ISOGEOMETRIC DISCRETIZATIONS IN STRUCTURAL DYNAMICS AND WAVE PROPAGATION

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Keywords: Isogeometric Analysis, NURBS, Structural Vibrations, Discrete Spectrum, Wave Propagation, Dispersion Relation

Abstract. We examine isogeometric discretizations in structural dynamics and wave propagation, and compare with classical p-method finite elements. In the case of structural dynamics we study free vibration (i.e., eigenvalue) problems as a basis for comparison, whereas in the case of wave propagation we investigate the time-harmonic case by way of classical dispersion analysis. We derive a fundamental relationship between free vibration and dispersion analysis that we articulate in terms of a “duality principle”. The results clearly show the superiority of the isogeometric approach utilizing the “k-method” of smooth refinement with NURBS over classical “p-method” finite elements, which are only C⁰-continuous.

We also illustrate the modeling capabilities of the isogeometric approach on more significant physical problems. In particular, we compare with experimental vibration results for the NASA Aluminum Testbed Cylinder. The three-dimensional isogeometric model in this case represents every geometrical feature exactly.
1 INTRODUCTION

Isogeometric Analysis, introduced by Hughes et al. [12], and expanded by Cottrell et al. [7] and Bazilevs et al. [1], is a new approach for the analysis of problems governed by partial differential equations. It is a generalization of classical finite element analysis (containing it as a special case) and has many features in common with it. However, it is more geometrically based and takes inspiration from Computer Aided Design (CAD). A primary goal of isogeometric analysis is to be geometrically precise no matter how coarse the discretization. Another goal is to simplify mesh refinement by eliminating the need for communication with the CAD geometry once the initial mesh is constructed. Moreover, it has also been shown in [7] that isogeometric analysis constitutes an important tool for studying structural vibrations, as it provides very good spectrum approximations for many cases of interest.

In this paper, we deal with structural dynamic and wave propagation problems, which are briefly presented in Section 2. For these problems, isogeometric analysis (which is outlined in Section 3) is compared to classical finite element analysis. For mathematical analysis, which is addressed in Section 4 in one space dimension, we introduce a “duality principle”, relating the two fields (free vibration and wave propagation) and unifying the theory. Numerical tests, which complement the theory, are presented in Section 5. They show the superiority of the “k-method” isogeometric approach (where smooth refinement is achieved using NURBS) over “p-method” finite elements (where only $C^0$-continuity is attained). In fact, we show that the so-called “optical branches” of spectra diverge with $p$ in the classical finite element method, a surprisingly negative result, which demonstrates that higher-order finite elements have no approximability for higher modes in vibration analysis, and explains the fragility of higher-order finite element methods in nonlinear and dynamic applications in which higher modes necessarily participate. However, optical branches are eliminated in the $k$-method isogeometric approach, a significant advantage and one that suggests its potential in the case where traditional higher-order finite element methods have failed.

In Section 6, we illustrate all the potential of isogeometric analysis for the the study of structural vibrations of complex real structures. In particular, we consider the NASA Aluminum Testbed Cylinder, whose geometrical features are exactly represented by a three-dimensional isogeometric model. Very good correlation of numerical and experimental results is observed.

2 STRUCTURAL VIBRATIONS AND WAVE PROPAGATION

In this section we briefly recall the main equations of structural vibrations and of wave propagation; for elaboration, see Chopra [5], Clough and Penzien [6], and Hughes [11] for structural vibrations; and Thompson and Pinsky [16], and Thompson [17] for wave propagation (note that in these reviews particular emphasis is on acoustics).

