\) or \(\Delta e : s_n \geq 0\); otherwise, for sufficiently high \(t\), the strain \(\varepsilon(t)\) falls into the zone described in Section 4.2 and therefore the SMPT1 scheme has no solution.

It follows immediately that the above limits differ in general from the correct limit \(s_{\text{exact}}^n\). Therefore, from the theoretical standpoint, it is expected that, for sufficiently large time steps, the first order accurate BE method grants lower error levels than the second order accurate midpoint methods. This can be appreciated also in the iso-error maps of Section 5.2, where only the DMPT2 scheme has a maximum error comparable with the BE method. A deep analysis on the good behaviour of the BE scheme for large time steps in relation to other methods is performed in [12].

4.4. Accuracy and stability

A classical second order expansion in the time step size \(\Delta t\) shows that the truncation error of the four midpoint methods behaves as \((\Delta t)^2\). In other words, given any set of data \([s_n, \alpha_n, \sigma_{y,n}, e_n, e_0^n]\), and a time step strain history \(\varepsilon(t), \ t \in [t_n, t_{n+1}]\), it holds

\[
(\Delta t)^{-1} \|s_{n+1}^{\text{exact}} - s_{n+1}^{\text{exact}}\| \leq C(\Delta t)^2 + o((\Delta t)^3)
\]

where the norm above is any fixed norm on the solution space, while \(s_{n+1}^{\text{exact}} = [s_{n+1}^{\text{exact}}, \sigma_{y,n+1}^{\text{exact}}, e_{n+1}^{\text{exact}}, e_{p,n+1}^{\text{exact}}]\) and \(s_{n+1} = [s_{n+1}, \alpha_{n+1}, \sigma_{y,n+1}, e_{n+1}, e_{p,n+1}\} represent, respectively, the exact solution and the solution obtained with the numerical scheme.

All the four schemes here considered are therefore second order accurate. See, for example, [1, 13] for the derivation of this result for the SMPT1 and SMPT2, respectively; we do not report here the results, which can be easily obtained with classical techniques.

Regarding the stability analysis of the methods, it proves necessary introducing the free Helmholtz energy norm. Let, here and in the sequel, \(S\) indicate the general triple \((s, \alpha, \sigma_y)\). Then, as indicated in [13], the free Helmholtz energy norm is defined by

\[
\|S\|_H^2 = (2G)^{-1} s : s + H_{\text{kin}}^{-1} \alpha : \alpha + H_{\text{iso}}^{-1} \sigma_y^2
\]

As shown for instance, in [13], such norm holds the important property that, if \(S_t\) and \(\tilde{S}_t\) are the flows generated for two different initial conditions \(S_0\) and \(\tilde{S}_0\), it holds

\[
\frac{d}{dt} \|S_t - \tilde{S}_t\|_H \leq 0 \quad \forall t > 0
\]

The flow generated by the system is therefore said to be contractive in the \(\|\cdot\|_H\) norm [3, 5, 13]. Then, the following stability results hold true for the four midpoint methods object of study.

**SMPT1** Applying the result of [1] to the present model, one has

**Proposition 4.2**

Let \(S_n, \tilde{S}_n\) represent two sets of variables at time \(t_n\), and let \(S_{n+1}, \tilde{S}_{n+1}\) be, respectively, the updated values of such quantities obtained applying the SMPT1 scheme.
Assume that both steps are purely plastic, i.e.

\[ \| s_n - \sigma_n \| = \sigma_{y,n}, \quad \| \tilde{s}_n - \tilde{\sigma}_n \| = \tilde{\sigma}_{y,n} \]

(54)

and that the increments of the plastic multipliers \( \lambda, \tilde{\lambda} \) over the interval are greater than zero. Then, it holds

\[ \| S_{n+1} - \tilde{S}_{n+1} \|_H \leq \| S_n - \tilde{S}_n \|_H \]

(55)

The above result shows that, during purely plastic steps, the algorithm is able to mimic the contractive property (53) of the system. Furthermore, from the theoretical viewpoint, bound (55) is a particularly strong stability result; indeed, it shows that the accumulated error of the scheme is dissipated, not only controlled. The above result, combined with the quadratic accuracy of the method, is sufficient to guarantee convergence for purely plastic load histories, as discussed in [1, 14].

Regarding general elastoplastic load histories, the following qualitative argument holds true. Given a general strain history in \([0, T]\), the number of switches between elastic and plastic phases is finite for the continuous solution. As a consequence, the same can be expected to happen at the limit for \( \Delta t \to 0 \) for any first order accurate and yield consistent numerical scheme. For a general loading history, it is therefore expected that the number \( N_{el} \) of mixed elastoplastic steps is bounded uniformly in \( \Delta t \), i.e. \( N_{el} \) does not explode when refining the timeline mesh. In such cases, result 4.2 is sufficient to prove convergence. On the other hand, although reasonable, the latter property on \( N_{el} \) is not in principle guaranteed. Therefore, at the present level, the theoretical stability result for the SMPT1 is not completely satisfactory.

**DMPT1**: Being the SMPT1 and DMPT1 schemes identical whenever a solution exists, for the DMPT1 there holds the same results and comments as for the SMPT1.

