A Phenomenological One-Dimensional Model Describing Stress-Induced Solid Phase Transformation with Permanent Inelasticity

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The use of shape memory alloys for a large number of applications in many fields of engineering has given rise to a great interest of researchers on an exhaustive modeling of their macroscopic behaviour.

Most of the models proposed in the literature, however, consider a perfect *pseudo-elastic* behaviour (i.e. no residual strains), which is experimentally proved to be only an approximation. This paper addresses a new one-dimensional model able to include also permanent inelasticity effects, whose main features are shown by many numerical tests.

1. INTRODUCTION

In last years, the research on the constitutive modeling of shape memory alloys (SMA) has greatly increased, due to the large number of SMA industrial applications in many branches of engineering [1, 2]. Accordingly, many models able to reproduce one or both of the SMA macroscopic behaviours (i.e. *pseudo-elasticity* and *shape-memory effect*) have been recently proposed in the literature (see e.g. [3–12]). Among them, the one proposed in [13] and improved in [14] seems to be very attractive for its potential. However, as in many other models present in the literature, residual inelastic strain effects are not included, while experimental results as the ones shown in Figure 1 (cf. [15]) prove them not to be negligible. Moreover, Figure 1 also highlights the need for the introduction of degradation effects.

It is in this framework that the present work takes its justification; in fact, we wish to propose a new phenomenological one-dimensional constitutive model, able to include permanent

inelasticity and degradation effects together with classical SMA macroscopic behaviours. In the following, an analytical description of the constitutive equations is presented and many numerical experiments are performed in order to show the features of the model.

2. 1D MODEL FOR STRESS-INDUCED SOLID PHASE TRANSFORMATION INCLUDING PERMANENT INELASTICITY

Starting from the SMA constitutive model discussed in [14] and limiting the development to a 1D situation, we describe now an extended set of constitutive equations, including a new internal variable able to activate and take into account the effect of residual inelastic strains.

2.1. Time-Continuous Frame

The total strain ε is assumed as control variable, the transformation strain ε^{tr} and the residual inelastic strain q as internal ones. The quantity ε^{tr} plays the role of describing the strain associated to the phase transformation as in [14], but here it should not be a fully reversible quantity, i.e., as the stress is driven to zero, the transformation strain should not be completely recovered, but it should tend to the value of the residual inelastic strain q . Moreover, the following condition is required:

$$|\varepsilon^{tr}| \leq \varepsilon_L, \quad (1)$$

where ε_L is a material parameter corresponding to the maximum transformation strain reached at the end of the transformation.

Assuming a small strain regime, the free energy function Ψ for a polycrystalline SMA material is expressed as the convex

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FIG. 1. Experimental results on a SMA wire. Cyclic tension test: stress versus strain up to 6% strain.

potential:

$$\Psi(\varepsilon, \varepsilon^{tr}, q) = \frac{1}{2}E(\varepsilon - \varepsilon^{tr})^2 + \beta\langle T - M_f \rangle |\varepsilon^{tr} - q| + \frac{1}{2}h(\varepsilon^{tr})^2 + \frac{1}{2}Hq^2 - A\varepsilon^{tr}q + \mathcal{J}_{\varepsilon_L}(\varepsilon^{tr}), \quad (2)$$

where E is the Young's modulus, β is a material parameter related to the dependence of the critical stress on the temperature, T is the absolute temperature, M_f is the temperature below which only martensite phase is stable, h , H and A are material parameters controlling, respectively, the slope of the linear stress-transformation strain relation, the saturation of the residual inelastic strain evolution and the bilinear coupling between ε^{tr} and q . In (2) we make use of the indicator function $\mathcal{J}_{\varepsilon_L}(\varepsilon^{tr})$:

$$\mathcal{J}_{\varepsilon_L}(\varepsilon^{tr}) = \begin{cases} 0 & \text{if } |\varepsilon^{tr}| < \varepsilon_L \\ +\infty & \text{otherwise,} \end{cases}$$

to satisfy the transformation strain constraint (1); moreover, we introduce the positive part($\bullet$) function:

$$\langle a \rangle = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Following standard arguments, we can derive from the free energy function Ψ the constitutive equations:

