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VARIATIONAL MULTISCALE RESIDUAL-DRIVEN TURBULENCE MODELING FOR LARGE EDDY SIMULATION OF INCOMPRESSIBLE FLOW

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Abstract. *The objectives of recent variational multiscale work in turbulence have been to capture all scales consistently and to avoid use of eddy viscosities altogether. This holds the promise of more accurate and efficient LES procedures. In this work, we describe a new variational multiscale formulation, which makes considerable progress toward these goals.*

1 SUMMARY

We begin by taking the view that the decomposition into coarse and fine scales is exact. For example, in the spectral case, the coarse-scale space consists of all Fourier modes beneath some cut-off wave number and the fine-scale space consists of all remaining Fourier modes. Consequently, the coarse-scale space has finite dimension whereas the fine-scale space is infinite dimensional. The derivation of the coarse- and fine-scale equations proceeds, first, by substituting the split of the exact solution into coarse and fine scales into the Navier-Stokes equations, then, second, by projecting this equation into the coarse- and fine-scale subspaces. The projection into coarse scales results in a finite dimensional system for the coarse-scale component of the solution, which depends parametrically on the fine-scale component. In the spectral case, in addition to the usual terms involving the coarse-scale component, only the cross-stress and Reynolds-stress terms involve the fine-scale component. In the case of non-orthogonal bases, even the linear terms give rise to coupling between coarse and fine scales. The coarse-scale component plays an analogous role to the filtered field in the classical approach, but has the advantage of avoiding all problems associated with homogeneity, commutativity, walls, compressibility, etc. The projection into fine scales results in an infinite-dimensional system for the fine-scale component of the solution, which depends parametrically on the coarse-scale component. We also assume the cut-off wave number is sufficiently large that the philosophy of LES is appropriate. For example, if there is a well-defined inertial sub-range, then we assume the cut-off wave number resides somewhere within it. This assumption enables us to further assume that the energy content in the fine scales is small compared with the coarse scales. This turns out to be important in our efforts to analytically represent the solution of the fine-scale equations. The strategy is to obtain approximate analytical expressions for the fine scales then substitute them into the coarse-scale equations which are, in turn, solved numerically. If the scale decomposition is performed in space and time, the *only* approximation in the procedure is the representation of the fine-scale solution. To provide a framework for the fine-scale approximation, we assume an infinite perturbation series expansion to treat the fine-scale nonlinear term in the fine-scale equation. By virtue of the smallness of the fine scales, this expansion is expected to converge rapidly under the circumstances described in many cases of practical interest. The remaining part of the fine-scale Navier-Stokes system is the *linearized* operator which is formally inverted through the use of a matrix Green's function. The combination of a perturbation series and Green's function provides an exact formal solution of the fine-scale Navier-Stokes equations. The driving force in these equations is the Navier-Stokes system residual computed from the coarse scales. This expresses the intuitively obvious fact that if the coarse scales constitute a good approximation to the solution of the problem, the coarse-scale residual will be small and the resulting fine-scale solution will be small as well. This is the case we have in mind and it provides a rational basis for assuming the perturbation series converges rapidly. Note that one cannot use such an argument on the original problem because in this case the perturbation series would almost definitely fail to converge. (If we could have used this argument, we would have solved the Navier-Stokes equations analytically! Unfortunately, this is not the case.) The formal solution of the fine-scale equations suggests various approximations may be employed in practical problem solving. We are tempted to use the word "modeling" because approximate analytical representations of the fine scales constitute the only approximation and hence may be thought of as the "modeling" component of the present approach, but we want to emphasize that this is very different from classical modeling ideas which are dominated by the *addition* of *ad hoc* eddy viscosities. We will present numerical results that demonstrate that eddy-viscosity terms are unnecessary in the present circumstances. There are two aspects to the approximation of the fine scales: 1) Approximation of the matrix Green's function for the linearized Navier-Stokes system; and 2) approximation of the nonlinearities represented by the perturbation series. The first and obvious thought for the latter aspect, nonlinearity, is to simply truncate the perturbation series. This idea is pursued in conjunction with some simple approximations of the Green's function. It turns out there is considerable experience in local scaling approximations of the Green's function based on the theory of stabilized methods; Hughes [3], Hughes *et al.* [4], Hughes and Sangalli [5], Hughes, Scovazzi and Franca [6]. The Green's function is typically approximated by locally defined algebraic operators (i.e., the "τ's" of stabilized methods) multiplied by local values of the coarse-scale residual.