2.1 Structural vibrations: natural frequencies and modes

Given a linear multi-degree-of-freedom structural system, the undamped, unforced equations of motion, which govern free vibrations, are

$$M\ddot{u} + Ku = 0,$$

where $M$ and $K$ are, respectively, the consistent mass and the stiffness matrices, $u = u(t)$ is the displacement vector and $\ddot{u} = \frac{d^2u}{dt^2}$ is the acceleration vector. The $n^{th}$ normal mode, $\phi_n$, is
obtained from \( u(t) \) by separation of variables as follows

\[
u(t) = \phi_n e^{i\omega_n t},\]

where \( \omega_n \) is the \( n^{th} \) natural frequency. Combining equations (1) and (2) leads to a generalized eigenproblem, which consists of finding the eigenvalue \( \omega_n^2 \) and the associated eigenvector \( \phi_n \neq 0 \) such that

\[
(K - \omega_n^2 M)\phi_n = 0.\]

(3)

The normal modes obtained from equation (3) are defined up to a multiplicative constant. Different ways of normalization have been proposed. The most widely used is \( \phi_n^T M \phi_n = 1 \).

Equation (1) describes the equations of motion for a discretized structural system (i.e., the number of degrees-of-freedom is finite). Such discretized systems are used to study the behavior of real continuous ones (with an infinite number of vibration modes) approximating only a finite number of modes (equal to the total number of the employed degrees-of-freedom), possibly all those which in practice are important when studying the dynamics of the real problem. So, a fundamental issue is how close the discrete modes are to the continuous ones, that is, how good is the spectrum approximation of the discrete method compared with the exact spectrum.

### 2.2 Wave propagation: the Helmholtz equation

The classical equation governing wave propagation is

\[
\nabla^2 u - \frac{1}{c^2} \ddot{u} = 0,
\]

(4)

where \( c \) is the wave propagation speed.

Assuming time-harmonic solutions, the linear wave equation (4) reduces to the Helmholtz equation in the frequency domain

\[
\nabla^2 u + k^2 u = 0,
\]

(5)

where \( k = \omega/c \) is the wavenumber and \( \omega \) is the angular frequency.

Solutions of the Helmholtz equation (5) are plane waves of frequency \( \omega \) traveling in the direction \( \mathbf{n} \) at a speed \( c \), which can be expressed as the time-harmonic wave train

\[
u(x, t) = Re\{A e^{i(kn \cdot x - \omega t)}\},
\]

(6)

where \( A = |A| e^{i\varphi} \) is a complex number with modulus \( |A| \) and phase \( \varphi \). The wavelength (with units of length) is defined by \( \lambda = 2\pi/k \), while the dual measure of period (with units of time) is defined by \( T = 2\pi/\omega \).

After discretization, equation (5) gives rise to

\[
(K - k^2 M)u = 0.
\]

(7)

The numerical solution of the above equation is a linear combination of plane waves having numerical wavenumber \( k^h \), where, in general, \( k^h \neq k \). Assuming a uniform discretization with elements of characteristic length \( h \), a fundamental issue is to determine the dispersion of a numerical method, that is, how close the discrete wavenumber \( k^h \) is to its continuous counterpart \( k \).
3 NURBS-BASED ISOGEOOMETRIC ANALYSIS

Non-Uniform Rational B-Splines (NURBS) are a standard tool for describing and modeling curves and surfaces in computer aided design and computer graphics (see Piegl and Tiller [14] and Rogers [15] for an extensive description of these functions and of their properties). In this work, we use NURBS as an analysis tool, referred to as “isogeometric analysis”, recently introduced by Hughes et al. [12]. The aim of this section is to present a brief overview of features and properties of this new approach, that we will apply to the problems introduced in the previous sections.

3.1 Isogeometric analysis

In the following we present a summary of the main features of isogeometric analysis. The interested reader may find more details and applications in [1, 7, 8, 12].

- A mesh for a NURBS patch is defined by the product of knot vectors.
- Knot spans subdivide the domain into “elements”.
- The support of each basis function consists of a small number of elements.
- The control points associated with the basis functions define the geometry.
- The isoparametric concept is invoked, that is, the unknown variables are represented in terms of the basis functions which define the geometry. The coefficients of the basis functions are the degrees-of-freedom, or control variables.
- Three different mesh refinement strategies are possible: analogues of classical $h$-refinement (by knot insertion) and $p$-refinement (by order elevation of the basis functions), and a new possibility referred to as $k$-refinement, which increases smoothness in addition to order.
- The element arrays constructed from isoparametric NURBS can be assembled into global arrays in the same way as finite elements (see Hughes [11], chapter 2).
- Dirichlet boundary conditions are applied to the control variables, in the same way as in finite elements. Neumann boundary conditions are satisfied naturally as in standard finite element formulations (see Hughes [11], chapters 1 and 2).

