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Analysis of kinematic linked interpolation methods for Reissner–Mindlin plate problems

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Abstract

The approximation to the solution of Reissner–Minidlin plate problem is considered in the framework of finite element technique. A general strategy, involving a linking operator between rotations and vertical displacements, is analyzed. An abstract convergence result is provided. Examples of elements falling into this framework are presented and shown to be stable and locking-free. Numerical tests detail the performances of the elements.

1. Introduction

The Reissner–Mindlin model is widely used by engineers to describe the behavior of an elastic plate loaded by transverse forces. The main feature of this model is that the shear stresses are taken into account, thus allowing to consider both thin and moderately thin plates (cf. [8,17]). Unfortunately, standard low-order finite elements usually fail the approximation, whenever the plate thickness becomes numerically small. Nowadays, the reason for this lack of convergence, called *shear locking phenomenon,* is well-understood (cf. [8,17], for instance). Roughly, as the thickness becomes smaller and smaller, the shear energy term imposes the Kirchhoff constraint, which is too severe for low-order elements. To overcome the shear locking, several alternative methods have been proposed and studied in recent years. Most of them are based on suitable mixed formulations, able to reduce the influence of the shear energy at the discrete level. For some of the formulations proposed, a rigorous mathematical analysis has been developed (cf. [1–3, 9–12,18,21], for instance). Following [23,24] in Refs. [4,22,26], a general technique to design finite element schemes which have the hope to perform well has been presented. The idea consists in improving the approximated deflection space by means of the rotational degrees of freedom. The vertical displacement discrete solution is thus appropriately linked to the rotation discrete solution. Other methods based on this philosophy have been developed in [5,6], for instance. Two of these schemes have already been mathematically analyzed in [16,19,20]. The aim of this paper is to give a general error analysis applicable to most methods using kinematic linked interpolation and described in the literature. Moreover, we propose a new element fitting this technique. The paper is organized as follows. In Section 2 the Reissner–Mindlin plate problem is briefly presented, along with a mixed variational formulation. Section 3 is devoted to establish the general error analysis for the kinematic linked interpolation procedure. Propositions 3.2 and 3.3 give an error bound under few reasonable hypotheses on the finite dimensional spaces involved in the discretization. In Section 4 examples of the methods investigated are considered and recognized to fit the general

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analysis of Section 3. In Section 5 results from numerical tests are reported and commented. Finally, Section 6 draws some conclusions and gives a sufficiently easy-to-handle recipe to design further methods using the kinematic linked interpolation technique.

Throughout the paper, C denotes a constant independent of h and t, not necessarily the same in each occurrence. Furthermore, we follow the standard notation of [10,15].

2. The Reissner–Mindlin model and the mixed formulation

Let us denote with $A = \Omega \times (-t/2, t/2)$ the region in $\mathbb{R}^3$ occupied by an undeformed elastic plate of thickness $t > 0$. The Reissner–Mindlin plate model (cf. [17]) describes the bending behavior of the plate in terms of the transverse displacement $w(t)$ and of the fiber rotations $\underline{\theta}(t)$, normal to the midplane Ω.

In the case of a clamped plate, the stationary problem consists in finding the couple $(\underline{\theta}(t), w(t))$ that minimizes the functional

$$\Pi_t(\underline{\theta}(t), w(t)) = \frac{1}{2} \int_\Omega \mathscr{C}\mathscr{E}\underline{\theta}(t) : \mathscr{E}\underline{\theta}(t) + \frac{\lambda t^{-2}}{2} \int_\Omega |\underline{\theta}(t) - \underline{\nabla} w(t)|^2 - \int_\Omega f w(t)\, \mathrm{d}x\, \mathrm{d}y \tag{1}$$

over the space $V = \Theta \times W = (H_0^1(\Omega))^2 \times H_0^1(\Omega)$. In (1) $\mathscr{C}$ is a positive-definite fourth-order symmetric tensor in which the Young's modulus E and the Poisson's ratio v enter. Furthermore, $\mathscr{E}\underline{\theta}(t)$ is the symmetric gradient of the field $\underline{\theta}(t)$ and $\lambda = Ek/2(1+v)$, with k shear correction factor (usually taken as 5/6). Accordingly, the first term in (1) is the plate bending energy, the second term is the shear energy, while the last one corresponds to the external energy. Korn's inequality assures that $a(\cdot,\cdot) = \int_\Omega \mathscr{C}\mathscr{E}(\cdot) : \mathscr{E}(\cdot)$ is a coercive form over Θ so that there exists a unique solution $(\underline{\theta}(t), w(t))$ in V of the following problem.

Problem P_t: For $t > 0$ fixed, find $(\underline{\theta}(t), w(t))$ in V such that

$$a(\underline{\theta}(t), \underline{\eta}) + \lambda t^{-2}(\underline{\theta}(t) - \underline{\nabla} w(t), \eta - \underline{\nabla} v) = \int_\Omega f v \quad \forall (\underline{\eta}, v) \in V. \tag{2}$$

It is well-known that a straightforward finite element discretization based on formulation (2) generally fails the approximation because of the shear locking phenomenon (cf. [8,17]). In fact, as the thickness becomes numerically small, the shear term in (1) imposes the Kirchhoff constraint which standard low-order elements cannot stand. Many methods have been proposed to overcome this undesirable lack of convergence. Several of them are based on a mixed formulation of problem (1). More precisely, let us introduce the scaled shear stress (cf. [10], for instance)

$$\underline{\gamma} = \lambda t^{-2}(\underline{\theta} - \underline{\nabla} w) \tag{3}$$

as independent unknown. Thus, the solution of the plate problem turns out to be the *critical point* of the functional

$$\tilde{\Pi}_t(\underline{\theta}, w, \underline{\gamma}) = \frac{1}{2} a(\underline{\theta}, \underline{\theta}) - \frac{\lambda^{-1} t^2}{2} \|\underline{\gamma}\|_{0,\Omega}^2 + (\underline{\gamma}, \underline{\theta} - \underline{\nabla} w) - (f, w) \tag{4}$$

on $V \times (L^2(\Omega))^2$. Hence, the mixed variational plate problem can be written as:

Problem $\tilde{P}_t$: For $t > 0$ fixed, find $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ in $V \times (L^2(\Omega))^2$ such that

$$\begin{aligned} &a(\underline{\theta}(t), \underline{\eta}) + (\underline{\gamma}(t), \underline{\eta} - \underline{\nabla} v) = (f, v) \quad \forall (\underline{\eta}, v) \in V, \\ &(\underline{s}, \underline{\theta}(t) - \underline{\nabla} w(t)) - \lambda^{-1} t^2 (\underline{\gamma}(t), s) = 0 \quad \forall \underline{s} \in (L^2(\Omega))^2. \end{aligned} \tag{5}$$

Following the notation of [10], let us now introduce the differential operators:

$$\underline{\mathrm{rot}}\colon\ \varphi \to \underline{\mathrm{rot}}\ \varphi = \left\{ \frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x} \right\}, \tag{6}$$

$$\mathrm{rot}\colon\ \underline{\chi} \to \mathrm{rot}\ \underline{\chi} = -\frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial x} \tag{7}$$

as well as the Hilbert space $\Gamma = H_0$ (rot; Ω) defined by

$$H_0(\mathrm{rot};\Omega) = \left\{ \underline{\chi} : \underline{\chi}(L^2(\Omega))^2,\ \ \mathrm{rot}\ \underline{\chi} \in L^2(\Omega),\ \ \underline{\chi}\cdot\underline{t} = 0 \text{ on } \partial\Omega \right\}, \tag{8}$$

$$\|\underline{\chi}\|^2_{H_0(\mathrm{rot};\Omega)} := \|\underline{\chi}\|^2_{0,\Omega} + \|\mathrm{rot}\underline{\chi}\|^2_{0,\Omega} \tag{9}$$

(here $\underline{t}$ is the unit tangent to $\partial\Omega$) and its dual space

$$\Gamma' = H^{-1}(\mathrm{div};\Omega) = \left\{ \underline{\gamma} : \underline{\gamma} \in (H^{-1}(\Omega))^2\ \ \mathrm{div}\,\underline{\gamma} \in H^{-1}(\Omega) \right\}. \tag{10}$$

The space Γ' is equipped with the norm

$$\|\underline{\gamma}\|^2_{\Gamma'} := \|\underline{\gamma}\|^2_{-1,\Omega} + \|\mathrm{div}\,\underline{\gamma}\|^2_{-1,\Omega}, \tag{11}$$

which is easily seen to be equivalent to the natural dual norm induced by $H_0(\mathrm{rot};\Omega)$.