**SMPT2**: For the SMPT2 it holds the following stronger result [5, 13, 15]

**Proposition 4.3**

Let \( S_n, \tilde{S}_n \) represent two sets of variables at time \( t_n \), and \( S_{n+1}, \tilde{S}_{n+1} \) the results obtained applying one step of the SMPT2 scheme, respectively. Then, it holds

\[ \| S_{n+1} - \tilde{S}_{n+1} \|_H \leq \| S_n - \tilde{S}_n \|_H \]

(56)

A numerical method for the integration of elastoplastic constitutive models for which Proposition 4.3 holds is said to be B-stable [3, 16]. Proposition 4.3, being valid also for mixed elastoplastic steps, is stronger than Proposition 4.2. In particular, recalling the quadratic accuracy of the method, the B-stability result (56) implies the convergence of the SMPT2 scheme for all types of loading histories.

**DMPT2**: Also for this scheme, the following B-stability result applies

**Proposition 4.4**

Let \( S_n, \tilde{S}_n \) represent two sets of variables at time \( t_n \), and \( S_{n+1}, \tilde{S}_{n+1} \) the results obtained applying one step of the DMPT2 scheme, respectively. Then, it holds

\[ \| S_{n+1} - \tilde{S}_{n+1} \|_H \leq \| S_n - \tilde{S}_n \|_H \]

(57)
Proof
It proves useful recalling that the DMPT2 scheme is equivalent to a combination of the SMPT2 scheme with the same return map algorithm correction of the BE method (see Remark 3.5). As proved in [13, 15], such return map algorithm correction is B-stable. In other words, let $\tilde{S}_{n+1}^{TR}$, $\tilde{S}_{n+1}^{TR}$ be two trial states and $S_{n+1}^{TR}$, $\tilde{S}_{n+1}^{TR}$ the respective solutions obtained with the return map plastic correction. Then it holds

$$\|S_{n+1} - \tilde{S}_{n+1}\|_H \leq \|S_{n+1}^{TR} - \tilde{S}_{n+1}^{TR}\|_H$$  \hspace{1cm} (58)

Therefore, the proof immediately follows joining Proposition 4.3 with the result in (58).\hfill \square

Due to Proposition 4.4 the DMPT2 method is B-stable and the same convergence considerations drawn with respect to the SMPT2 scheme hold.

We conclude this section summarizing the properties of the analysed schemes in the following prospectus:

<table>
<thead>
<tr>
<th>Property</th>
<th>Scheme</th>
<th>B-st.</th>
<th>$O(\Delta t^2)$</th>
<th>Yield cons.</th>
<th>Exact $\Delta \to \infty$</th>
<th>Exist. sol.</th>
<th>Symm. Tg.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BE</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td></td>
<td>SMPT1</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td></td>
<td>SMPT2</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td></td>
<td>DMPT1</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
<tr>
<td></td>
<td>DMPT2</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

which reports acronyms related to the following properties:

- B-stability.
- 2nd order accuracy.
- Yield consistency.
- Exactness for infinitely long time steps in case of perfect plasticity.
- Existence of solution.
- Symmetry of the consistent tangent operator.

It is noted that the DMPT2 scheme is the only yield consistent, quadratic and B-stable scheme within the considered schemes.

Remark 4.1
An important issue to properly compare different schemes is the computational cost. However, a rigorous analysis of computational time and costs strongly depends on the optimality of the implementation [17, 18]. Therefore, in order to have a rough indication on the computational bulk of the four methods, we calculated by hand the number of operations needed to compute a time marching step, assuming to follow the algorithmical schemes as given in this contribution, i.e. without any further optimization. The related costs, indicated with $\kappa$ may be ordered as

$$\kappa_{BE} \leq \kappa_{SMPT2} \leq \kappa_{DMPT1} \leq \kappa_{SMPT1} \approx \kappa_{DMPT2}$$
with a ratio in the number of operations between the BE scheme and the DMPT2 scheme of about \( \frac{1}{2} \). The (slightly) higher cost of the SMPT1 over the DMPT1 mainly depends on the computation of the coefficients of the second order equation for \( \dot{\lambda} \).

5. NUMERICAL TESTS

In this section a set of numerical examples is presented. The aim is to compare numerically, under various circumstances and settings, the SMPT and DMPT midpoint methods in terms of error. To this end in all presented tests we adopt as reference algorithm the BE method.

The numerical tests are divided into three parts. In the first one, a total of three pointwise mixed stress–strain loading histories are considered. The object of this test is to study the convergence of the method to the exact solution with respect to the computation of the stress and of the strain in a general non-proportional loading history. In the second part, a set of iso-error maps are presented. Iso-error maps permit to investigate the error produced by each algorithm in the computation of the stress, with respect to increasing time discretization amplitudes. The third test regards an initial boundary-value problem, solved with a finite element strategy. The task is to analyse the reliability of the different integration schemes implemented into a finite element code. The aforementioned tests lay out a comparison of the presented methods and determine the optimal one for practical engineering computation.

The pointwise stress–strain loading histories are solved with the CE-Driver [19], while initial boundary-value problem is modelled and solved using FEAP [20]. The error plots and the iso-error maps are obtained using Matlab. Each test set-up is given in detail at the beginning of the pertinent section.

In the analysis two sets of material constants are adopted. The two sets are labelled, respectively, Material 1 [5] and Material 2 [21] and are reported concisely in Table III.