Finally, it is important to remark that in structural analysis NURBS elements represent all rigid body motions and constant strain states exactly (see Hughes [11]). Consequently, structures assembled from compatible NURBS elements pass standard “patch tests” (see Hughes [11], chapters 3 and 4, for a description of patch tests).

3.2 Linear and nonlinear parameterizations

When dealing with NURBS, an important issue is the choice of the parameterization to be used. Take as an example a 1D domain: the simplest (and more natural) option is to employ a linear parameterization, but in some situations a nonlinear choice can be more suitable.

The isogeometric procedure originally proposed by Hughes et al. [12] is based on a distribution of control points which leads to a linear parameterization (i.e., constant Jacobian determinant), but in Cottrell et al. [7] it has been shown that when studying structural vibrations
a nonlinear parameterization such that the control points are uniformly spaced gives better results. In Figure 1, we show the 1D distribution of 21 control points obtained for the two cases using cubic NURBS (top), along with plots of the corresponding parameterization $x = x(\xi)$ and Jacobian $J(\xi) = \frac{dx}{d\xi}$ (bottom).

![Figure 1: Linear parameterization versus a nonlinear one with uniformly-spaced control points (cubic NURBS, 21 control points). Top: distribution of control points; dots correspond to linear parameterization control points and asterisks to uniformly-spaced control points. Bottom: plot of the parameterization (left) and of its Jacobian (right) for the two cases.](image)

3.3 $k$-method and $p$-method

We conclude this section on isogeometric analysis by briefly pointing out what we mean in this paper with the terms “$k$-method” and “$p$-method”. Referring to the already cited $k$- and $p$-refinement strategies, we call the $k$-method an analysis method exploiting the full continuity across the elements allowed by NURBS basis functions (i.e., $C^{p-1}$ for a degree $p$ NURBS). In the following we will simply label this method as “NURBS”. Instead, we call the $p$-method an analysis method where only $C^0$-continuity is enforced across elements (this can be obtained with isogeometric analysis by repeating the knots of a degree $p$ NURBS $p - 1$ times). This approach, used in combination with a linear parameterization, is equivalent to classical $h$-$p$ finite element methods, and in the following we will simply label it as “FEM”.

4 ANALYTICAL RESULTS IN ONE DIMENSION

In this section, we sketch some analytical computations for finding the discrete spectrum for structural vibrations and the dispersion relation for wave propagation. In particular, we discuss the similarity between the two frameworks. We first deal with the case of an approximation with linear elements, for which $k$- and $p$-methods coincide. Then, we briefly discuss the extension of the results to higher order (quadratic, for simplicity) approximations, for both methods. For more details on this subject we refer to the work in preparation, Hughes et al. [13].
4.1 Analytical computation of discrete spectrum for linear approximation

Following Hughes [11], we consider the generalized (discrete) eigenproblem associated with (3) on the one-dimensional domain \((0, L)\). We take here the linear approximation and employ a uniform mesh of nodes (control points) \(0 = x_0 < x_1 < \ldots < x_A < \ldots < x_{N+1} = L\). The number of elements is \(n_{el} = N + 1\) and the mesh-size is \(h = L/n_{el}\). Considering homogeneous Dirichlet (fixed-fixed) boundary conditions, the discrete system is

\[
\frac{1}{h} (\phi_{A-1} - 2\phi_A + \phi_{A+1}) + \frac{h(\omega h)^2}{6} (\phi_{A-1} + 4\phi_A + \phi_{A+1}) = 0, \quad A = 1, \ldots, N, \tag{8}
\]

\[
\phi_0 = \phi_{N+1} = 0. \tag{9}
\]

The total number of degrees-of-freedom is \(N\). Equations (8) form a linear homogeneous recurrence relation of order 2, whose solutions (without consideration of the boundary conditions (9)) are linear combinations of the exponentials \(\phi_A = e^{\pm i\omega h}\), where \(i = \sqrt{-1}\) is the imaginary unit.