We are now ready to claim that the following proposition holds (for a proof see [10], for instance).

Proposition 2.1. *Given $t > 0$ and $f \in H^{-1}(\Omega)$, there exists a unique triple $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ in $V \times (L^2(\Omega))^2$ satisfying Eq.* (5). *Moreover, the following estimate holds*:

$$\|\underline{\theta}(t)\|_1 + \|w(t)\|_1 + \|\underline{\gamma}(t)\|_{\Gamma'} + t\|\underline{\gamma}(t)\|_0 \leqslant C\|f\|_{-1}. \tag{12}$$

We also recall here the following regularity result (cf. [3]).

Proposition 2.2. *Suppose Ω convex and $f \in L^2(\Omega)$. Let $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ be the solution of problem* (5). *Then, the following estimate holds*:

$$\|\underline{\theta}(t)\|_2 + \|w(t)\|_2 + \|\underline{\gamma}(t)\|_{H(\mathrm{div})} + t\|\underline{\gamma}(t)\|_1 \leqslant C\|f\|_0, \tag{13}$$

where

$$\|\underline{\gamma}(t)\|^2_{H(\mathrm{div})} = \|\underline{\gamma}(t)\|^2_0 + \|\mathrm{div}\ \underline{\gamma}(t)\|^2_0. \tag{14}$$

3. Kinematic linked interpolation methods: a general result

From now on, for the sake of simplicity and without loss of generality, we choose $\lambda = 1$. Hence, the mixed variational problem reads as follows:

Problem $\tilde{P}_t$: For $t > 0$ fixed, find $(\underline{\theta}, w, \underline{\gamma})$ in $V \times (L^2(\Omega))^2$ such that

$$\begin{aligned} &a(\underline{\theta}, \underline{\eta}) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla} v) = (f, v) \quad \forall(\underline{\eta}, v) \in V, \\ &(\underline{s}, \underline{\theta} - \underline{\nabla} w) - t^2(\underline{\gamma}, \underline{s}) = 0 \quad \forall \underline{s} \in (L^2(\Omega))^2. \end{aligned} \tag{15}$$

Let us now introduce a sequence $\{\mathscr{T}_h\}_{h>0}$ of partitioning of Ω into elements K (triangles or quadrilaterals) whose diameter h_K is bounded by h. We also suppose the regularity of $\{\mathscr{T}_h\}_{h>0}$ (cf. [15]), in the sense that there exists a constant $\sigma > 0$ such that

$$h_K \leqslant \sigma \rho_K \quad \forall K \in \bigcup_{h>0} \mathscr{T}_h, \tag{16}$$

where h_K is the diameter of K and ρ_K is the maximum diameter of the circles contained in K.

A standard discretization of problem (15) consists in choosing finite dimensional spaces $\Theta_h \subset \Theta$, $W_h \subset W$ and $\Gamma_h \subset (L^2(\Omega))^2$ and in considering the discrete problem

$$\begin{aligned}&\text{Find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \text{ in } \Theta_h \times W_h \times \Gamma_h \text{ such that}\\ &a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla} v) = (f, v) \quad \forall (\underline{\eta}, v) \in \Theta_h \times W_h,\\ &(\underline{s}, \underline{\theta}_h - \underline{\nabla} w_h) - t^2 (\underline{\gamma}_h, \underline{s}) = 0 \quad \forall \underline{s} \in \Gamma_h.\end{aligned} \tag{17}$$

However, a different approach can be used in the discretization procedure. For example, it is possible to "augment" the deflection space by means of the rotational degrees of freedom. More precisely, a suitable linear and bounded operator $L : \Theta_h \to H^1_0(\Omega)$ is defined and the new finite element space

$$V_h = \left\{ (\underline{\eta}_h, v_h + L\underline{\eta}_h) : \ \underline{\eta}_h \in \Theta_h, v_h \in W_h \right\} \tag{18}$$

is selected to approximate the kinematic unknowns. Hence, the discretized problem turns out to be

$$\begin{aligned}&\text{Find } (\underline{\theta}_h, w_h^*; \underline{\gamma}_h) \text{ in } V_h \times \Gamma_h \text{ such that}\\ &a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla} v) = (f, v) \quad \forall (\underline{\eta}, v) \in V_h\\ &(\underline{s}, \underline{\theta}_h - \underline{\nabla} w_h^*) - t^2 (\underline{\gamma}_h, \underline{s}) = 0 \quad \forall \underline{s} \in \Gamma_h.\end{aligned} \tag{19}$$

Due to (18), problem (19) is obviously equivalent to the following:

$$\begin{aligned}&\text{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \text{ in } \Theta_h \times W_h \times \Gamma_h \text{ such that}\\ &a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}(v + L\underline{\eta})) = (f, v + L\underline{\eta}) \quad \forall (\underline{\eta}, v) \in \Theta_h \times W_h,\\ &(\underline{s}, \underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)) - t^2 (\underline{\gamma}_h, \underline{s}) = 0 \quad \forall \underline{s} \in \Gamma_h\end{aligned} \ . \tag{20}$$

Remarks 3.1. Although the linking operator L is defined only on Θ_h, a natural extension to Θ is provided by any interpolation operator $(\cdot)_{\mathrm{II}} : \Theta \to \Theta_h$, simply by setting

$$L\underline{\eta} := L\underline{\eta}_{\mathrm{II}} \quad \forall \underline{\eta} \in \Theta.$$

To perform a general error analysis for a method based on formulation (20), let us first introduce a mesh-dependent norm for the shear stresses

$$\|\underline{\xi}\|_h^2 := \sum_{K \in \mathscr{T}_h} h_K^2 \|\underline{\xi}\|_{0,K}^2 \quad \forall \underline{\xi} \in (L^2(\Omega))^2 \tag{21}$$

and, given $(\underline{\theta}, w, \underline{\xi}) \in \Theta \times W \times (L^2(\Omega))^2$, set

$$|||\underline{\theta}, w, \underline{\xi}|||^2 := \|\underline{\theta}\|_1^2 + \|w\|_1^2 + \|\underline{\xi}\|_h^2 + t^2 \|\underline{\xi}\|_0^2. \tag{22}$$

Furthermore, let us assume as true the following two hypotheses involving only the discretization spaces:

$$\|P_h \underline{\nabla} v_h\|_0 \geqslant c \|\underline{\nabla} v_h\|_0 \quad \forall v_h \in W_h, \tag{23}$$

$$\sup_{(\underline{\eta}, v)} \frac{(\underline{s}, \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1} \geqslant \beta \|\underline{s}\|_h \quad \forall \underline{s} \in \Gamma_h, \tag{24}$$

where β is positive constant independent of h and the supremum is made over the discrete space $\Theta \times W_h$. Moreover, P_h is the L^2-projection operator onto Γ_h and c is a positive constant independent of h. It is interesting to notice that, due to hypothesis (23), *for $t > 0$ fixed*, problem (20) has a unique solution. *From now on, we always suppose that the hypotheses* (23) *and* (24) *are both fulfilled.*

Remarks 3.2. If one has

$$\underline{\nabla} W_h \subseteq \Gamma_h, \tag{25}$$

then condition (23) is certainly fulfilled with $c = 1$. In practice, it is the above condition (25) which is required, rather than the more general (23). Furthermore, note that, being the projection operator P_h nonexpansive, condition, (23) holds for $c \leqslant 1$. Moreover, let us set

$$\mathscr{A}(\underline{\theta}, w, \gamma; \underline{\eta}, v, \underline{s}) := a(\underline{\theta}, \underline{\eta}) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla} v) - (\underline{s}, \underline{\theta} - \underline{\nabla} w) + t^2(\underline{\gamma}, \underline{s}) \tag{26}$$

and

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, \underline{s}_h) := a(\underline{\theta}_h, \underline{\eta}_h) + (\underline{\gamma}_h, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) - (\underline{s}_h, \underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)) + t^2(\underline{\gamma}_h, \underline{s}_h). \tag{27}$$