Finally, we recall that the Young Modulus \( E \) and the Poisson ratio \( \nu \) uniquely determine the constants \( K \) and \( G \) as follows:

\[
K = \frac{E}{3(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}
\]

5.1. Mixed stress–strain loading histories

In this section a number of three biaxial non-proportional stress–strain loading histories are considered. The loading histories, which are portrayed in Figure 11 are obtained assuming to control two strain components and requiring that all the stress components not corresponding to the two controlled strains are identically equal to zero. In particular, the following are the controlled strain

| Table III. Material constants adopted for the numerical tests. |
|-------------------|----------------|-----------------|-----------------|-----------------|
| \( E \) (MPa) | \( \nu \) (dimensionless) | \( \sigma_{y,0} \) (MPa) | \( H_{iso} \) (MPa) | \( H_{kin} \) (MPa) |
| Material 1 | \( 3 \times 10^4 \) | 0.3 | 3 | 0 | 0 |
| Material 2 | \( 7 \times 10^3 \) | 0.3 | 24.3 | 225 | 50 |

Figure 11. Pointwise stress–strain tests. Mixed stress–strain load histories for Problem 1, Problem 2 and Problem 3.

components:

Problem 1: $\varepsilon_{11} \quad \varepsilon_{12}$
Problem 2: $\varepsilon_{11} \quad \varepsilon_{22}$
Problem 3: $\varepsilon_{11} \quad \varepsilon_{12}$
It is noted that Problem 1 and Problem 3 are set controlling the same strain components, which evolve according to different loading histories as reported in Figure 11. In particular, the controlled strain components vary proportionally to

$$\varepsilon_{y,\text{mono}} = \sqrt{\frac{3}{2}} \varepsilon_{y,0}$$

which represents the first yielding strain in a uniaxial strain-controlled loading history.

5.1.1. Instantaneous error. The computation of the instantaneous error permits to evaluate the error granted by an algorithm in the stress–strain computation for the solution of a loading history at each integration instant. The error is evaluated separately for the stress and the strain, in order to appreciate the dependence of the error on the time step size $\Delta t$. To this end, the following relative norms are introduced:

$$E_n^\sigma = \frac{\|\sigma_n - \sigma_n^{\text{ex}}\|}{\sigma_{y,n}}, \quad E_n^e = 2G \frac{\|\varepsilon_n - \varepsilon_n^{\text{ex}}\|}{\sigma_{y,n}}$$

where $\| \cdot \|$ indicates the usual Euclidean norm and $\sigma_{y,n}$ is the yield surface radius at time $t_n$. In Equations (60), $\sigma_n^{\text{ex}}$ and $\varepsilon_n^{\text{ex}}$ are, respectively, the stress and the strain reference solution at time $t_n$ computed using the BE scheme with a very fine time discretization, corresponding to $\Delta t = 10^{-5}$ s. The quantities $\sigma_n$ and $\varepsilon_n$ represent the stress and strain tensors, respectively, computed adopting the prescribed time discretizations ($\Delta t = 0.1, 0.05, 0.025$ s).

In Figure 12 the stress and strain instantaneous error plots are presented. In each diagram an error curve, corresponding, respectively, to the BE, SMPT1, SMPT2, DMPT1 and DMPT2 method, is reported. Figure 13 shows the same curves only for the SMPT1, DMPT1 and DMPT2 schemes for a clearer visualization and comparison. The mentioned plots refer to the instantaneous stress and strain errors (60) using Material 1 for Problem 2.

Figures 14 and 15 report systematically the same error curves, using Material 2 for Problem 1. Only a limited number of instantaneous error graphs are presented since they provide sufficient information for comparative purposes. The graphs reported in Figures 12–15 imply the following considerations:

- The BE method is linearly accurate, i.e. the error is proportional to the step size $\Delta t$.
- The SMPT1, DMPT1 and DMPT2 methods are quadratically accurate, i.e. the error is proportional to the square of $\Delta t$.
- The SMPT1 and the DMPT1 methods are equivalent in terms of error. This leads to deduce that no problem of existence of solution is encountered in the simulations carried out during the present pointwise test cases involving the SMPT1 algorithm.
- The SMPT2 method does not show a sharp second order accuracy. The presence of spurious oscillation in the error curves, especially at the time instants following the change of direction in the driving input (cf. Figure 11), makes it harder to recognize a pure quadratic pattern rate of decrease of the instantaneous error with respect to the step size. This phenomenon can be explained as follows. Being the SMPT2 method endpoint inconsistent, mixed elastoplastic steps occur frequently during a loading history. This leads to a linear order of accuracy since during such steps, due to the discontinuous nature of the model, the method looses quadratic accuracy and grants simply first order accuracy. In the following section this behaviour is
readily appreciated by means of measuring the total average error produced by the method over the loading history.

- The DMPT2 scheme results are comparable to the SMPT1 and DMPT1 for Material 1 (cf. Figure 13), while it is the most performing for the simulation involving Material 2 (cf. Figure 15).

5.1.2. Total error. The scaled $\ell^1$ norm in time of the absolute stress and strain error

$$E_T^\sigma = \frac{\Delta t}{T} \sum_{n=1}^{N} \frac{\|\sigma_n - \sigma_n^{ex}\|}{\sigma_{y,n}}, \quad E_T^\varepsilon = 2G \frac{\Delta t}{T} \sum_{n=1}^{N} \frac{\|\varepsilon_n - \varepsilon_n^{ex}\|}{\sigma_{y,n}}$$

leads to the average total error corresponding to a pointwise loading history. In Figure 16 we plot the total stress and strain error versus the number of time steps in logarithmic scale for the five methods BE, SMPT1, SMPT2, DMPT1 and DMPT2, for Problem 1 with Material 2.
Figure 13. Pointwise stress–strain tests: Problem 2 with Material 1. Stress and strain instantaneous error for $\Delta t = 0.1, 0.025 \text{ s}$. 

comparison purposes, also the total error corresponding to the \( \text{ESC}^2 \) method, a recently developed exponential-based integration algorithm [8], is reported in the same figure.

