



# Analysis of mixed finite elements for laminated composite plates

F. Auricchio<sup>a</sup>, C. Lovadina<sup>b,\*</sup>, E. Sacco<sup>c</sup>

<sup>a</sup> *Dipartimento di Meccanica Strutturale, Università di Pavia, Italy*

<sup>b</sup> *Dipartimento di Ingegneria Meccanica e Strutturale, Università di Trento, Via Mesiano 77, I-38050, Trento, Italy*

<sup>c</sup> *Dipartimento di Meccanica, Strutture, A&T Università di Cassino, Italy*

Received 30 November 1999; received in revised form 18 July 2000

---

## Abstract

The approximation to the solution of a laminated plate problem is considered. The main result consists in proving that a performant scheme can be obtained by simply coupling a standard membrane element for the in-plane displacements with a robust method for plate bending problem, both elements related to the case of homogeneous material. A rigorous convergence analysis is developed for the case of clamped boundary conditions. Numerical results are presented, showing the accordance between the theoretical results and the actual computations. © 2001 Elsevier Science B.V. All rights reserved.

---

## 1. Introduction

Laminated composite plates are nowadays widely used in engineering practice, especially in space, automobile and civil applications [18]. The main motivation for this growing interest is related to the improved ratio between performances and weight associated with laminated plates with respect to the case of homogeneous ones. However, the behaviour of laminated plates is more complex, because:

- they are anisotropic materials;
- they present significant shear deformation in the thickness direction;
- they exhibit extension-bending coupling.

In literature, many different models have been presented. The simplest one is surely the Classical Laminated Plate Theory (CLPT), which is based on the Kirchoff hypotheses (cf. e.g. [22] for more details). However, due to the fact that shear deformations are not taken into account, the model approximation is quite poor. Moreover, in the framework of finite element procedures (cf. [7,15]), CLPT models require  $C^1$ -conforming schemes, which are quite expensive from the computational point of view.

According to these considerations, models arising from the Reissner–Mindlin assumptions (called First-order Shear-Deformation Theory: FSDT) are often preferred. The main FSDT features are related to the possibility of using  $C^0$ -conforming methods and the ability to capture shear deformations, which makes it possible to consider also the case of moderately thick plates (cf. e.g. [4]).

In the present paper, we develop an analysis of mixed finite element schemes relative to a FSDT model. Our main result is that a performant method can be obtained by simply taking a standard membrane element for the in-plane displacements and a robust plate bending element, both elements related to

---

\* Corresponding author.

E-mail address: lovadina@ing.unitn.it (C. Lovadina).

homogeneous plates (cf. [2,3,10,11,17,19] for example). We remark that this conclusion is not straightforward, since the laminated structure exhibits an extension-bending coupling (i.e. in-plane displacements arise even for vertical loads).

The paper is organized as follows. In Section 2 we briefly introduce the laminated plate model, together with the variational formulation that will be used as the starting point of the discretization procedure. We establish an existence and uniqueness result for the continuous solution, mainly by recognizing that the variational formulation of the problem is well posed. In Section 3 we develop our error bounds for mixed finite element methods in an abstract framework, as we want to give an analysis covering as many schemes as possible. In Section 4 we present two examples of applications of our theory. Section 5 is devoted to present numerical experiments, showing that the predicted behaviours are in fact met by actual computations.

In what follows, we will use standard notations (cf. [9,13]). Moreover,  $C$  will be a constant independent of  $t$  and  $h$ , not necessarily the same in each occurrence. Finally, for the continuous solution and the applied loads, we will suppose the necessary regularity.