Under the assumption \(\omega h < \sqrt{12}\), \(\omega h\) is real and uniquely determined in \(0 \leq \omega h \leq \pi\). The relation between \(\omega h\) and \(\omega h^h\) is given by

\[
\omega h = \sqrt{\frac{6}{\cos(\omega h)} - \cos(\omega h)} \tag{10}
\]

Taking into account the boundary conditions (9), \(\omega\) has to be restricted to the values \(\pi/L, \ 2\pi/L, \ldots, n\pi/L, \ldots, N\pi/L\). The \(n^{th}\) discrete vibrating mode is indeed

\[
\phi_A = C e^{iAn\pi h/L} - e^{-iAn\pi h/L} \equiv C \sin(A\pi h/L). \tag{11}
\]

As is known, the \(n^{th}\) discrete mode is the nodal interpolant of the \(n^{th}\) exact mode, which is associated to the exact natural frequency \(\omega\). The quantity \(\frac{\omega h}{\omega} - 1 = \omega h^h - \omega\) represents the relative error of the frequency. The plot of

\[
\frac{\omega h}{\omega} = \frac{1}{\omega h} \sqrt{\frac{6}{\cos(\omega h)} - \cos(\omega h)} \tag{12}
\]

is given in Figure 2 (the abscissa \(n/N\) is equivalent to \(\omega h\)).

4.2 Dispersion analysis for linear approximation

We discuss here the discrete dispersion relation for linear approximation. Consider the Helmholtz equation (5) on the infinite domain (line), and introduce the numerical grid \(x_A = hA, A \in \mathbb{Z}\). The resulting stencil is

\[
\frac{1}{h} (u_{A-1} - 2u_A + u_{A+1}) + k^2 \frac{h}{6} (u_{A-1} + 4u_A + u_{A+1}) = 0, \quad \forall A \in \mathbb{Z}. \tag{13}
\]

The standard mathematical analysis in this context consists of comparing the wavenumbers of the exact and discrete solutions. We recall that the exact solutions of the Helmholtz equation are linear combinations of \(u(x) = e^{\pm iAx}\). Also, the discrete solutions, that is, solutions of the stencil equation (13), can be written as exponentials, exactly as we have discussed in the
previous section. Now, following the notation common in the context of dispersion analysis, we write the solutions as

$$u_h(x_A) \equiv u_A = e^{\pm ik^h h A}, \quad (14)$$

denoting by $k^h$ the discrete wave number, which is uniquely determined under the condition $0 \leq Re(k^h h) \leq \pi$. Restricting to the case of real $k^h h$, inserting (14) into (13), the relation between $k$ and $k^h$ is obtained. In particular, the quantity $k^h / k$ (which is related to the dispersion error $(k^h - k) / k = k^h / k - 1$) can be obtained in terms of $k^h h$,

$$\frac{k^h}{k} = \frac{k^h h}{\sqrt{6 \frac{1 - \cos(k^h h)}{2 + \cos(k^h h)}}}. \quad (15)$$

### 4.3 The Duality Principle for linear approximation

Note that (15) is the reciprocal of expression (12), written in terms of different quantities: $\omega$ corresponds to $k^h$ while $\omega^h$ corresponds to $k$. The analytical computation of the discrete spectrum is truly equivalent to the dispersion analysis: switching from one language to the other is just a matter of exchanging notation. Therefore the plot of $k^h / k$, which is the reciprocal of the quantity usually considered in the literature, is the same as the plot of $\omega^h / \omega$. In Section 5 we will present analytical results using the language of spectrum analysis, but they are also applicable to dispersion analysis.

### 4.4 Analytical results for the $p$-method with quadratic elements

We have seen in the previous section that, in the case of linear elements, the spectrum and the dispersion analysis are, from the mathematical viewpoint, the same. This is true for the $p$-method with higher-order elements as well, even though the analysis becomes more technical. Here, we only discuss the main features of the case of quadratic elements, referring to [13] for details.