The quadratic convergence of the SMPT1, DMPT1 and DMPT2 methods, with respect to the linear one of the BE algorithm is evident. Still the SMPT1 and the DMPT1 methods produce the same error. The \( \text{ESC}^2 \) scheme is quadratically accurate.

The SMPT2 method shows quadratic accuracy but not sharply. In fact, as the time step size decreases, the accuracy order tends to decrease down to a linear rate, as can be observed in Figure 16. This can be explained recalling the arguments discussed above for the accuracy order decrease phenomenon observed in the pointwise tests.

It is noted that the DMPT2 method exhibits the lowest error levels within the midpoint methods set.

The \( \text{ESC}^2 \) method as can be checked in the previously mentioned figures produce competitive error levels even though it is to be remarked that this method implies a higher computational cost with respect to the midpoint schemes.
Figure 14. Pointwise stress–strain tests: Problem 1 with Material 2. Stress and strain instantaneous error for $\Delta t = 0.1, 0.025$ s.

5.2. Iso-error maps

Iso-error maps are commonly adopted in the literature [1, 5, 22, 23] as a systematic tool to test the precision of integration algorithms for elastoplastic models. In particular, they render a clear evaluation of the integration algorithm accuracy when a large time discretization is adopted. Error maps are plotted as a result of particular mixed stress–strain loading histories which are piecewise linear in time.

Each loading history is set up by controlling, for example, the $\varepsilon_{11}$ and $\varepsilon_{22}$ strain components and requiring all the remaining stresses to be equal to zero. The evolution in time of the controlled quantities is piecewise linear and can be divided into two distinct phases: Phase 1 and Phase 2, defined as follows (see Table IV). Phase 1 consists of a purely elastic path and proceeds from the zero stress and strain state (State 0) to a specific state on the yield surface (State 1) given in terms of the yield strain components $\varepsilon_{11,y}$ and $\varepsilon_{22,y}$. Phase 2 is a purely plastic path which starts from State 1 and leads to a final state (State 2) given in terms of the strain increments $\Delta_{\varepsilon_{11}}$ and $\Delta_{\varepsilon_{22}}$.  

As commonly done, we consider three different choices of State 1, corresponding to plane states of stress on the yield surface [5], labelled A, B and C, respectively, represented in Figure 17 and corresponding to uniaxial, biaxial and pure shear states. Each State 1 is expressed in Table V in terms of the quantity $\varepsilon_{y,\text{mono}}$ defined in (59) according to

$$\varepsilon_{11} = \varepsilon_{11,\text{y}}$$

$$\varepsilon_{22} = \varepsilon_{22,\text{y}}$$

For each choice of State 1, we then consider a State 2 defined as

$$\varepsilon_{11} = \varepsilon_{11,\text{y}} + \Delta \varepsilon_{11}$$

$$\varepsilon_{22} = \varepsilon_{22,\text{y}} + \Delta \varepsilon_{22}$$
Table IV. Benchmark mixed stress–strain history for iso-error maps computation.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>$\varepsilon_{11}$</th>
<th>$\varepsilon_{22}$</th>
<th>$\sigma_{33}$</th>
<th>$\sigma_{12}$</th>
<th>$\sigma_{13}$</th>
<th>$\sigma_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 0</td>
<td>$t = 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 1</td>
<td>$t = 1$</td>
<td>$\varepsilon_{11,y}$</td>
<td>$\varepsilon_{22,y}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 2</td>
<td>$t = 2$</td>
<td>$\varepsilon_{y,11} + \Delta \varepsilon_{11}$</td>
<td>$\varepsilon_{y,22} + \Delta \varepsilon_{22}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 16. Pointwise stress–strain tests: Problem 1 with Material 2. Stress (upper diagram) and strain (lower diagram) total error versus number of steps per second.

Figure 17. Plane stress von-Mises yield surface representation in principal stresses plane.
State 1 choices for iso-error map plots.

Table V. Iso-error maps. Choices for the State 1 point on the yield surface.

<table>
<thead>
<tr>
<th>State 1</th>
<th>$\varepsilon_{11,y}$</th>
<th>$\varepsilon_{22,y}$</th>
<th>$\sigma_{33}$</th>
<th>$\sigma_{12}$</th>
<th>$\sigma_{13}$</th>
<th>$\sigma_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1 A</td>
<td>$\varepsilon_{y,\text{mono}}$</td>
<td>$-\nu \varepsilon_{y,\text{mono}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 1 B</td>
<td>$(1-\nu) \varepsilon_{y,\text{mono}}$</td>
<td>$\varepsilon_{y,\text{mono}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 1 C</td>
<td>$(1+\nu)^{3} \varepsilon_{y,\text{mono}}$</td>
<td>$-\frac{(1+\nu)}{3} \varepsilon_{y,\text{mono}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We solve a total of $120 \times 120$ mixed stress–strain histories for each State 1, corresponding to the following sets of normalized strain increments (see Figures 18–20)

$$\frac{\Delta \varepsilon_{11}}{\varepsilon_{11,y}} = -6.0, -5.9, \ldots, 6.0$$

$$\frac{\Delta \varepsilon_{22}}{\varepsilon_{22,y}} = -6.0, -5.9, \ldots, 6.0$$

This subdivision leads to a total of 14,400 computed mixed stress–strain loading histories and to an equal number of calculated relative error values. According to [2, 5], as an error measure, we adopt the following expression:

$$E_{\text{iso}}^{\sigma} = \frac{\|\sigma - \sigma^{\text{ex}}\|}{\|\sigma^{\text{ex}}\|}$$ (62)
where $\sigma$ is the final stress tensor, computed adopting a single time step discretization between State 1 and State 2, whereas $\sigma^\text{ex}$ corresponds to an ‘exact’ solution adopting a very fine time step between State 1 and State 2. All the calculations refer to Material 2 [5].
Figure 19. Iso-error maps for yield surface State 1 BE and indication of the maximum stress error level.