**2. The laminated composite plate model and the mixed formulation**

Let us denote with  $A = \Omega \times (-t/2, t/2)$  the region in  $\mathbf{R}^3$  occupied by an undeformed elastic plate of thickness  $t > 0$  and midplane  $\Omega$ . We suppose that the plate is formed by  $n$  superposed orthotropic layers of equal thickness  $t/n$  and with material axes arbitrarily oriented in the  $x$ - $y$  plane. Thus, the  $k$ th layer ( $1 \leq k \leq n$ ) is positioned in  $\Omega \times (z_{k-1}, z_k)$ , where  $z_0 = -t/2$  and  $z_k = -t/2 + k(t/n)$ . Moreover we will work in the framework of the FSDT (cf. [22]). Restricting ourselves to the case of a clamped plate subjected to a vertical load  $f$ , the functional to be considered reads as follows:

$$\begin{aligned} \Pi = & \frac{1}{2} \left( \int_{\Omega} \mathcal{A}_n \underline{\underline{\varepsilon}}(\underline{\underline{u}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{u}}^*) \, d\Omega + 2 \int_{\Omega} \mathcal{B}_n \underline{\underline{\varepsilon}}(\underline{\underline{u}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}^*) \, d\Omega + \int_{\Omega} \mathcal{D}_n \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}^*) \, d\Omega \right) \\ & + \int_{\Omega} \underline{\underline{s}}^* \cdot (\underline{\underline{\theta}}^* - \nabla w^*) \, d\Omega - \frac{1}{2} \int_{\Omega} \mathcal{H}_n^{-1} \underline{\underline{s}}^* \cdot \underline{\underline{s}}^* \, d\Omega - \int_{\Omega} f w^* \, d\Omega. \end{aligned} \tag{1}$$

Above,  $\underline{\underline{u}}^*$  are the in-plane displacements,  $\underline{\underline{\theta}}^*$  the rotations,  $w^*$  the vertical displacement field, and  $\underline{\underline{s}}^*$  are the resultant shear stresses. Moreover,  $\mathcal{A}_n$ ,  $\mathcal{B}_n$  and  $\mathcal{D}_n$  are fourth-order tensors defined by

$$\mathcal{A}_n \underline{\underline{\tau}} = \sum_{k=1}^n (z_k - z_{k-1}) \mathcal{C}_k \underline{\underline{\tau}}, \tag{2}$$

$$\mathcal{B}_n \underline{\underline{\tau}} = \frac{1}{2} \sum_{k=1}^n (z_k^2 - z_{k-1}^2) \mathcal{C}_k \underline{\underline{\tau}}, \tag{3}$$

$$\mathcal{D}_n \underline{\underline{\tau}} = \frac{1}{3} \sum_{k=1}^n (z_k^3 - z_{k-1}^3) \mathcal{C}_k \underline{\underline{\tau}}, \tag{4}$$

where  $\underline{\underline{\tau}}$  is a second-order tensor (which in our case will always be symmetric). Moreover,  $\mathcal{C}_k$  is a fourth-order symmetric and positive definite tensor.

Finally,  $\mathcal{H}_n$  is a second-order tensor, defined by

$$\mathcal{H}_n \underline{\underline{\tau}} = \kappa \sum_{k=1}^n (z_k - z_{k-1}) \mathcal{S}_k \underline{\underline{\tau}}, \tag{5}$$

where  $\underline{\underline{\tau}}$  is a given vector and  $\mathcal{S}_k$  is a second-order symmetric and positive definite tensor.

**Remark 2.1.** The coefficient  $\kappa$  in (5) is called shear correction factor, and, in the case of homogeneous plates, it is known a priori and it is taken as  $5/6$ . For a laminated structure this value is not the optimal one, in general, and it is not known a priori. However, in the following we suppose to know the value of the shear correction factor  $\kappa$ , also for the composite plate.

**Remark 2.2.** The functional (1) reveals an interesting feature of laminated composites: the bending behaviour and the membrane one are coupled through the term involving the tensor  $\mathcal{B}_n$ . Thus, even though the plate is acted upon a vertical load, in-plane displacements can occur.