We start by focusing on dispersion analysis. Therefore, we take into consideration the Helmholtz equation on the infinite line. The corresponding stencil equation is different for the element-endpoint degrees-of-freedom (indexed by integer $A$) and the bubble (internal to the element) degrees-of-freedom (indexed by $A + 1/2$). However, the common way to proceed consists in eliminating the bubble degrees-of-freedom, obtaining a system of equations for $u_A$ only:

$$\frac{1}{3h} \left[ \left( \frac{30 + 2(k^h)^2}{10 - (k^h)^2} \right) u_{A-1} + \left( \frac{-60 + 16(k^h)^2}{10 - (k^h)^2} \right) u_A + \left( \frac{30 + 2(k^h)^2}{10 - (k^h)^2} \right) u_{A+1} \right]$$

$$+ k^2 \frac{h}{30} \left[ \left( \frac{5(k^h)^2}{40 - 4(k^h)^2} \right) u_{A-1} + \left( \frac{200 - 15(k^h)^2}{20 - 2(k^h)^2} \right) u_A + \left( \frac{5(k^h)^2}{40 - 4(k^h)^2} \right) u_{A+1} \right] = 0, \quad (16)$$

Observe that (16) is a homogeneous linear recurrence equation of order 2, as for the linear case (13). Then, we know that its solutions can be written as $u_A = C_- e^{-ik^h h A} + C_+ e^{ik^h h A}$, for a suitable $k^h$ which satisfies

$$\cos(k^h h) = \frac{3k^4h^4 - 104k^2h^2 + 240}{k^4h^4 + 16k^2h^2 + 240}. \quad (17)$$

For any $k^h h$, condition (17) gives two values of $kh > 0$. The selection of the meaningful branch (depending on $0 < k^h h < \pi$ or $\pi < k^h h < 2\pi$) is discussed in [13].

7
The situation is analogous for the spectrum analysis. Consider the eigenproblem (3) on the domain \((0, L)\), subdivided into \(n_{el}\) elements. Then we have \(N = 2n_{el} - 1\) d.o.f.'s. As for the Helmholtz equation, we proceed by eliminating the bubble degrees-of-freedom, arriving at the equation

\[
\frac{1}{3h} \left[ \left( \frac{30 + 2(\omega h)^2}{10 - (\omega h)^2} \right) \phi_{A-1} + \left( \frac{-60 + 16(\omega h)^2}{10 - (\omega h)^2} \right) \phi_A + \left( \frac{30 + 2(\omega h)^2}{10 - (\omega h)^2} \right) \phi_{A+1} \right] + \frac{(\omega h)^2 h}{30} \left[ \left( \frac{5(\omega h)^2}{40 - 4(\omega h)^2} \right) (\phi_{A-1} + \phi_{A+1}) + \left( \frac{200 - 15(\omega h)^2}{20 - 2(\omega h)^2} \right) \phi_A \right] = 0,
\]

for all \(A = 1, \ldots, N\), which is the counterpart of (16). We can now reason as for the linear case and use the results for the dispersion analysis of this section, recalling the correspondence \(\omega^h \leftrightarrow k\) and \(\omega \leftrightarrow k^h\). It follows that \(\phi_A = C_- e^{-i\omega h A} + C_+ e^{i\omega h A}\) describes a vibrating mode if the boundary conditions are fulfilled. This condition determines \(n_{el} - 1\) vibrating modes in the first branch \((0 < \omega h < \pi)\), one mode corresponding to \(\omega h = \pi\), and \(n_{el} - 1\) modes in the second branch \((\pi < \omega h < 2\pi)\). The endpoint-element degrees-of-freedom of the \(n^{th}\) vibrating mode are \(\phi_A = C \sin(An\pi h/L)\).

The relation between \(\omega h\) and \(\omega^h h\) is

\[
\cos(\omega h) = \frac{3 (\omega h)^4 - 104 (\omega h)^2 + 240}{(\omega h)^4 + 16 (\omega h)^2 + 240}.
\]

(19)

This relation is plotted in Figure 3 (left) (the abscissa \(n/N\) is equivalent to \(\omega h/2\pi\)). Observe that the first half of the numerical spectrum (corresponding to \(0 < \omega h < \pi\)) lies on the lower branch (acoustical), while the second half \((\pi < \omega h < 2\pi)\) lies on the higher branch (optical). The reason for this is explained in [13].