Hence, the continuous problem (15) can be rewritten as

$$\begin{aligned} &\text{find } (\underline{\theta}, w, \underline{\gamma}) \text{ in } \Theta \times W \times (L^2(\Omega))^2 \text{ such that} \\ &\mathscr{A}(\underline{\theta}, w, \underline{\gamma}; \underline{\eta}, v, \underline{s}) = (f, v) \quad \forall(\underline{\eta}, v, \underline{s}) \in \Theta \times W \times (L^2(\Omega))^2, \end{aligned} \tag{28}$$

while the discretized one reads as follows:

$$\begin{aligned} &\text{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \text{ in } \Theta_h \times W_h \times \Gamma_h \text{ such that} \\ &\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}, v, \underline{s}) = (f, v + L\underline{\eta}) \quad \forall(\underline{\eta}, v, \underline{s}) \in \Theta_h \times W_h \times \Gamma_h. \end{aligned} \tag{29}$$

We are now ready to prove our stability result. The technique used is similar to that involved in [13,19]. However, for the sake of completeness and for the convenience of the reader, we developed it in detail.

Proposition 3.1. *Given* $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h$, *there exists* $(\underline{\eta}_h, v_h, \underline{s}_h) \in \Theta_h \times W_h \times \Gamma_h$ *such that*

$$|||\underline{\eta}_h, v_h, \underline{s}_h||| \leqslant C|||\underline{\theta}_h, w_h, \underline{\gamma}_h||| \tag{30}$$

and

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, v_h, \underline{s}_h) \geqslant C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||^2. \tag{31}$$

Proof. Let us $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ be given in $\theta_h \times W_h \times \Gamma_h$. The proof is performed in three steps.

(i) Let us first choose $(\underline{\eta}_1, v_1, \underline{s}_1) \in \Theta_h \times W_h \times \Gamma_h$ such that $\underline{\eta}_1 = \underline{\theta}_h, v_1 = w_h$ and $\underline{s}_1 = \underline{\gamma}_h$. It is obvious that

$$|||\underline{\eta}_1, v_1, \underline{s}_1||| = |||\underline{\theta}_h, w_h, \underline{\gamma}_h|||. \tag{32}$$

Furthermore, it holds

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{s}_1) = a(\underline{\theta}_h, \underline{\theta}_h) + t^2\|\underline{\gamma}_h\|_0^2. \tag{33}$$

By Korn's inequality it follows that

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{s}_1) \geqslant C_1\left(\|\underline{\theta}_h\|_1^2 + t^2\underline{\gamma}_h\|_0^2\right). \tag{34}$$

(ii) Let us first notice that from Eq. (24) it follows that there exists $(\underline{\eta}_2, v_2) \in \Theta_h \times W_h$ such that

$$\|\underline{\eta}_2\|_1 + \|v_2\|_1 \leqslant C\|\underline{\gamma}_h\|_h \tag{35}$$

and

$$(\underline{\gamma}_h, \underline{\eta}_2 - \underline{\nabla}(v_2 + L\underline{\eta}_2)) = \|\underline{\gamma}_h\|_h^2. \tag{36}$$

Choose now $(\underline{\eta}_2, v_2, \underline{s}_2) \in \Theta_h \times W_h \times \Gamma_h$, such that $\underline{s}_2 = 0$. On the one hand, due to estimate (35), the following inequality holds:

$$|||\underline{\eta}_2, v_2, \underline{s}_2||| \leqslant C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||. \tag{37}$$

On the other hand

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{s}_2) = a(\underline{\theta}_h, \underline{\eta}_2) + (\underline{\gamma}_h, \underline{\eta}_2 - \underline{\nabla}(v_2 + L\underline{\eta}_2)), \tag{38}$$

due to (36)

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{s}_2) = a(\underline{\theta}_h, \underline{\eta}_2) + \|\underline{\gamma}_h\|_h^2. \tag{39}$$

To control the first term in the right-hand side of Eq. (39), we note that

$$a(\underline{\theta}_h, \underline{\eta}_2) \geqslant -\frac{M}{2\delta}\|\underline{\theta}_h\|_1^2 - \frac{\delta M}{2}\|\underline{\eta}_2\|_1^2 \geqslant -\frac{M}{2\delta}\|\underline{\theta}_h\|_1^2 - \frac{\delta C M}{2}\|\underline{\gamma}_h\|_h^2.$$

Taking δ sufficiently small, we get

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{s}_2) \geqslant C_2\|\underline{\gamma}_h\|_h^2 - C_3\|\underline{\theta}_h\|_1^2. \tag{40}$$

(iii) Choose $(\underline{\eta}_3, v_3, \underline{s}_3) \in \Theta_h \times W_h \times \Gamma_h$ such that $\underline{\eta}_3 = 0, v_3 = 0$ and $\underline{s}_3 = P_h\underline{\nabla} w_h$. We have

$$|||\underline{\eta}_3, v_3, \underline{s}_3|||^2 = \|P_h\underline{\nabla} w_h\|_h^2 \leqslant C\|\underline{\nabla} w_h\|_0^2 \leqslant C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||^2. \tag{41}$$

It holds

$$\begin{aligned}
\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_3, v_3, \underline{s}_3) &= -(P_h\underline{\nabla} w_h, \underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)) + t^2(\underline{\gamma}_h, P_h\underline{\nabla} w_h) \\
&= \|P_h\underline{\nabla} w_h\|_0^2 - (P_h\underline{\nabla} w_h, \underline{\theta}_h - \underline{\nabla} L\underline{\theta}_h) + t^2(\underline{\gamma}_h, P_h\underline{\nabla} w_h) \\
&\geqslant \left(1 - \frac{\delta}{2}\right)\|P_h\underline{\nabla} w_h\|_0^2 - \frac{1}{2\delta}\|\underline{\theta}_h - \underline{\nabla} L\underline{\theta}_h\|_0^2 + t^2(\underline{\gamma}_h, P_h\underline{\nabla} w_h) \\
&\geqslant \left(1 - \frac{\delta}{2}\right)\|P_h\underline{\nabla} w_h\|_0^2 - \frac{C}{2\delta}\|\underline{\theta}_h\|_1^2 + t^2(\underline{\gamma}_h, P_h\underline{\nabla} w_h).
\end{aligned} \tag{42}$$

Moreover, one has

$$t^2(\underline{\gamma}_h, P_h\underline{\nabla} w_h) \geqslant t^2\left(-\frac{1}{2\varepsilon}\|\underline{\gamma}_h\|_0^2 - \frac{\varepsilon}{2}\|P_h\underline{\nabla} w_h\|_0^2\right). \tag{43}$$

By (42) and (43), using (23) and taking δ and ε sufficiently small, one finally gets

$$\mathscr{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_3, v_3, \underline{s}_3) \geqslant C_4\|\underline{\nabla} w_h\|_0^2 - C_5\|\underline{\theta}_h\|_1^2 - C_6 t^2\|\underline{\gamma}_h\|_0^2. \tag{44}$$

Now it only suffices to take a suitable linear combination of $\{\underline{\eta}_i, v_i, \underline{s}_i\}_{i=1}^3$ so that by (32), (34), (37), (40), (41) and (44) it follows that (30) and (31) hold. The proof is then complete. □

We are now ready to perform the error analysis.