An outline of the presentation is summarized as follows: we begin by presenting the mathematical details of the variational multiscale theory. This represents our general approach to LES-style turbulence modeling and is independent of the specifics of the discrete spaces utilized to represent the coarse scales. The relationship between this version of the variational multiscale method and classical stabilized methods is delineated. It is noted that that the variational multiscale method includes additional terms. Both conceptually and from the point of view of actual implementation, stabilized methods may be viewed as historical stepping stones leading to the more coherent variational multiscale formulation. We then present our numerical studies of forced isotropic turbulence at $Re_\lambda = 165$ and $Re_\lambda = \infty$. (Re_λ is the Taylor microscale Reynolds number.) We begin with a description of the approximation spaces consisting of NURBS elements (non-uniform rational B-splines, see, e.g., Rogers [13], Piegl and Tiller [12], Farin [2], and Cohen, Riesenfeld and Elber [1]). In the case of the rectilinear geometry considered, NURBS reduce to B-splines, which have been advocated for turbulence calculations previously (see Kravchenko, Moin and Moser [7], Shariff and Moser [14], Kravchenko, Moin and Shariff [8], and Kwok, Moser and Jiménez [9]). We employ trivariate linear, quadratic, and cubic NURBS with periodic boundary conditions. Linear trivariate NURBS turn out to be identical to trilinear hexahedral finite elements, but the higher-order NURBS are different than classical higher-order finite elements. We perform a dispersion error analysis for NURBS versus classical finite elements on simple, linear, one-dimensional advective and diffusive model problems, and conclude that NURBS have better approximation properties than classical finite elements. We employ meshes of 32^3, 64^3, 128^3, and 256^3 to explore convergence with mesh refinement (h-convergence). We also examine the behavior of increasing order from linear to cubic on fixed meshes (k-convergence). In the case of $Re_\lambda = 165$, we compare with the DNS spectral results of Langford and Moser [10]. Energy spectra and third-order structure functions are presented. Sample energy spectra results are presented in Figure 1. In the case of $Re_\lambda = \infty$ we also clearly see the development of an inertial subrange. We present results for turbulent channel flows at $Re_\tau = 395$. (Re_τ is the wall-friction Reynolds number.) We employ meshes of 32^3 and 64^3. This time the mesh is graded in the wall-normal direction to better capture the boundary layer. Again, we consider convergence from the h- and k-refinement perspectives. A striking result is how much better quadratic elements are than linear elements. For a mesh of 64^3, the quadratic and cubic results are essentially identical to the DNS results of Moser, Kim and Mansour [11] for first- and second-order statistics (see Figure 2), and for a mesh of 32^3 they are in close agreement. We close with conclusions and suggested future directions for research.

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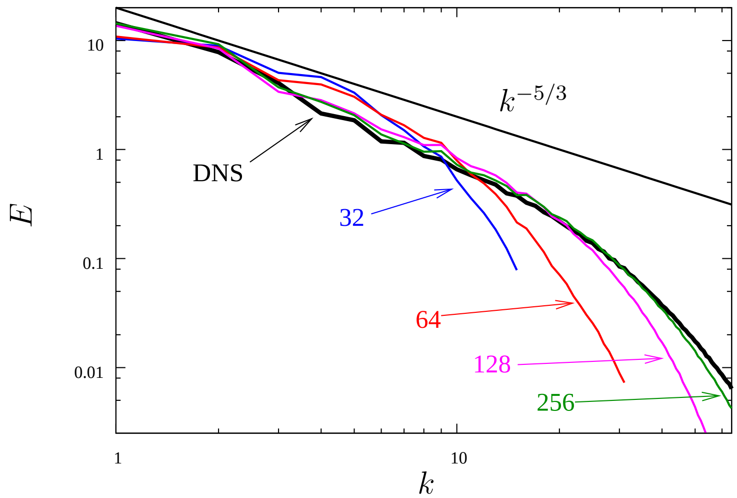
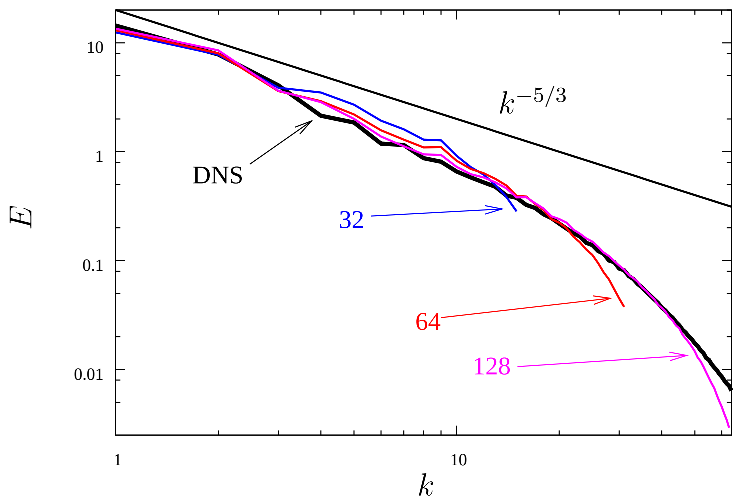
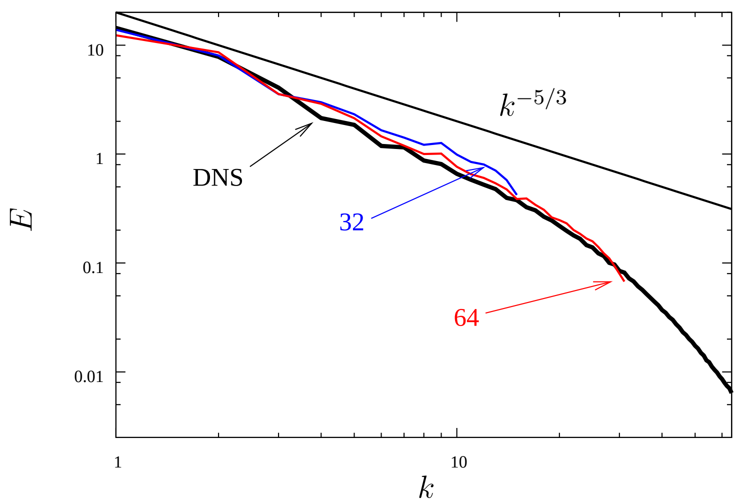