$$\begin{cases} \sigma = \frac{\partial \Psi}{\partial \varepsilon} = E(\varepsilon - \varepsilon^{tr}), \\ X = -\frac{\partial \Psi}{\partial \varepsilon^{tr}} = \sigma - \beta\langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{|\varepsilon^{tr} - q|} - h\varepsilon^{tr} \\ \quad + Aq - \gamma \frac{\varepsilon^{tr}}{|\varepsilon^{tr}|}, \\ Q = -\frac{\partial \Psi}{\partial q} = \beta\langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{|\varepsilon^{tr} - q|} - Hq + A\varepsilon^{tr}, \end{cases} \quad (3)$$

where σ is the stress, while X and Q are thermodynamic stress-like quantities associated respectively to the transformation strain ε^{tr} and to the residual inelastic strain q . The variable γ is the result of the subdifferential of the indicator function and is defined as:

$$\begin{cases} \gamma = 0 & \text{if } |\varepsilon^{tr}| < \varepsilon_L \\ \gamma \geq 0 & \text{if } |\varepsilon^{tr}| = \varepsilon_L. \end{cases}$$

We now introduce a limit function F defined as:

$$F(X, Q) = |X| + \kappa|Q| - R, \quad (4)$$

where κ is a scaling modulus for the inelastic effect and R is a material parameter defining the radius of the elastic domain and controlling the width of the hysteresis loops (cf. [14]). Considering an associative framework, the flow rules for internal variables take the form:

$$\begin{cases} \dot{\varepsilon}^{tr} = \dot{\zeta} \frac{\partial F}{\partial X} = \dot{\zeta} \frac{X}{|X|}, \\ \dot{q} = \dot{\zeta} \frac{\partial F}{\partial Q} = \dot{\zeta} \kappa \frac{Q}{|Q|}. \end{cases} \quad (5)$$

We finally complete the model by the classical Kuhn-Tucker conditions:

$$\begin{cases} \dot{\zeta} \geq 0 \\ F \leq 0 \\ \dot{\zeta} F = 0. \end{cases} \quad (6)$$

Remark 1.

The choice of the parameter κ controls the entity of the inelastic effect and it is not completely free. For simplicity, we show how to compute an upper bound for this parameter when $h = H = A = 0$ MPa and $|\varepsilon^{tr}| < \varepsilon_L$. In this case the constitutive equations for X and Q take the form:

$$\begin{cases} X = \sigma - \beta\langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{|\varepsilon^{tr} - q|}, \\ Q = \beta\langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{|\varepsilon^{tr} - q|}; \end{cases}$$

substituting in the expression (4) for the limit function and taking into account the second Kuhn-Tucker condition (6)₂, we get:

$$F = |X| + \kappa\beta\langle T - M_f \rangle - R \leq 0.$$

In order to satisfy this inequality, since $|X| \geq 0$, it is necessary to have:

$$\kappa\beta\langle T - M_f \rangle - R \leq 0,$$

that leads to the following upper bound for the parameter κ :

$$\kappa \leq \frac{R}{\beta\langle T - M_f \rangle}.$$

We finally highlight that the choice $\kappa = 0$ leads to recover the model without permanent inelasticity discussed in [14]. In fact, setting $\kappa = 0$ in (5)₂, we get $\dot{q} = 0$, which means that q does not evolve and is always equal to its initial value, i.e. $q \equiv 0$.

Remark 2.

In many experimental studies it has been observed that SMA materials show a not indefinitely evolving inelasticity, that saturates after reaching a certain level. In our model we can introduce such an effect by choosing a non-zero H parameter. We start noting that the evolution of q necessarily stops when $Q = 0$. Then, considering for simplicity the case $A = 0$ MPa, for this condition we can obtain a priori the saturated value of q from (3)₃, i.e.:

$$q_{\max}^{\pm} = \pm \frac{\beta \langle T - M_f \rangle}{H},$$

which is dependent only on the temperature.

On the other hand, for $A \neq 0$ MPa the condition $Q = 0$ leads to a relation from which it is not possible to obtain a simple a priori expression for $q_{\max}^{\pm}$, due to the coupling between q and ε^{tr} . In such a case, the value of $q_{\max}^{\pm}$ will not be dependent only on the temperature.

Moreover, even if sometimes experimental results show that the loading rate can influence the maximum level of permanent inelastic strain, we stress here that such dependence is not reproduced by the present model.

We finally remark that for $T < M_f$ (i.e. in the case of stable martensite phase) an evolution for q is obtained only for a non-zero value of A .