Proposition 3.2. *Let* $(\underline{\theta}, w, \underline{\gamma})$ *be the solution of problem* (28). *Let* $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ be *the solution of the discretized problem* (29). *Given* $(\underline{\theta}_{\mathrm{II}}, w_{\mathrm{I}}, \underline{\gamma}^*) \in \Theta_h \times w_h \times \Gamma_h$ *the following error estimate holds*:

$$\begin{aligned}
&\|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + t\|\underline{\gamma} - \underline{\gamma}_h\|_0 \\
&\quad \leqslant C\Big(\|\underline{\theta} - \underline{\theta}_{\mathrm{II}}\|_1 + \|w - w_{\mathrm{I}}\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + \|\underline{\gamma} - \underline{\gamma}^*\|_h + t\|\underline{\gamma} - \underline{\gamma}^*\|_0\Big) \\
&\qquad + C\sup_{\underline{s}\in\Gamma_h}\frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{\mathrm{II}} - \underline{\nabla}(w - w_{\mathrm{I}} - L\underline{\theta}_{\mathrm{II}}))}{\|\underline{s}\|_h + t\|\underline{s}\|_0}.
\end{aligned} \tag{45}$$

Proof. By Proposition 3.1, given $(\underline{\theta}_h - \underline{\theta}_{\mathrm{II}}, w_h - w_{\mathrm{I}}, \underline{\gamma}_h - \underline{\gamma}^*) \in \Theta_h \times W_h \times \Gamma_h$, there exists $(\underline{\eta}_h, v_h, \underline{s}_h) \in \Theta_h \times W_h \times \Gamma_h$ such that

$$|||\underline{\eta}_h, v_h, \underline{s}_h||| \leqslant C|||\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*||| \tag{46}$$

and

$$\mathscr{A}_h(\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*; \underline{\eta}_h, v_h, \underline{s}_h) \geqslant C|||\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2. \tag{47}$$

By recalling (28) and (29) and noting that $(\underline{\theta}, w, \underline{\gamma})$ (resp. $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$) is the solution of the continuous (resp. discretized) problem, one obtains from (47)

$$\begin{aligned} C|||\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2 \leqslant\; & a(\underline{\theta} - \underline{\theta}_{II}, \underline{\eta}_h) + (\underline{\gamma} - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\ & - (\underline{s}_h, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II})) + t^2(\underline{\gamma} - \underline{\gamma}^*, \underline{s}_h). \end{aligned} \tag{48}$$

Due to the continuity of the bilinear forms involved in estimate (48) it follows:

$$\begin{aligned} C|||\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2 \leqslant\; & C\Big(\|\underline{\theta} - \underline{\theta}_{II}\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|\Gamma' + t\|\underline{\gamma} - \underline{\gamma}^*\|_0\Big)|||\underline{\eta}_h, v_h, \underline{s}_h||| \\ & + C|||\underline{\eta}_h, v_h, \underline{s}_h||| \sup_{\underline{s}\in\Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}))}{\|\underline{s}\|_h + t\|\underline{s}\|_0}. \end{aligned} \tag{49}$$

Therefore, from (46) one obtains

$$\begin{aligned} & C|||\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*||| \\ & \leqslant C\Big(\|\underline{\theta} - \underline{\theta}_{II}\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + t\|\underline{\gamma} - \underline{\gamma}^*\|_0\Big) + C \sup_{\underline{s}\in\Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}))}{\|\underline{s}\|_h + t\|\underline{s}\|_0}, \end{aligned} \tag{50}$$

and the estimate (45) follows from the triangle inequality. □

Remarks 3.3. Proposition 3.2 provides an error estimate for the methods based on a kinematic linked interpolation operator whenever one chooses $(\underline{\theta}_{II}, w_I, \underline{\gamma}^*)$ as "good" interpolating functions of the continuous solution of the problem. Furthermore, the supremum in estimate (45) arises from the non-consistency of the discrete problem, due to the presence of the linking operator L.

Remarks 3.4. To bound the supremum appearing in (45) it is obviously necessary to explicitly define the operator L and to study its features. In the next section, we give some examples in which the consistency error is of the same rate, in terms of powers of h, as the interpolation error given by the choice of the discretization spaces. In these cases Proposition 3.2 leads to an optimal error estimate for the kinematic unknowns.

Let us now see how the last term in the right-hand side of (45) can be treated. To make our argument easier, let us *first* consider the limit case of a "zero thickness" plate (i.e. $t = 0$). Accordingly,

$$\underline{\theta}_{II} = (\underline{\nabla} w)_{II}, \tag{51}$$

so that we have to study the term

$$\sup_{\underline{s}\in\Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L(\underline{\nabla} w)_{II}))}{\|\underline{s}\|_h}. \tag{52}$$

In particular, let us suppose that both $(\cdot)_I$ and $(\cdot)_{II}$ are Lagrange interpolation operators over the continuous piecewise linear functions. By standard arguments, we obtain that

$$(\underline{s}, \underline{\theta} - \underline{\theta}_{II}) \leqslant C\|\underline{s}\|_h h^{-1}\|\underline{\theta} - \underline{\theta}_{II}\|_0 \leqslant Ch|\underline{\theta}|_2\|\underline{s}\|_h. \tag{53}$$

The troubles arise whenever we try to bound the term involving the gradient. In fact, due to Poincaré inequality,

$$(\underline{s}, \underline{\nabla}(w - w_{\mathrm{I}} - L(\underline{\nabla} w)_{\mathrm{II}})) \leqslant C h^{-1} \|w - \Pi w\|_1 \|\underline{s}\|_h, \tag{54}$$

where we have set

$$\Pi w := w_{\mathrm{I}} + L(\underline{\nabla} w)_{\mathrm{II}}. \tag{55}$$

Hence, to make $h^{-1}\|w - \Pi w\|_1$ to be $\mathrm{O}(h)$, which is of course what we hope, we need better approximation features for the operator Π than those met by the Lagrange operator. More precisely, it is natural to require that

$$\Pi \text{ is a } P_2 - \text{invariant operator},$$

so that by Bramble–Hilbert lemma it holds

$$h^{-1}\|w - \Pi w\|_1 \leqslant C h |w|_3 \tag{56}$$

and our goal is obtained.

In general, if $\|v - v_{\mathrm{I}}\|_1 = \mathrm{O}(h^k)$, we wish to select an operator L in such a way that $\|v - \Pi v\|_1 = \mathrm{O}(h^{k+1})$, where Π is defined by (55).

Let us consider the case of arbitrary thickness $t \geqslant 0$ and suppose to have for the continuous solution enough regularity to be able to well-define everything that follows.

Since

$$t^2 \underline{\gamma} = \underline{\theta} - \underline{\nabla} w$$

we have that

$$\underline{\nabla} L \underline{\theta}_{\mathrm{II}} = t^2 \underline{\nabla} L \underline{\gamma}_{\mathrm{II}} + \underline{\nabla} L(\underline{\nabla} w)_{\mathrm{II}}. \tag{57}$$

Hence,

$$-(\underline{s}, -\underline{\nabla}(w - w_{\mathrm{I}} - L\underline{\theta}_{\mathrm{II}})) = -(\underline{s}, -\underline{\nabla}(w - w_{\mathrm{I}} - L(\underline{\nabla} w)_{\mathrm{II}}) + t^2(\underline{s}, \underline{\nabla} L \underline{\gamma}_{\mathrm{II}}). \tag{58}$$

It follows that

$$(\underline{s}, \underline{\theta} - \underline{\theta}_{\mathrm{II}} - \underline{\nabla}(w - w_{\mathrm{I}} - L\underline{\theta}_{\mathrm{II}})) \leqslant C\Big(h^{-1}\|\underline{\theta} - \underline{\theta}_{\mathrm{II}}\|_0 + h^{-1}\|w - \Pi w\|_1 + t\|\underline{\nabla} L \underline{\gamma}_{\mathrm{II}}\|_0\Big)(\|\underline{s}\|_h + t\|\underline{s}\|_0). \tag{59}$$

Remarks 3.5. The term $h^{-1}\|\underline{\theta} - \underline{\theta}_{\mathrm{II}}\|_0$ causes no trouble, since the L^2-norm, and not the H^1-norm, is involved.

Finally, supposing to have the necessary regularity, we can summarize what has been done so far in the following proposition.