The total error range is subdivided into 10 equally spaced levels according to which the iso-curves are drawn in Figures 18–20. Each iso-curve is indicated by a proper error label while the thick continuous line represents the couples of strain increment values corresponding to proportional loading histories starting from State 1. For the sake of completeness, aside from each map we also
Figure 20. Iso-error maps for yield surface State 1 C and indication of the maximum stress error level.

report the maximum error value computed on the grid adopted for the computation of the iso-error map. Observing Figures 18–20 we can derive the following conclusions:

- As expected from the considerations in Section 4.3, in all the three States 1 A, B, C the first order BE method shows significant lower error levels, for large time steps, with respect to
the SMPT1, DMPT1 and SMPT2 schemes. On the other hand, the DMPT2 method gives error levels which are comparable to those of the BE method. In particular, while the DMPT2 scheme performance for large time steps is completely satisfactory, the other three MPT schemes show relative error levels up to 100%.

- The five analysed schemes grant zero error for the loading histories corresponding to a strain increment indicated by the dotted lines in Figure 18. This is due to the fact that these paths correspond to proportional loadings, i.e. loading histories for which the normalized stress tensor does not vary during the process. Being the starting point State 1 \( A \) not an intersection of the yield surface with one of its axes of symmetry, in such state the above behaviour is registered only for strain paths corresponding to the \textit{outward} normal direction. Instead, in States 1 \( B \), \( C \), the BE, SMPT1, DMPT1, and DMPT2 schemes maintain such exactness property also for proportional loading histories in the opposite direction. This observation does not hold for the SMPT2 method due to the intrinsic inconsistency of the algorithm.

5.3. Initial boundary-value problem

We consider an initial boundary-value problem regarding the elongation of a rectangular strip with a circular hole in the centre, assuming plane strain regime \([5]\). The strip possesses three planes of symmetry mutually orthogonal. Then, as can be seen in Figure 21(a), only a quarter of it is sufficient to define its geometry. The lengths referred to Figure 21(a) are

\[
B = 100 \text{ mm}, \quad H = 180 \text{ mm}, \quad B_0 = 50 \text{ mm}
\]

while the thickness is 10 mm.

Initially the strip is undeformed and unstressed. The problem loading history results symmetric with respect to the three symmetry planes of the strip, hence only a single quarter of the domain
may be studied, referring to the following equivalent problem. The loading history is composed of a first phase (1 s), in which the strip is stretched assigning the horizontal top side a vertical displacement up to a value $\delta_{\text{max}}$ and a second phase (1 s) in which the imposed displacement is set back to zero. The controlled displacement varies linearly in time as reported in Figure 22 and $\delta_{\text{max}} = 1 \text{ mm}$. The boundary conditions due to symmetry impose to block the horizontal displacement on the left side and to block the vertical displacement on the bottom side of the strip. The circular side and the vertical right side are stress-free. The analysis refer to Material 2.

The problem is modelled and solved with the finite element code FEAP [20], in which the considered integration methods are implemented. The mesh is composed of $N_{\text{el}}$ finite elements (displacement-based four-node SOLID2D elements adopting a four point Gauss quadrature rule [20]). The following results refer to a mesh with $N_{\text{el}} = 192$ as can be observed in Figure 21(b).

The comparison between the methods is carried out evaluating the following $L^2$ norm error on stress and strain

$$
\tilde{E}^\sigma = \sqrt{\frac{\int_\Omega \| \sigma_n - \sigma_n^{\text{ex}} \|^2}{\int_\Omega \| \sigma_n^{\text{ex}} \|^2}} = \sqrt{\frac{\sum_{n=1}^{N_{\text{el}}} \sum_{q=1}^{4} w_{nq} \| \sigma_{nq} - \sigma_{nq}^{\text{ex}} \|^2}{\sum_{n=1}^{N_{\text{el}}} \sum_{q=1}^{4} w_{nq} \| \sigma_{nq}^{\text{ex}} \|^2}}
$$

(63)

$$
\tilde{E}^\epsilon = \sqrt{\frac{\int_\Omega \| \epsilon_n - \epsilon_n^{\text{ex}} \|^2}{\int_\Omega \| \epsilon_n^{\text{ex}} \|^2}} = \sqrt{\frac{\sum_{n=1}^{N_{\text{el}}} \sum_{q=1}^{4} w_{nq} \| \epsilon_{nq} - \epsilon_{nq}^{\text{ex}} \|^2}{\sum_{n=1}^{N_{\text{el}}} \sum_{q=1}^{4} w_{nq} \| \epsilon_{nq}^{\text{ex}} \|^2}}
$$

(64)

The quantities $\sigma_n$ and $\epsilon_n$ are, respectively, the stress and strain tensors at time $t_n$, calculated with the integration scheme adopting a prescribed time step $\Delta t$, while $\sigma_n^{\text{ex}}$ and $\epsilon_n^{\text{ex}}$ are the corresponding reference solutions (BE scheme) using a time step $\Delta t = 10^{-5} \text{ s}$. In the following the time step size is, respectively, $\Delta t_1 = 0.05 \text{ s}$, $\Delta t_2 = 0.025 \text{ s}$, $\Delta t_3 = 0.0125 \text{ s}$. The norms (63) and (64) are evaluated at four different instants of the loading history, corresponding to $t_n = 0.5$, 1.0, 1.5, 2.0 s as can be observed in Figure 22. The results are summarized in Table VI.
Inspecting Table VI, the following issues may be deduced:

- The BE method, which serves as a reference, presents linear accuracy.
- The SMPT1 and the DMPT1 are quadratically accurate and produce the same error levels. This equals to say that the simulation under investigation did not involve non-converging cases for the SMPT1.
- The SMPT2 method still loses its theoretical quadratic accuracy. As already pointed out, the reason for this occurrence can be traced in the endpoint inconsistency of the method.
- The DMPT2 integration scheme has a second order accuracy and again presents the lowest error levels. The DMPT2 and the SMPT1/DMPT1 grant comparable levels of error.