2.2. Time-Discrete Frame

We now introduce a subdivision of the time interval of interest $[0, t_f]$ and solve the evolution problem over the generic interval $[t_n, t_{n+1}]$ with $t_{n+1} > t_n$. Note that in the following, for the sake of notation simplicity, we drop the subindex $n + 1$ for all the variables computed at time $t = t_{n+1}$. We remark that we assume the temperature to be assigned at each time step in order to treat only the mechanical part of the problem.

2.2.1. Time Integration

Supposed known the solution at time t_n and the value of the total strain at time t_{n+1} , an implicit backward Euler scheme is employed to integrate the flow rules for the internal variables. In order to differentiate in zero the free energy Ψ and the limit function F , we substitute the terms $|\varepsilon^{tr} - q|$ and $|Q|$ with $\overline{|\varepsilon^{tr} - q|}$ and $\overline{|Q|}$, where $\overline{|\bullet|}$ is a regularized absolute value defined as:

$$\overline{|a|} = \sqrt{|a|^2 + \delta} - \sqrt{\delta},$$

with δ user-defined parameter controlling the regularization smoothness. We stress that, as we introduce a regularization only to obtain always differentiable functions and to guarantee the numerical robustness of the model, the parameter δ can be chosen to be even very small (e.g., 10^{-8}).

Hence, the time-discrete constitutive model can be written as:

$$\begin{cases} \sigma = E(\varepsilon - \varepsilon^{tr}), \\ X = \sigma - \beta \langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{\sqrt{|\varepsilon^{tr} - q|^2 + \delta}} - h\varepsilon^{tr} + Aq - \gamma \frac{\varepsilon^{tr}}{|\varepsilon^{tr}|}, \\ Q = \beta \langle T - M_f \rangle \frac{\varepsilon^{tr} - q}{\sqrt{|\varepsilon^{tr} - q|^2 + \delta}} - Hq + A\varepsilon^{tr}, \\ \varepsilon^{tr} = \varepsilon_n^{tr} + \Delta\zeta \frac{X}{|X|}, \\ q = q_n + \Delta\zeta \kappa \frac{Q}{\sqrt{|Q|^2 + \delta}}, \\ F = |X| + \kappa \overline{|Q|} - R, \end{cases} \quad (7)$$

with the requirements:

$$\begin{cases} \gamma \geq 0, \\ |\varepsilon^{tr}| \leq \varepsilon_L, \\ \Delta\zeta \geq 0, \quad F \leq 0, \quad \Delta\zeta F = 0, \end{cases} \quad (8)$$

where $\Delta\zeta = \zeta - \zeta_n = \int_{t_n}^{t_{n+1}} \dot{\zeta} dt$ is the time-integrated consistency parameter.

2.2.2. Solution

The solution of the discrete model is performed by means of an elastic-predictor inelastic-corrector *return map* procedure as in classical plasticity problems [16]. An elastic trial state is evaluated for frozen internal variables and a trial value of the limit function is computed in order to verify the admissibility of the trial state; if this is not verified, the step is inelastic and the evolution equations are integrated.

As in [14], we distinguish two inelastic phases in our model: a non-saturated phase ($|\varepsilon^{tr}| < \varepsilon_L$, $\gamma = 0$) and a saturated one ($|\varepsilon^{tr}| = \varepsilon_L$, $\gamma \geq 0$). In our solution procedure we assume to be in a non-saturated phase, and at convergence we check if our assumption is violated. If the non-saturated solution is not admissible, then we search for a new solution under saturated conditions.

At each inelastic step, the nonlinear system constituted by equations (7) has to be solved. As the aim of this paper is showing the model behaviour (and not focusing on algorithmic problems), we find a solution to the system using the function *fsolve* implemented in the optimization toolbox of the program MATLAB[®].

3. NUMERICAL RESULTS

In this Section we perform a number of stress-driven tests in order to show the model behaviour when reproducing main macroscopic SMA features. Unless otherwise stated, in all the performed tests we consider the material properties specified in Table 1.

TABLE 1
Material parameters

Parameter	Value	Unit
E	$5 \cdot 10^4$	MPa
$\beta \langle T - M_f \rangle$	150	MPa
h	1000	MPa
R	50	MPa
ε_L	4	%
δ	10^{-8}	—

The aim of the following numerical experiments is to investigate the different effects controlled by each model parameter. For every test, we plot the stress input history, the output stress-strain curve and the output histories for the internal variables (ε^{tr} and q) and the associated stress-like quantities (X and Q).

- Test 1: tension cycles with $H = 0$ MPa, $A = 0$ MPa and $\kappa = 2\%$.