Proposition 3.3. *Let* $(\underline{\theta}, w, \underline{\gamma})$ *be the solution of problem* (28). *Let* $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ *be the solution of the discretized problem* (29). *Suppose that both* (23) *and* (24) *hold. Suppose moreover to have a scalar interpolating operator* $(\cdot)_{\mathrm{I}}$ *taking values on* W_h, *and another one* (*vectorial*) $(\cdot)_{\mathrm{II}}$ *taking values on* Θ_h. *Setting* $\Pi w := w_{\mathrm{I}} + L(\underline{\nabla} w)_{\mathrm{II}}$ (cf. (55)), *the following error estimate holds*:

$$\begin{aligned} &\|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + t\|\underline{\gamma} - \underline{\gamma}_h\|_0 \\ &\quad \leqslant C\big(\|\underline{\theta} - \underline{\theta}_{\mathrm{II}}\|_1 + h^{-1}\|\underline{\theta} - \underline{\theta}_{\mathrm{II}}\|_0 + \|w - w_{\mathrm{I}}\|_1 + h^{-1}\|w - \Pi w\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} \\ &\qquad + t(\|\underline{\gamma} - \underline{\gamma}^*\|_0 + |L\underline{\gamma}_{\mathrm{II}}|_1)\big) \end{aligned} \tag{60}$$

4. Some example of elements

The aim of this section is to present and study some elements designed following the kinematic linked interpolation technique. We wish to remark that the analysis for the quadrilateral element is performed only in the rectangular case.

4.1. A low-order element

We first present a finite element already proposed in [6,22] and theoretically studied in [19,20]. Following [22], let us set:

$$\begin{aligned} \Theta_h &= \{\underline{\eta}_h \in \Theta : \underline{\eta}_{h|T} \in (P_1(T) \oplus B_3(T))^2 \;\; \forall T \in \mathscr{T}_h\}, \\ W_h &= \{v_h \in W : v_{h|T} \in P_1(T) \;\; \forall T \in \mathscr{T}_h\}, \\ \Gamma_h &= \{\underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|T} \in (P_0(T))^2 \;\; \forall T \in \mathscr{T}_h\}, \end{aligned} \tag{61}$$

where $P_r(T)$ is the space of polynomials defined on T of degree at most r and $B_3(T)$ is the space of cubic bubbles defined on T. To define the linear operator L, let us first introduce for each $T \in \mathscr{T}_h$ the functions.

$$\varphi_i = \lambda_j \lambda_k. \tag{62}$$

In (62) $\{\lambda_i\}_{1\leqslant i\leqslant 3}$ are the barycentric coordinates of the triangle T and the indices (i, j, k) form a permutation of the set (1, 2, 3). In a sense, the function φ_i is a sort of edge bubble relatively to the edge e_i of T. Let us now set

$$\mathrm{EB}_2(T) = \mathrm{Span}\ \{\varphi_i\}_{1\leq i\leqslant 3} \tag{63}$$

The operator L is locally defined (cf. [26]) as

$$L_{|T}\underline{\eta}_h = \sum_{i=1}^{3} \alpha_i \varphi_i \in \mathrm{EB}_2(T), \tag{64}$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla} L \underline{\eta}_h) \cdot \underline{\tau}_i = \text{constant along each } e_i \tag{65}$$

with $\underline{\tau}_i$ the vector tangential to the edge e_i.

For this element it is obvious that

$$\underline{\nabla} W_h \subseteq \Gamma_h,$$

so that condition (23) is met (cf. Remark 3.2). Moreover, in [19] it has been shown that the inf–sup condition (24) is met, by simply taking suitable rotational bubble functions.

As far as the linear operator L is concerned, in [19] it has also been proven that

- $L : \Theta \to W_h$ is well-defined, bounded and that

  $$|L\underline{\eta}|_1 \leqslant Ch|\underline{\eta}|_1 \forall \underline{\eta} \in \Theta_h.$$

- If $(\cdot)_{\mathrm{I}}$ and $(\cdot)_{\mathrm{II}}$ are the usual Lagrange interpolation operators, setting

  $$\Pi v := v_{\mathrm{I}} + L(\underline{\nabla} v)_{\mathrm{II}},$$

  we have that $\Pi : H^3(\Omega) \to H^1(\Omega)$ is a P_2-invariant operator, so that

  $$\|v - \Pi v\|_1 \leqslant Ch^2 |v|_3.$$

Hence, one can invoke Proposition 3.3 to conclude that the method is first-order convergent in the kinematic variables, uniformly in the plate thickness.

Remarks 4.1. This scheme has a strong connection with another one based on the MITC philosophy (cf. [9,11,12] for a detailed analysis of such a family of plate elements). By means of this connection, in the paper [20] an error analysis (using different techniques from those employed in [19]) has been developed.

4.2. A new quadratic element

We wish to present now a new element, which can be considered as the quadratic version of the previous method. More precisely, let us select

$$\begin{aligned} \Theta_h &= \{\underline{\eta}_h \in \Theta : \underline{\eta}_{h|T} \in P_2(T)^2 \oplus (P_1(T)^2 \oplus \underline{\nabla} B_3(T)) b_T \ \ \forall T \in \mathscr{T}_h\}, \\ W_h &= \{v_h \in W : v_{h|T} \in P_2(T) \oplus B_3(T) \ \ \forall T \in \mathscr{T}_h\}, \\ \Gamma_h &= \{\underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|T} \in P_1(T)^2 \oplus \underline{\nabla} B_3(T) \ \ \forall T \in \mathscr{T}_h\}, \end{aligned} \tag{66}$$

where $b_T = 27\lambda_1\lambda_2\lambda_3$. To define the linear operator L, let us first introduce for each $T \in \mathscr{T}_h$ the functions

$$\varphi_i = \lambda_j\lambda_k(\lambda_k - \lambda_j), \tag{67}$$

where the indices (i, j, k) form a permutation of the set (1, 2, 3).

Let us now set

$$\mathrm{EB}_3(T) = \mathrm{Span}\{\varphi_i\}_{1 \leqslant i \leqslant 3}. \tag{68}$$

The operator L is locally defined as

$$L_{|T}\underline{\eta}_h = \sum_{i=1}^{3} \alpha_i \varphi_i \in \mathrm{EB}_3(T), \tag{69}$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla} L \underline{\eta}_h) \cdot \underline{\tau}_i = \text{linear along each } e_i. \tag{70}$$

Remarks 4.2. Choice (67) may seem ambiguous: exchanging k with j the function φ_i changes sign. As a result, in building the global operator L by means of (69), one may have constructed a linear operator which does not take values in $H_0^1(\Omega)$. However, it is easily realized that this drawback is overcome by simply choosing the φ_i functions in a consistent way between each two adjacent triangles.

Remarks 4.3. A reduction in the number of internal degrees of freedom for rotations and shears can be performed in connection with a Partial Selective Reduced Integration scheme (cf. [2,14,18]). See [7] for details and other methods based on this idea.

Notice that also for this choice

$$\underline{\nabla} W_h \subseteq \Gamma_h.$$

Let us now turn our attention to the features of the operator L. It is easily seen that the operator L is well-defined by condition (70). In fact, let us notice that each $\underline{\eta}_h \in \Theta_h$ can be locally (on each triangle T) split into

$$\underline{\eta}_h = \underline{\eta}_1 + \underline{\eta}_2 + \underline{\eta}_\mathrm{b},$$

where $\underline{\eta}_1$ is the linear part of $\underline{\eta}_h$, $\underline{\eta}_\mathrm{b}$ is a bubble-type function, while $\underline{\eta}_2$ is the "purely quadratic" part of $\underline{\eta}_h$ (in the sense that it is a quadratic vector function vanishing at the vertices of T). Hence, condition (70) leads to solve, since $\underline{\eta}_1 \cdot \underline{\tau}_i$ is already linear and $\underline{\eta}_\mathrm{b} \cdot \underline{\tau}_i = 0$,

$$(\underline{\eta}_2 - \alpha_i \underline{\nabla}\varphi_i) \cdot \underline{\tau}_i = b_i + c_i s \quad \text{along each } e_i, \tag{71}$$

where s is a local coordinate on e_i running from -1 up to 1. Integrating along e_i one obtains

$$\int_{e_i} \underline{\eta}_2 \cdot \underline{\tau}_i = b_i |e_i| \quad \text{along each } e_i, \tag{72}$$

where $|e_i|$ is the length of the edge. From (72) we get b_i

$$b_i = \frac{1}{|e_i|}\int_{e_i} \underline{\eta}_2 \cdot \underline{\tau}_i := \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}. \tag{73}$$