Table VI. Boundary-value problem, Stress and strain $L^2$ norm errors with Material 2 at $t = 0.5$, 1.0, 1.5, 2.0 s.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\Delta t$</th>
<th>$\Delta E^\sigma$</th>
<th>$\Delta E^\tau$</th>
<th>$\Delta E^\sigma$</th>
<th>$\Delta E^\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.5$ s</td>
<td>$\Delta t_1$</td>
<td>$1.74 \times 10^{-3}$</td>
<td>$1.29 \times 10^{-4}$</td>
<td>$1.02 \times 10^{-2}$</td>
<td>$1.29 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_2$</td>
<td>$9.23 \times 10^{-4}$</td>
<td>$3.12 \times 10^{-5}$</td>
<td>$46.61 \times 10^{-3}$</td>
<td>$3.12 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_3$</td>
<td>$4.76 \times 10^{-4}$</td>
<td>$9.02 \times 10^{-6}$</td>
<td>$2.82 \times 10^{-3}$</td>
<td>$9.02 \times 10^{-6}$</td>
</tr>
<tr>
<td>$1.0$ s</td>
<td>$\Delta t_1$</td>
<td>$2.16 \times 10^{-3}$</td>
<td>$2.86 \times 10^{-4}$</td>
<td>$8.84 \times 10^{-3}$</td>
<td>$2.86 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_2$</td>
<td>$1.17 \times 10^{-3}$</td>
<td>$7.06 \times 10^{-5}$</td>
<td>$6.86 \times 10^{-3}$</td>
<td>$7.06 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_3$</td>
<td>$6.03 \times 10^{-4}$</td>
<td>$2.24 \times 10^{-5}$</td>
<td>$1.13 \times 10^{-3}$</td>
<td>$2.24 \times 10^{-5}$</td>
</tr>
<tr>
<td>$1.5$ s</td>
<td>$\Delta t_1$</td>
<td>$2.94 \times 10^{-3}$</td>
<td>$1.19 \times 10^{-4}$</td>
<td>$1.53 \times 10^{-2}$</td>
<td>$1.19 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_2$</td>
<td>$1.51 \times 10^{-3}$</td>
<td>$3.01 \times 10^{-5}$</td>
<td>$7.59 \times 10^{-3}$</td>
<td>$3.01 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_3$</td>
<td>$7.67 \times 10^{-4}$</td>
<td>$7.80 \times 10^{-6}$</td>
<td>$3.52 \times 10^{-3}$</td>
<td>$7.80 \times 10^{-6}$</td>
</tr>
<tr>
<td>$2.0$ s</td>
<td>$\Delta t_1$</td>
<td>$2.36 \times 10^{-4}$</td>
<td>$2.78 \times 10^{-4}$</td>
<td>$1.12 \times 10^{-2}$</td>
<td>$2.78 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_2$</td>
<td>$1.19 \times 10^{-5}$</td>
<td>$5.73 \times 10^{-5}$</td>
<td>$4.97 \times 10^{-2}$</td>
<td>$5.73 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_3$</td>
<td>$5.97 \times 10^{-5}$</td>
<td>$1.65 \times 10^{-5}$</td>
<td>$1.38 \times 10^{-3}$</td>
<td>$1.65 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
6. CONCLUDING REMARKS

We developed a complete comparison of the generalized midpoint return mapping schemes for the classical $J_2$ plasticity model with linear isotropic and kinematic hardening.

Extending and completing the existing results in the literature, we first addressed theoretical properties such as the existence of solution, accuracy, stability and yield consistency of the methods. Afterwards we developed an extensive set of numerical tests, based on pointwise stress–strain loading histories, iso-error maps and initial boundary-value problems.

Both on the basis of the theory and the tests, the DMPT2 methods seem, at least for the model here investigated, undoubtedly the best performing generalized midpoint method. First of all, it is the only method among the four considered which is both B-stable and yield consistent. Moreover, the DMPT2 method performs considerably better than the other three schemes in all the considered tests: pointwise mixed loading histories, iso-error maps and boundary-value problems. Finally, it is worth to note that the DMPT2 scheme is based on the combination of radial return maps, which is an advantage for the extension to more complicated models.

It is also noted that the SMPT2, due to its B-stability and symmetry of the tangent operator, is one of the most popular schemes in the scientific literature. On the other hand, the tests presented here show that, due to its endpoint yield inconsistency, such method essentially loses the quadratical convergence, which is the main advantage of midpoint-type methods.

APPENDIX A

In this section we present the exact statements and proofs for two previously mentioned results. In the following proofs, we will indicate two tensors $A, B$ as parallel, or equivalently use the mathematical sign $A \parallel B$, if $A$ and $B$ are equal up to a multiplicative constant.