2.1. Analysis of the model

We first begin by setting (cf. (2)–(5))

$$\begin{aligned} \mathcal{A}_{\underline{\underline{\tau}}} &:= t^{-1} \mathcal{A}_n \underline{\underline{\tau}} = t^{-1} \sum_{k=1}^n (z_k - z_{k-1}) \mathcal{C}_k \underline{\underline{\tau}}, & \mathcal{B}_{\underline{\underline{\tau}}} &:= t^{-2} \mathcal{B}_n \underline{\underline{\tau}} = \frac{t^{-2}}{2} \sum_{k=1}^n (z_k^2 - z_{k-1}^2) \mathcal{C}_k \underline{\underline{\tau}}, \\ \mathcal{D}_{\underline{\underline{\tau}}} &:= t^{-3} \mathcal{D}_n \underline{\underline{\tau}} = \frac{t^{-3}}{3} \sum_{k=1}^n (z_k^3 - z_{k-1}^3) \mathcal{C}_k \underline{\underline{\tau}}, & \mathcal{H}_{\underline{\underline{\tau}}} &:= t^{-1} \mathcal{H}_n \underline{\underline{\tau}} = t^{-1} \kappa \sum_{k=1}^n (z_k - z_{k-1}) \mathcal{S}_k \underline{\underline{\tau}}. \end{aligned} \tag{6}$$

Hence, the functional to be considered is easily recognized to be (cf. (1))

$$\begin{aligned} \Pi_t &= \frac{1}{2} \left( t \int_{\Omega} \mathcal{A}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{u}}^*) \, d\Omega + 2t^2 \int_{\Omega} \mathcal{B}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}^*) \, d\Omega + t^3 \int_{\Omega} \mathcal{D}_{\underline{\underline{\varepsilon}}}(\underline{\underline{\theta}}^*) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}^*) \, d\Omega \right) \\ &+ \int_{\Omega} \underline{\underline{s}}^* \cdot (\underline{\underline{\theta}}^* - \nabla w^*) \, d\Omega - \frac{t^{-1}}{2} \int_{\Omega} \mathcal{H}^{-1} \underline{\underline{s}}^* \cdot \underline{\underline{s}}^* \, d\Omega - \int_{\Omega} f w^* \, d\Omega. \end{aligned} \tag{7}$$

In order to study the behaviour of the solution when  $t$  is small, we decide to scale the unknowns and the load in a proper way. We thus set

$$\underline{\underline{u}} = t^{-1} \underline{\underline{u}}^*, \quad \underline{\underline{\theta}} = \underline{\underline{\theta}}^*, \quad w = w^*, \quad \underline{\underline{s}} = t^{-3} \underline{\underline{s}}^*, \tag{8}$$

and we choose the load as

$$f = t^3 g. \tag{9}$$

Inserting these expressions in (7) and dividing by  $t^3$ , we finally obtain the functional

$$\begin{aligned} \Pi_t(\underline{\underline{u}}, \underline{\underline{\theta}}, w, \underline{\underline{s}}) &= \frac{1}{2} \left( \int_{\Omega} \mathcal{A}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}) : \underline{\underline{\varepsilon}}(\underline{\underline{u}}) \, d\Omega + 2 \int_{\Omega} \mathcal{B}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}) \, d\Omega + \int_{\Omega} \mathcal{D}_{\underline{\underline{\varepsilon}}}(\underline{\underline{\theta}}) : \underline{\underline{\varepsilon}}(\underline{\underline{\theta}}) \, d\Omega \right) \\ &+ \int_{\Omega} \underline{\underline{s}} \cdot (\underline{\underline{\theta}} - \nabla w) \, d\Omega - \frac{t^2}{2} \int_{\Omega} \mathcal{H}^{-1} \underline{\underline{s}} \cdot \underline{\underline{s}} \, d\Omega - \int_{\Omega} g w \, d\Omega. \end{aligned} \tag{10}$$

Setting  $U = H_0^1(\Omega)^2$ ,  $\Theta = H_0^1(\Omega)^2$ ,  $W = H_0^1(\Omega)$ , and  $S = L^2(\Omega)^2$ , we are thus led to consider the variational problem.