To continue, multiply Eq. (71) by s and integrate along the side. It is easy to see that one has

$$c_i = 0.$$

Therefore,

$$\alpha_i \underline{\nabla}\varphi_i \cdot \underline{\tau}_i = \underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}. \tag{74}$$

Notice that Eq. (74) is uniquely solvable. In fact, $\underline{\nabla}\varphi_i \cdot \underline{\tau}_i$ is an even quadratic polynomial with respect to the local coordinate s and it has zero mean value on e_i. Furthermore, $\underline{\eta}_2 \cdot \underline{\tau}_i$ is an even quadratic polynomial in s, so that $\underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}$ is even in s and it has zero mean value on e_i. It follows that $\underline{\nabla}\varphi_i \cdot \underline{\tau}_i$ and $\underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}$ differ by a multiplicative constant α_i. Moreover, $L\underline{\eta}_h$ is defined by the purely quadratic part of $\underline{\eta}_h$ only. Finally, from (74) by an easy scaling argument we get

$$|\alpha_i| \|\underline{\nabla}\varphi_i \cdot \underline{\tau}_i\|_{L^\infty(e_i)} \leqslant C|\underline{\eta}_2|_{1,T}, \tag{75}$$

such that

$$|\alpha_i| \leqslant Ch_T|\underline{\eta}_2|_{1,T}. \tag{76}$$

Hence,

$$|L\underline{\eta}_h|_{1,T} \leqslant C\sum_{i=1}^{3}|\alpha_i| \leqslant Ch_T|\underline{\eta}_2|_{1,T} \leqslant Ch_T|\underline{\eta}_h - \underline{\eta}_1 - \underline{\eta}_{\mathrm{b}}|_{1,T} \leqslant Ch_T^2|\underline{\eta}_h|_{2,T}. \tag{77}$$

We can now verify the inf–sup condition (24). Again, given $\underline{s} \in \Gamma_h$, let us choose $\underline{\eta}_h$ such >that its restriction on T is $\underline{\eta}_h = h_T^2 b_T \underline{s}$ and select $v_h = 0$. Note that this is admissible due to choice (66). Since $L\underline{\eta}_h = 0$, it is straightforward to have

$$(\underline{s}, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) = (\underline{s}, \underline{\eta}_h) \geqslant C\|\underline{s}\|_h^2 \tag{78}$$

and

$$\|\underline{\eta}_h\|_1 \leqslant C\|\underline{s}\|_h, \tag{79}$$

from which the inf–sup condition easily follows.

It remains to treat the term $\|w - \Pi w\|_1$. Let us choose as $(\cdot)_{\mathrm{II}}$ the standard Lagrange interpolation operator over the continuous piecewise quadratic functions. Moreover, let $(\cdot)_{\mathrm{I}}$ be the Lagrange interpolation operator over the space W_h. Setting

$$\Pi v = v_{\mathrm{I}} + L(\underline{\nabla} v)_{\mathrm{II}}$$

we wish to have that Π is P_3-invariant, i.e. $\Pi v = v$ for every $v \in P_3$. We proceed, as usual, locally on each element T. We first note that the following direct sum splitting holds true:

$$P_3(T) = P_2(T) \oplus B_3(T) \oplus \mathrm{EB}_3(T). \tag{80}$$

If $v \in P_2(T)$ then $v_{\mathrm{I}} = v$ and

$$L(\underline{\nabla} v)_{\mathrm{II}} = L(\underline{\nabla} v) = 0,$$

since $\underline{\nabla} v$ is already linear. Hence, $\Pi v = v$ in this case. Take now $v \in B_3(T)$. We have $v_{\mathrm{I}} = v$. Moreover, we have $\underline{\nabla} v \in P_2(T)^2$ so that

$$(\underline{\nabla} v)_{\mathrm{II}} = \underline{\nabla} v.$$

Consider the equation

$$(\underline{\nabla} v - \underline{\nabla}(L(\underline{\nabla} v)_{\mathrm{II}})) \cdot \underline{\tau}_i = \text{linear along each } e_i. \tag{81}$$

Since $\underline{\nabla} v \cdot \underline{\tau}_i = 0$ the only solution of (81) is $L(\underline{\nabla} v)_{\mathrm{II}} = 0$ and also in this case we have $\Pi v = v$. Finally, choose $v \in \mathrm{EB}_3(T)$. Note that $v_{\mathrm{I}} = 0$, and by unicity the equation

$$(\underline{\nabla} v - \underline{\nabla}(L(\underline{\nabla} v)_{\mathrm{II}})) \cdot \underline{\tau}_i = \text{linear along each } e_i \tag{82}$$

has the solution

$$L(\underline{\nabla} v)_{\mathrm{II}} = v. \tag{83}$$

Hence, $\Pi v = v_{\mathrm{I}} + L(\underline{\nabla} v)_{\mathrm{II}} = L(\underline{\nabla} v)_{\mathrm{II}} = v$ and Π is actually P_3-invariant. It follows that

$$\|w - \Pi w\|_1 \leqslant Ch^3 |w|_4.$$

As far as $\underline{\gamma}^*$ is concerned, choose it as the L^2-projection of $\underline{\gamma}$ onto Γ_h. Furthermore, note that by estimate (77) it holds

$$\|L\underline{\gamma}_{\mathrm{II}}\|_1 \leqslant Ch^2 |\underline{\gamma}|_2.$$

From standard interpolation theory and Proposition 3.3 we can then conclude that for the method under consideration we have the estimate

$$\|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + t\|\underline{\gamma} - \underline{\gamma}_h\|_0 \leqslant Ch^2, \tag{84}$$

where the constant C above depends on $|\underline{\theta}|_3$, $|w|_4$ and $|\underline{\gamma}|_2$.

4.3. A quadrilateral element

We now consider the element presented in [5]. We first set

$$\Gamma_h = \{\underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|K} = (a + b\eta, c + d\xi) \ \ \forall K \in \mathscr{T}_h\}, \tag{85}$$

where (ξ, η) are the standard isoparametric coordinates of K.
We then select

$$\Theta_h = \{\underline{\eta}_h \in \Theta : \underline{\eta}_{h|K} \in Q_1(K)^2 \oplus \Gamma_h b_4 \ \ \forall K \in \mathscr{T}_h\}, \tag{86}$$

where $Q_r(T)$ is the space of polynomials defined on K of degree at most r in each isoparametric coordinate ξ and ξ, while $b_4 = (1 - \xi^2)(1 - \eta^2)$. Finally, we take

$$W_h = \{v_h \in W : v_{h|K} \in Q_1(K) \quad \forall K \in \mathscr{T}_h\}. \tag{87}$$

To define the linear operator L, let us introduce for each $K \in \mathscr{T}_h$ the functions

$$\varphi_i = \lambda_j \lambda_k \lambda_m. \tag{88}$$

In (88) $\{\lambda_i\}_{1 \leqslant i \leqslant 4}$ are the equations of the sides of K and the indices (i, j, k, m) form a permutation of the set (1, 2, 3, 4). Again, the function φ_i is a sort of edge bubble relatively to the edge e_i of K. Let us now set

$$\mathrm{EB}(K) = \mathrm{Span}\{\varphi_i\}_{1 \leqslant i \leqslant 4}. \tag{89}$$

The operator L is locally defined (cf. [5]) as

$$L_{|K} \underline{\eta}_h = \sum_{i=1}^{4} \alpha_i \varphi_i \in \mathrm{EB}(K), \tag{90}$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla} L \underline{\eta}_h) \cdot \underline{\tau}_i = \text{constant along each } e_i. \tag{91}$$

We will not develop the details of the analysis for this method, since it is similar to that of the methods presented above. Instead, we wish to remark that *in the rectangular case* it is easy to show that:

- It holds

$$\underline{\nabla} W_h \subseteq \Gamma_h.$$

- $Q_1(K) \oplus \mathrm{EB}(K)$ is the space of the classical "serendipity" element (cf. [17]), so that in particular

$$P_2(K) \subseteq Q_1(K) \oplus \mathrm{EB}(K).$$

Furthermore, it is not hard to show, following the guidelines of the other sections, that for this quadrilateral element one has:

- $L : \Theta \to W_h$ is well-defined, bounded and it holds

$$|L\underline{\eta}|_1 \leqslant Ch|\underline{\eta}|_1 \quad \forall \underline{\eta} \in \Theta_h.$$

- If $(\cdot)_{\mathrm{I}}$ and $(\cdot)_{\mathrm{II}}$ are the usual Lagrange interpolation operators, setting

$$\Pi v := v_{\mathrm{I}} + L(\underline{\nabla} v)_{\mathrm{II}},$$

we have that $\Pi : H^3(\Omega) \to H^1(\Omega)$ is a P_2-invariant operator (cf. [7]), so that

$$\|v - \Pi v\|_1 \leqslant Ch^2 |v|_3.$$

Therefore, applying Proposition 3.3, it is easily seen that the method is first-order convergent, uniformly in the thickness.