In conclusion, in this case there is also duality between dispersion and spectrum analysis, and in particular (17) corresponds to (19).

4.5 Analytical results for the \(k\)-method with quadratic elements

For the \(k\)-method, with higher-order elements, there is also a strict relation between spectrum and dispersion analysis. However, it is more difficult to show the connection analytically in a rigorous manner. We only give here some comments, for the case of quadratic elements, and postpone to [13] for a deeper study of this subject.

Considering the Helmholtz equation on the infinite line, the stencil equation for linear parametrization is

\[
\frac{1}{6h} (u_{A-2} + 2u_{A-1} - 6u_A + 2u_{A+1} + u_{A+2}) + \frac{k^2 h}{12} (u_{A-2} + 26u_{A-1} + 66u_A + 26u_{A+1} + u_{A+2}) = 0, \quad \forall A \in \mathbb{Z}.
\]

(20)

which is an homogeneous recurrence relation of order 4. Its solutions can be written as linear combinations of \(e^{\pm ik_{1}^h A}\) and \(e^{\pm ik_{2}^h A}\) when \(k_{1}^h\) and \(k_{2}^h\) are distinct. This is analogous to the case of linear approximation, but now the space of discrete solutions has dimension 4, unlike the space of exact solutions which has dimension 2. It is found (see [13]) that two discrete solutions, say \(e^{\pm ik_{1}^h A}\), are approximations of the physical solutions \(e^{\pm ik h A}\), while the other two are spurious numerical solutions (evanescent waves). However, from the practical point of view, the spurious solutions do not pollute the numerical solution obtained by the \(k\)-method.
Therefore, what is meaningful for the dispersion analysis is the study of the relation between $k^h_1$ and $k$.

On the other hand, when one considers the eigenproblem (3) on $(0, L)$, the discrete equations at the boundary are different from the interior equations (stencil). The resulting recurrence relation is more difficult to study and compare with (20). However, it can be seen numerically ([13] and Figure 2) that the spectrum analysis is in fact closely related to the dispersion analysis, in the spirit of the duality principle that we discussed above.

5 NUMERICAL TESTS

We now report on the results of some numerical tests we performed using the $k$-method (i.e., NURBS) and the $p$-method (i.e., FEM) on a 1D vibration problem (i.e., the problem of the longitudinal structural vibrations of a rod) to obtain the numerical spectra. In the previous section, this has been pointed out to be equivalent to the classical time-harmonic dispersion analysis for 1D wave propagation (“duality principle”).

Before comparing NURBS and FEM results, we briefly remark that the following plots regarding NURBS are obtained using a nonlinear parameterization (as described in section 3.2) in order to avoid the so-called “outlier frequencies”. These are spurious frequencies (or discrete optical branches) that show up when a linear parameterization is employed. For a more detailed discussion on the appearance of such outliers and on the way to eliminate them, the reader is referred to [7].

In Figure 2, we present on the same plot the numerical and the analytical spectra obtained with the $k$-method for $p = 1, ..., 3$ (all the shown numerical spectra have been computed employing 1000 control points). It is possible to observe that they are practically coincident.

In Figure 3, we present the numerical and analytical spectra obtained with the $p$-method for $p = 2$ (left) and for $p = 3$ (right). In this case, the numerical spectra consist of the union of the different analytical branches arising for $p$-method.

Finally, Figure 4 shows a comparison of $k$-method and $p$-method numerical spectra for $p = 1, ..., 4$ (we recall that for $p = 1$ the two methods coincide). Here, the superiority of the isogeometric approach is clearly revealed, as one can see that optical branches of spectra diverge with $p$ in the classical finite element method. This negative result shows that even higher-order finite elements have no approximability for higher modes in vibration analysis, and explains the fragility of higher-order finite element methods in nonlinear and dynamic applications in which higher modes necessarily participate.