The first performed numerical experiment consists of tension cycles reaching a maximum stress of $\sigma_{\max} = 300$ MPa. The choice of a non-zero parameter κ gives rise to a permanent inelasticity phenomenon. Figures 2 and 3 refer to a single tension cycle, while Figures 4 and 5 refer to ten tension cycles. A significant evolution of q , which represents the level of transformation strain that is not recovered during the unloading phase to $\sigma = 0$, is observed.

- Test 2: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$.

In this test we verify what is highlighted by the second Remark of Section 2. The second plot in Figure 7 shows that the internal variable q does not exceed the threshold $\beta \langle T - M_f \rangle / H = 150 / (1.5 \cdot 10^4) = 1\%$, which is also the value of residual strain shown by the stable stress-strain cycles of Figure 6.

FIG. 2. Test 1: single tension cycle with $H = 0$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

- Test 3: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 4\%$.

This numerical experiment is similar to the previous one, but here the parameter κ is taken equal to the double as before, implying a greater inelastic effect, reaching its maximum value ($q = 1\%$) in half the time as before (see Figures 8 and 9).

- Test 4: tension and compression cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$.

In order to properly assess the behaviour of this model it is convenient to test it under cycles both in tension and compression. In Figures 10 and 11 we report the results of a loading history constituted by ten tension cycles followed by a compression one, while in Figures 12 and 13 the results of a loading history of ten tension and ten compression cycles; the good performance of the model are confirmed by both of these tests.

- Test 5: twenty tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$.

This experiment is aimed at investigating the effect on the model of the parameter A coupling the two internal variables. As shown by Figure 14 (right), a value of A different from zero results in shifting down the loops. This sort of degradation effect is an important feature of the model as an analogous phenomenon is observed in experimental tests, as proved by Figure 1.

- Test 6: tension cycle with $T < M_f$ and strain recovery by heating; $H = 0$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$.

The goal of this last test is to study the behaviour of the model when reproducing the shape-memory effect. The input consists in a tension cycle with a maximum stress $\sigma_{\max} = 100$ MPa at a temperature $T < M_f$ (which implies $\beta \langle T - M_f \rangle = 0$ MPa), followed by a heating process at a constant zero stress up to a temperature such that $\beta \langle T - M_f \rangle = 100$ MPa. Figures 16 and 17 show that even in this case a inelastic effect is activated, so that we observe only a partial shape-recovery.

FIG. 3. Test 1: single tension cycle with $H = 0$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Output histories for ϵ^{tr} , q , X and Q .

FIG. 4. Test 1: ten tension cycles with $H = 0$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 5. Test 1: ten tension cycles with $H = 0$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Output histories for ϵ^{tr} , q , X and Q .

FIG. 6. Test 2: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 7. Test 2: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Output histories for ϵ^{tr} , q , X and Q .

FIG. 8. Test 3: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 4\%$. Input stress history and stress-strain output.

FIG. 9. Test 3: ten tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 4\%$. Output histories for ϵ^{tr} , q , X and Q .

FIG. 10. Test 4: ten tension and one compression cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 11. Test 4: ten tension and one compression cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Output histories for ε^{tr} , q , X and Q .

FIG. 12. Test 4: ten tension and ten compression cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 13. Test 4: ten tension and ten compression cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 0$ MPa and $\kappa = 2\%$. Output histories for ε^{tr} , q , X and Q .

FIG. 14. Test 5: twenty tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 15. Test 5: twenty tension cycles with $H = 1.5 \cdot 10^4$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$. Output histories for ε^{tr} , q , X and Q .

FIG. 16. Test 6: tension cycle with $T < M_f$ and strain recovery by heating; $H = 0$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$. Input stress history and stress-strain output.

FIG. 17. Test 6: tension cycle with $T < M_f$ and strain recovery by heating; $H = 0$ MPa, $A = 10^3$ MPa and $\kappa = 2\%$. Output histories for ε^{tr} , q , X and Q .

4. CONCLUSIONS

In this paper, we have proposed a new 1D constitutive model for describing the macroscopic behaviour of SMA. As compared with the model proposed in [13, 14] and taken as a basis for the development of this new one, it is capable of including also permanent inelasticity, which can be described both with a saturating or a non-saturating evolution; degradation effects can be included as well. The good behaviour of the model has been proved by means of a number of numerical tests.

The development of a 3D version of the presented model and of a detailed algorithmic strategy for its solution will be the subject of future communications.

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