5. Numerical tests

The aim of this section is to present some numerical tests showing the behaviour of the interpolating schemes previously considered. For simplicity, we consider only the case of a clamped plate. However, we remark that in the papers [4–6,22], extensive numerical results for other boundary conditions have been presented, confirming that the linked interpolation technique properly behaves. The schemes have been implemented into the finite element analysis program (FEAP) (cf. [25]) and their performances have been checked on a model problem for which the exact solution is explicitly known (cf. [14]). This allows to compute the discrete solution error. The model problem consists in a clamped square plate $\Omega = [0,1] \times [0,1]$, subject to the transverse load

$$\begin{aligned} f(x,y) = \frac{E}{12(1-\nu^2)}\Big[&12y(y-1)(5x^2-5x+1)\Big(2y^2(y-1)^2 + x(x-1)(5y^2-5y+1)\Big) \\ &+ 12x(x-1)(5y^2-5y+1)\Big(2x^2(x-1)^2 + y(y-1)(5x^2-5x+1)\Big)\Big]. \end{aligned} \tag{92}$$

The exact solution is given by

$$\theta_1(x,y) = y^3(y-1)^3 x^2 (x-1)^2 (2x-1), \tag{93}$$

$$\theta_2(x,y) = x^3(x-1)^3 y^2 (y-1)^2 (2y-1), \tag{94}$$

$$\begin{aligned} w(x,y) = &\frac{1}{3}x^3(x-1)^3 y^3 (y-1)^3 \\ &- \frac{2t^2}{5(1-\nu)}\Big[y^3(y-1)^3 x(x-1)(5x^2-5x+1) + x^3(x-1)^3 y(y-1)(5y^2-5y+1)\Big]. \end{aligned} \tag{95}$$

The error of a discrete solution is measured through the discrete relative rotation error E_θ and the discrete relative displacement error E_w, defined as

$$E_\theta^2 = \frac{\sum_{N_i}\left[(\theta_{h1}(N_i) - (\theta_1(N_i))^2 + (\theta_{h2}(N_i) - (\theta_2(N_i))^2\right]}{\sum_{N_i}\left[(\theta_1(N_i))^2 + (\theta_2(N_i))^2\right]}, \tag{96}$$

$$E_w^2 = \frac{\sum_{N_i}(w_h(N_i) - w(N_i))^2}{\sum_{N_i} w(N_i)^2}. \tag{97}$$

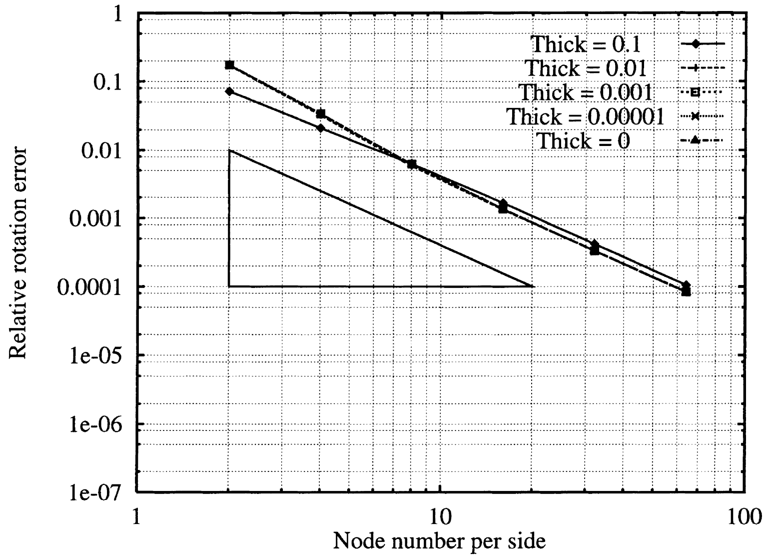

Fig. 1. T3-LIM element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

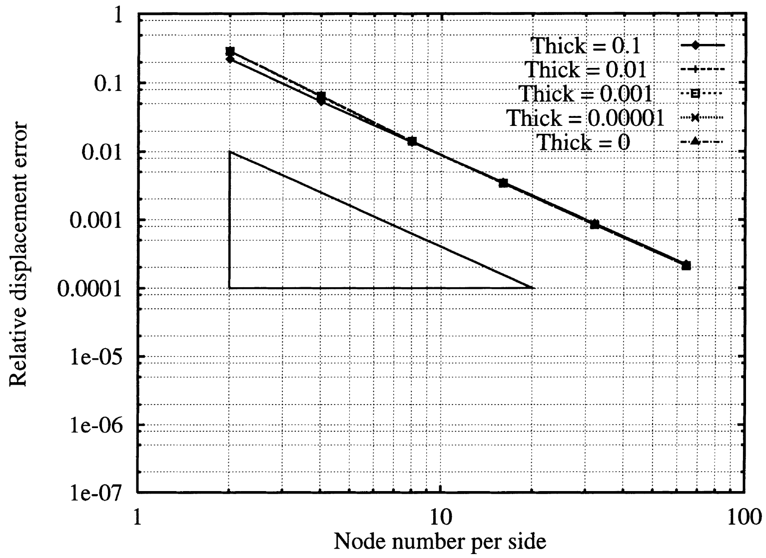

Fig. 2. T3-LIM element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

For simplicity, the summations are performed on all the nodes N_i relative to global interpolation parameters (that is, in the error evaluation the internal parameters associated with bubble functions are neglected). Let us also notice that the above error measures can be seen as discrete L^2-type errors and that a h^{k+1} convergence rate in these norms actually means a h^k convergence rate in the H^1 energy-type norm. Moreover, all the analyses are performed using regular meshes and discretizing only one quarter of the plate, due to symmetry considerations.

In Figs. 1–6 the relative rotation error and the relative displacement error versus the number of nodes per side are plotted for the three elements presented in the previous section. In particular, the triangular element of Section 4.1 has been labeled by T3-LIM, the quadratic triangular element of Section 4.2 by T9-LIM and, finally, the quadrilateral element of Section 4.3 by Q4-LIM. To check the robustness of the

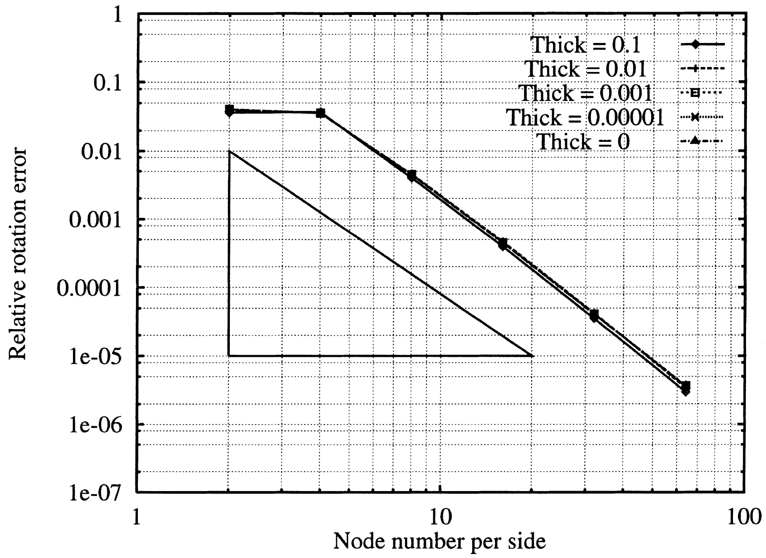

Fig. 3. T9-LIM element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^3 convergence rate in the L^2 error norm, corresponding to a h^2 convergence rate in the H^1 energy-type norm.