**Proposition A.1**

For the DMPT2 scheme it holds

$$\| \Sigma_{n+1} \| = \sigma_{y,n+1}$$

at the end of all plastic time steps, i.e. whenever $\dot{\lambda}_1 + \dot{\lambda}_2 > 0$.

**Proof**

The result is trivial whenever $\dot{\lambda}_1 = 0$. In such case either (41) holds, and the step is purely elastic, or it does not. If the step is purely elastic, there is nothing to prove. Otherwise, i.e. if (41) is not satisfied, then the radial return map projection (42)–(43) applied in step 2 guarantees that the strict yield condition (A1) holds.

Conversely, if $\dot{\lambda}_1 > 0$, it has to be proved that the condition

$$\| \Sigma_{n+1}^{\text{TR}} \| < \sigma_{y,n+1}^{\text{TR}} = \sigma_{y,n} + 2 \dot{\lambda}_1 H_{iso}$$

is never satisfied. Indeed, if (A2) holds, the scheme stops and we end up with a plastic step in which (A1) is not satisfied. Conversely, if (A2) does not hold, the presence of the return map projection (42)–(43) in the scheme guarantees that (A1) is satisfied. We therefore have to
show that
\[
\| \Sigma^{TR}_{n+1} \| \geq \sigma_{y,n} + 2\lambda_1 H_{iso} \quad (A3)
\]
whenever (29) does not hold, i.e. whenever \( \Sigma^{}_{n+1/2} \) is out of the yield surface.

We start observing that, due to the identities (32) and (30)\(_4\), \( \lambda_1 \) and \( \Sigma^{}_{n+1/2} \) are uniquely defined by \( \Sigma^{TR}_{t+1/2} \), \( \sigma_{y,n} \) and the material constants. In particular, \( \Sigma^{}_{n+1/2} \) and the right-hand side of (A3) do not depend on \( \Sigma_n \) once \( \Sigma^{TR}_{n+1/2} \) is assigned.

Let \( \Sigma^*_n \) represent the radial projection of \( \Sigma^{TR}_{n+1/2} \) on the yield surface at time \( t_n \), i.e. at the start of the time step. Note that, from the definition of radial projection and recalling (30)\(_4\), (31), one finds
\[
\Sigma^*_n \parallel \Sigma^{TR}_{n+1/2} \parallel \Sigma^{}_{n+1/2} \quad (A4)
\]
Therefore, due to (A4), \( \Sigma^*_n \) represents also the radial projection of \( 2\Sigma^{}_{n+1/2} \) on the yield surface at time \( t_n \). As a consequence, observing that any admissible \( \Sigma_n \) lays in the yield surface at time \( t_n \), we get
\[
\| \Sigma^{TR}_{n+1} \| = \| 2\Sigma^{}_{n+1/2} - \Sigma_n \| \geq 2\| \Sigma^{}_{n+1/2} - \Sigma^*_n \| \quad (A5)
\]
Due to (A5), recalling that \( \Sigma^{}_{n+1/2} \) and the right-hand side of (A3) do not depend on \( \Sigma_n \), it is sufficient to prove (A3) for the particular case \( \Sigma_n = \Sigma^*_n \); any other choice of \( \Sigma_n \) would give in (A3) the same right-hand side and a larger left-hand side.

Assume now that \( \Sigma_n = \Sigma^*_n \). In such case, from (A4), it follows immediately that \( \Sigma^{}_{n+1/2} \) and \( \Sigma^{}_{n+1/2} = \Sigma^*_n \) are parallel. As a consequence, it holds
\[
\| \Sigma^{TR}_{n+1} \|^2 = \| 2\Sigma^{}_{n+1/2} - \Sigma_n \|^2 = (2\| \Sigma^{}_{n+1/2} \| - \| \Sigma_n \|)^2 \quad (A6)
\]
Observing that by hypothesis \( \lambda_1 > 0 \), the return map projection applied in step 1 of the scheme guarantees
\[
\| \Sigma^{}_{n+1/2} \| = \sigma_{y,n+1/2} = \sigma_{y,n} + \lambda_1 H_{iso} \quad (A7)
\]
From (A6), (A7) and noting that by definition \( \| \Sigma^*_n \| = \sigma_{y,n} \), it follows:
\[
\| \Sigma^{TR}_{n+1} \|^2 = [2(\sigma_{y,n} + \lambda_1 H_{iso}) - \sigma_{y,n}]^2 = (\sigma_{y,n} + 2\lambda_1 H_{iso})^2 \quad (A8)
\]
The proposition is proved. \( \square \)

**Proposition A.2**

The SMPT1 and DMPT1 methods are equivalent, provided both schemes have a solution.

It has to be to proved that, given the same initial history variables at time \( t_n \), the corresponding values obtained at time \( t_{n+1} \) for the two schemes are the same. From (16)\(_3\) and (30)\(_4\) it is noted that
\[
n^\text{SMPT1}_{n+1/2} = n^\text{DMPT1}_{n+1/2} \quad (A9)
\]
because both tensors are parallel to \( \Sigma^{TR}_{n+1/2} = s^{}_{n+1/2} - \alpha_n \) and have unitary norm. Given (A9), the solutions of the two systems (17) and (38), combined with the same yield condition \( F(\Sigma^{}_{n+1}) = 0 \), are the same. \( \square \)
In this appendix we report the consistent tangent operators for the SMPT and DMPT methods in concise form. Further details on the derivation can be found in [24].

To make notation more clear, the subscripts of all history variables evaluated at time $t_{n+1}$ are omitted for brevity. Quantities evaluated either at $t_n$ or at $t_{n+z}$ are specified by the relative subscript.