(a) C^0-continuous linear NURBS

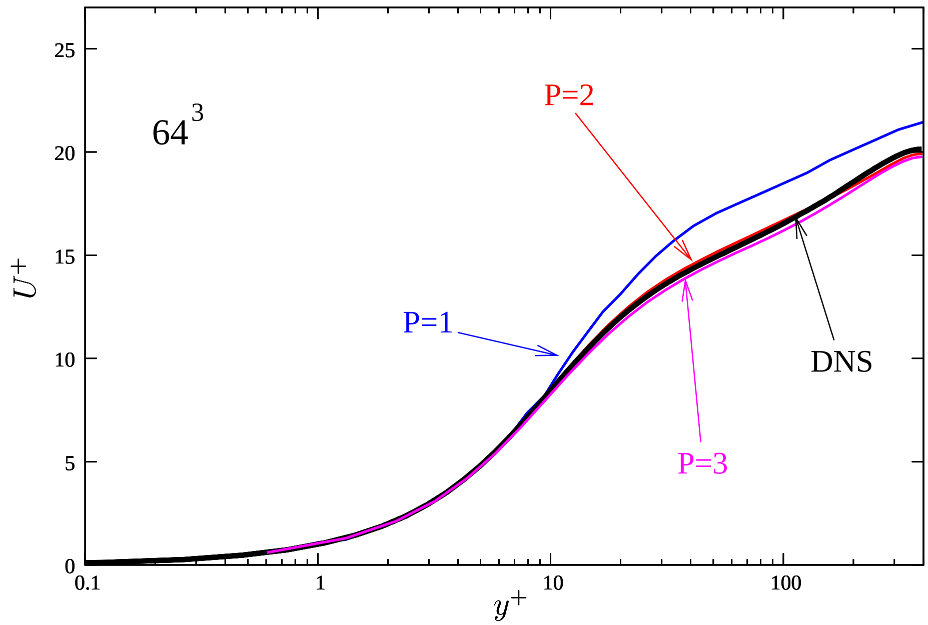


(b) C^1-continuous quadratic NURBS

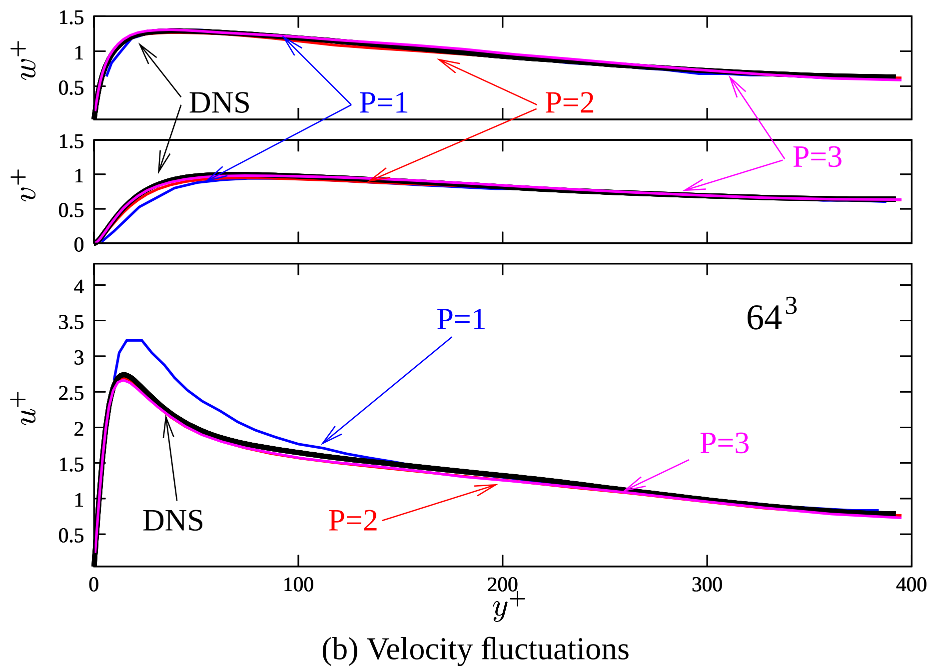


(c) C^2-continuous cubic NURBS

Figure 1: Energy spectra for $h-$refinement. $Re_\lambda = 165$.



(a) Mean stream-wise velocity



(b) Velocity fluctuations

Figure 2: Turbulent channel flow at $Re_\tau = 395$ computed on a mesh of 64^3 elements: k-refinement interpretation of results.