6 NASA ALUMINUM TESTBED CYLINDER

In this section, we present an application of isogeometric analysis which aims at showing the potential of this approach for the exact modeling of complex real structures and the study of structural vibration frequencies and modes.

The NASA Aluminum Testbed Cylinder (ATC) is shown in Figure 5. An isogeometric model (see Figure 6) was constructed from design drawings. There are three distinct members composing the framework (see Figure 6, right): twenty-four identical, prismatic stringers (see Figure 7); nine identical main ribs (see Figures 8-10); and two end ribs, which are mirror images of each other (see Figure 11). Note that every geometrical feature is exactly represented in the model. It is also interesting to note that control meshes amount to typical trilinear hexahedral meshes (see Figure 10). This suggests that hexahedral finite element mesh generators (see, e.g., CUBIT [3]) may be useful in building isogeometric NURBS models. The stringer-main rib and
stringer-end rib junctions are shown in Figure 12. Note that there are gaps between the stringer and the ribs in the notch regions. Experimental vibration data has been obtained for the frame and skin assembly (see [4, 9, 10] for details and reference computational results). The Automated Multi- Level Substructuring (AMLS) eigensolver (see Bennighof and Lehoucq [2]) was used in our calculations of the full ATC structure. Results for the frame and skin assembly are presented in Figure 13. The numerical results lie above the experimental results. Moreover, as an example, contour plots of the first Love mode are reported in Figure 14. The mesh consisted of 228,936 rational quadratic elements and 2,219,184 degrees-of-freedom. The cost of array formation and assembly was commensurate with standard quadratic finite elements.
7 CONCLUSIONS

In this paper we have analyzed isogeometric discretizations in structural dynamics and wave propagation and compared them with classical $p$-method finite elements. In the case of structural dynamics we have considered free vibration problems as a basis for comparison, whereas in the case of wave propagation we have studied the time-harmonic case by means of classical dispersion analysis. In particular, we have discussed a “duality principle” between free vibration and dispersion analysis, connecting the approximation properties of the different methods for these two problems. The results obtained clearly show the superiority of the $k$-method isogeometric approach employing NURBS over the classical $C^0$-continuous $p$-method. In particular, we showed that “optical branches” of spectra diverge with $p$ in the classical finite element method, while they are eliminated in the $k$-method isogeometric approach, suggesting a great potential in studying problems where traditional higher-order finite element methods are known to fail. We have also illustrated the modeling capabilities of the isogeometric approach on a significant physical problem, the NASA Aluminum Testbed Cylinder. In this example we have shown the possibility of using a three-dimensional isogeometric model to represent exactly every geometrical feature of a complex real structure. Moreover, we have computed numerical structural frequencies and modes, which have been compared with experimental vibration results showing a good correlation.

REFERENCES


Figure 7: NASA ATC. Isogeometric model of the longitudinal stringer. Sample meshes.

Figure 8: NASA ATC. Isogeometric model of the main rib.

Figure 9: NASA ATC. Mesh 1 (left) of a typical 15° main rib segment encapsulates the geometry exactly. Mesh 2 (center) is formed by refining selectively to make the aspect ratios of elements more uniform throughout the mesh. Mesh 3 (right) is created by uniform $h$-refinement of mesh 2.


Figure 10: NASA ATC. Detail of the “notch” region in the main rib. The control net is on the left and the exact geometry is on the right.

Figure 11: NASA ATC. Mesh 1 (left) of a typical 15° segment of an end rib. Mesh 2 (center) is created by selective $h$-refinement of Mesh 1. Mesh 3 (right) is created by uniform $h$-refinement of mesh 2.

Figure 12: NASA ATC. Stringer–main rib (top) and stringer–end rib (bottom) junctions.


Figure 13: NASA ATC. Comparison of numerical and experimental frequency results for the frame and skin assembly (left). Relative errors (right). The mean error over the first 14 modes is 7.7% and the maximum error is 13.7%.

Figure 14: NASA ATC. Calculated first Love mode of the frame and skin assembly. The color contours represent the ovalization (left) and the axial displacement (right) of the assembly.