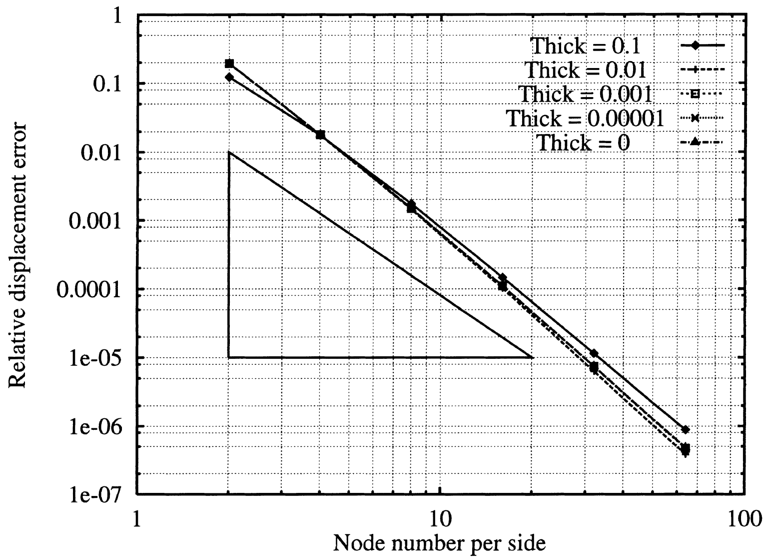

Fig. 4. T9-LIM element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^3 convergence rate in the L^2 error norm, corresponding to a h^2 convergence rate in the H^1 energy-type norm.

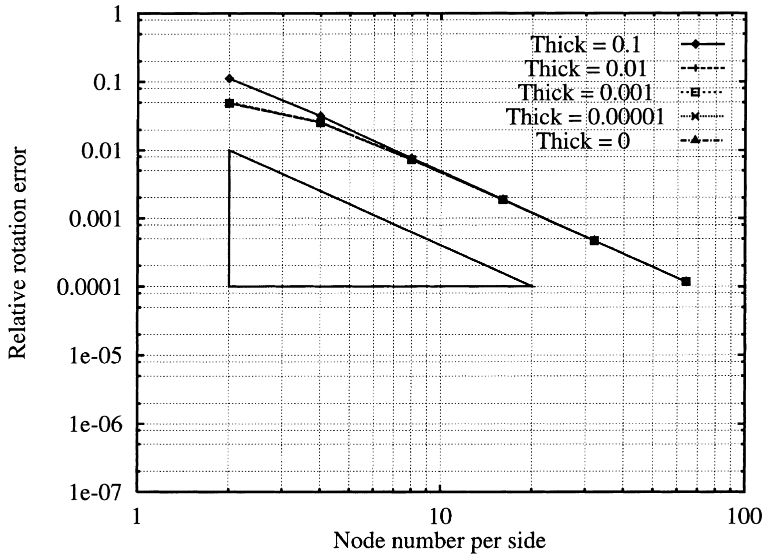

Fig. 5. Q4-LIM element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

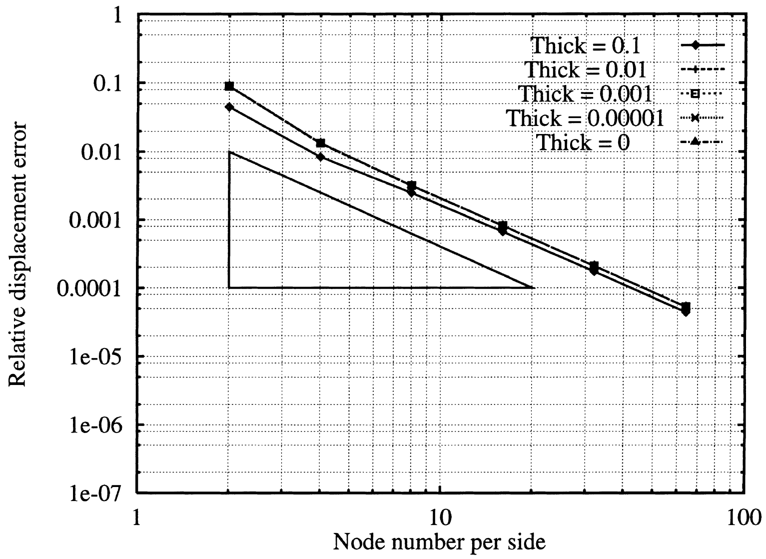

Fig. 6. Q4-LIM element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

elements with respect to the thickness parameter t (i.e. whether they suffer from locking or not), the numerical tests have been performed by choosing the following values of thickness:

$$t = 10^{-1}, \quad t = 10^{-2}, \quad t = 10^{-3}, \quad t = 10^{-5}, \quad t = 0.$$

The latter case corresponds to neglecting the shear energy (i.e. the Kirchhoff constraint is imposed). It is interesting to observe that:

- all the elements show the appropriate convergence rate for both rotations and vertical displacements (in the figures slopes showing the theoretical L^2 convergence rates are also reported);

- the presented methods are all fully insensitive to the variation of thickness, in such a way that the error graphics for different choices of t are nearly superposed. As a consequence, all the elements are actually locking-free and they can be used for both thick and thin plate problems;
- the element considered are all able to exactly solve also the case $t = 0$, which corresponds to enforce the Kirchhoff constraint.

6. Conclusions

A general analysis of mixed methods for plates based on a linking operator between rotations and vertical displacements has been developed. The main result is drawn in the error estimate detailed in Proposition 3.3 of Section 3. We remark again that the technique used for getting the error analysis is in the spirit of [13,19]. In Section 4 some elements have been presented and theoretically studied, while in Section 5 numerical tests have been reported. Although a complete numerical analysis would require a comparison with the performances of other well-established plate elements (i.e. the ones presented in [9,11], for instance), we believe that the numerical tests in the present paper are sufficient to describe the behaviour of our methods.

As final conclusion, we wish to outline a recipe for the development of $\mathrm{O}(h^r)$ elements based on kinematic linked operators.

We propose to choose an approximation triple $(\Theta_h, W_h, \Gamma_h)$ and a linear operator L for problem (20) such that the properties listed below are met.

1. The following approximation features hold:

$$\inf_{\underline{\eta}\in\Theta_h} \|\underline{\theta} - \underline{\eta}\|_1 = \mathrm{O}(h^r),$$
$$\inf_{v\in W_h} \|w - v\|_1 = \mathrm{O}(h^r),$$
$$\inf_{\underline{s}\in\Gamma_h} \|\underline{\gamma} - \underline{s}\|_0 = \mathrm{O}(h^r);$$

2. One has

$$\|P_h\underline{\nabla} v_h\|_0 \geqslant c\|\underline{\nabla} v_h\|_0 \quad \forall v_h \in W_h,$$

where P_h is the L^2-projection operator onto Γ_h and c is a positive constant independent of h. In practice, however, one would like to have the easier-to-verify condition

$$\underline{\nabla} W_h \subseteq \Gamma_h.$$

3. It holds the inf–sup condition

$$\sup_{(\underline{\eta},v)} \frac{(\underline{s}, \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1} \geqslant \beta\|\underline{s}\|_h \quad \forall \underline{s} \in \Gamma_h,$$

where β is a positive constant independent of h, the supremum is made over the discrete space $\Theta_h \times W_h$ and the mesh-dependent norm $\|\cdot\|_h$ is defined by (21).

4. The operator L is linear, bounded in H^1-norm and such that

$$|L\underline{\xi}_{\mathrm{II}}|_1 = \mathrm{O}(h^r),$$

for every sufficiently regular function $\underline{\zeta}$. Moreover, the operator

$$\Pi v := v_{\mathrm{I}} + L(\underline{\nabla} v)_{\mathrm{II}}$$

is P_{r+1}-invariant. Above, $(\cdot)_{\mathrm{I}}$ is an interpolating operator onto W_h, while $(\cdot)_{\mathrm{II}}$ is an interpolating operator onto Θ_h.

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