### B.1. SMPT1

Taking into account the volumetric part of the stress, from Equations (1)–(3) it is found that the tangent operator has the following expression:

$$
\mathbb{D}_{\text{disc}}^{\alpha} = \frac{\partial \sigma}{\partial \varepsilon} = \frac{\partial s}{\partial e} \mathbb{I}_{\text{dev}} + K(I \otimes I) \tag{B1}
$$

where $\mathbb{I}$ is the fourth order identity tensor and

$$
\mathbb{I}_{\text{dev}} = \mathbb{I} - \frac{1}{3}(I \otimes I) \tag{B2}
$$

Recalling that

$$
\frac{\partial s}{\partial e} = \frac{\partial s}{\partial e} \frac{\partial e}{\partial e} = \frac{\partial s}{\partial e} \mathbb{I}_{\text{dev}} \tag{B3}
$$

one finds

$$
\frac{\partial s}{\partial e} = 2G(\mathbb{I} - n_{n+z} \otimes \frac{d \lambda}{de} + \kappa \mathbb{I}_n) \tag{B4}
$$

with

$$
\frac{d \lambda}{de} = \beta \left( 2G \mathbb{I} - \frac{Y \lambda}{\|\Sigma_{n+z}\|} n_{n+z} \otimes n_{n+z} \right) \Sigma \tag{B5}
$$

$$
\mathbb{I}_n = \mathbb{I} - n_{n+z} \otimes n_{n+z} \tag{B6}
$$

and

$$
\beta = [H_{\text{iso}} \sigma_{y,n+1} + (2G + H_{\text{kin}})(\Sigma_{n+1} : n_{n+z})]^{-1} \tag{B7}
$$

$$
\kappa = -\frac{2G \lambda}{\|\Sigma_{n+z}\|^2 + 2G \lambda - H_{\text{kin}} \lambda} \tag{B8}
$$

It is noted that the operator (B1) is not symmetric.

### B.2. SMPT2

In this case we get

$$
\mathbb{D}_{\text{disc}} = K(I \otimes I) + 2G(1 - C)\mathbb{I}_{\text{dev}} + [2G(C - A)]n_{n+z} \otimes n_{n+z} \tag{B8}
$$
with

\[ A = \frac{2G}{2G + H_{\text{iso}} + H_{\text{kin}}} \]

\[ C = \frac{2G \lambda}{\|\Sigma^{\text{TR}}_{n+2}\|} \]

It is noted that the operator (B8) is symmetric.

**B.3. DMPT1**

In this case it holds

\[ \frac{\partial s}{\partial e} = 2G \left( \mathbb{I} - \mathbf{n}_{n+2} \otimes \frac{\partial \lambda_2}{\partial e} - \lambda_2 \frac{\partial \mathbf{n}_{n+2}}{\partial e} \right) \quad \text{(B9)} \]

where

\[ \frac{\partial \lambda_2}{\partial e} = \theta_1^{-1} \left[ 2G \boldsymbol{\Sigma} - (2G + H_{\text{kin}}) \lambda_2 \frac{\partial \mathbf{n}_{n+2}}{\partial e} \right] \quad \text{(B10)} \]

\[ \frac{\partial \mathbf{n}_{n+2}}{\partial e} = 2G \mathbf{x}(1 - C)\|\Sigma_{n+2}\|^{-1} \mathbf{n} \quad \text{(B11)} \]

with

\[ \theta_1 = [H_{\text{iso}} \sigma_{y,n+1} + (2G + H_{\text{kin}})(\Sigma : \mathbf{n}_{n+2})] \]

\[ C = (2G + H_{\text{kin}}) \lambda_1 \|\Sigma_{n+2}\|^{-1} \]

\[ A = \frac{2G + H_{\text{kin}}}{2G_1} \]

It is noted that the operator (B9) is not symmetric.

**B.4. DMPT2**

It is found that

\[ \frac{\partial s}{\partial e} = 2G \left( \mathbb{I} - \mathbf{n} \otimes \frac{\partial \lambda_2}{\partial e} - \lambda_2 \frac{\partial \mathbf{n}}{\partial e} \right) \quad \text{(B12)} \]

where

\[ \frac{\partial \lambda_2}{\partial e} = \frac{1}{2G + H_{\text{iso}} + H_{\text{kin}}} \left( \frac{\partial \Sigma^{\text{TR}}}{\partial e} \mathbf{n} \right) \quad \text{(B13)} \]
\[
\frac{\partial \mathbf{n}}{\partial \epsilon} = \| \mathbf{\Sigma}^{\text{TR}} \|^{-1} \mathbf{n} \left( \frac{\partial \mathbf{\Sigma}^{\text{TR}}}{\partial \epsilon} \right) \tag{B14}
\]

\[
\frac{\partial \mathbf{\Sigma}^{\text{TR}}}{\partial \epsilon} = [2G(1 - C) \mathbb{1} + 2G(C - A) \mathbf{n}_{n+z} \otimes \mathbf{n}_{n+z}] \tag{B15}
\]

with

\[
\mathbb{1}_n = \mathbb{1} - \mathbf{n} \otimes \mathbf{n}
\]

\[
C = (2G + H_{\text{kin}}) \mathbb{1} \mathbb{1}_1 \left\| \mathbf{\Sigma}_{n+z}^{\text{TR}} \right\|^{-1}
\]

\[
A = \frac{2G + H_{\text{kin}}}{2G_1}
\]

It is noted that operator (B12) is not symmetric.

REFERENCES