**Problem 2.1.** Find  $(\underline{\underline{u}}, \underline{\underline{\theta}}, w; \underline{\underline{s}}) \in U \times \Theta \times W \times S$ , solution of the following system:

$$\begin{aligned} (\mathcal{A}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}), \underline{\underline{\varepsilon}}(\underline{\underline{v}})) + (\mathcal{B}_{\underline{\underline{\varepsilon}}}(\underline{\underline{v}}), \underline{\underline{\varepsilon}}(\underline{\underline{\theta}})) &= 0 & \forall \underline{\underline{v}} \in U, \\ (\mathcal{B}_{\underline{\underline{\varepsilon}}}(\underline{\underline{u}}), \underline{\underline{\varepsilon}}(\underline{\underline{\eta}})) + (\mathcal{D}_{\underline{\underline{\varepsilon}}}(\underline{\underline{\theta}}), \underline{\underline{\varepsilon}}(\underline{\underline{\eta}})) + (\underline{\underline{s}}, \underline{\underline{\eta}} - \nabla \zeta) &= (g, \zeta) & \forall (\underline{\underline{\eta}}, \zeta) \in \Theta \times W, \\ (\underline{\underline{r}}, \underline{\underline{\theta}} - \nabla w) - t^2 (\mathcal{H}^{-1} \underline{\underline{s}}, \underline{\underline{r}}) &= 0 & \forall \underline{\underline{r}} \in S. \end{aligned} \tag{11}$$

We will show that Problem 2.1 admits a unique solution in  $U \times \Theta \times W \times S$ . To this end, we need to establish the following lemma.

**Lemma 2.1.** *There exist positive constants  $C_1$  and  $C_2$ , independent of  $t$ , such that*

$$C_1 \left( \|\underline{\underline{\tau}}\|_0^2 + \|\underline{\underline{\sigma}}\|_0^2 \right) \leq (\mathcal{A}\underline{\underline{\tau}}, \underline{\underline{\tau}}) + 2(\mathcal{B}\underline{\underline{\tau}}, \underline{\underline{\sigma}}) + (\mathcal{D}\underline{\underline{\sigma}}, \underline{\underline{\sigma}}) \leq C_2 \left( \|\underline{\underline{\tau}}\|_0^2 + \|\underline{\underline{\sigma}}\|_0^2 \right), \tag{12}$$

for every  $\underline{\underline{\tau}}, \underline{\underline{\sigma}} \in L^2(\Omega)_s^4, L^2(\Omega)_s^4$  denoting the space of symmetric tensors with components in  $L^2(\Omega)$ .

**Proof.** We first introduce the linear operator

$$\mathcal{L} : L^2(\Omega)_s^4 \times L^2(\Omega)_s^4 \rightarrow L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$$

defined by

$$\mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = [\mathcal{A}\underline{\underline{\tau}} + \mathcal{B}\underline{\underline{\sigma}}, \mathcal{B}\underline{\underline{\tau}} + \mathcal{D}\underline{\underline{\sigma}}], \tag{13}$$

for each  $[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$ . Notice that we have

$$\left( \mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}], [\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \right) = (\mathcal{A}\underline{\underline{\tau}}, \underline{\underline{\tau}}) + 2(\mathcal{B}\underline{\underline{\tau}}, \underline{\underline{\sigma}}) + (\mathcal{D}\underline{\underline{\sigma}}, \underline{\underline{\sigma}}), \tag{14}$$

because of the symmetry of the operator  $\mathcal{B}$ . Moreover, if one defines, for  $1 \leq k \leq n$ ,

$$\mathcal{L}_k[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = \left[ t^{-1}(z_k - z_{k-1})\mathcal{C}_k\underline{\underline{\tau}} + \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\mathcal{C}_k\underline{\underline{\sigma}}, \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\mathcal{C}_k\underline{\underline{\tau}} + \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3)\mathcal{C}_k\underline{\underline{\sigma}} \right], \tag{15}$$

then it is easily recognized that it holds (cf. (6))

$$\mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = \sum_{k=1}^n \mathcal{L}_k[\underline{\underline{\tau}}, \underline{\underline{\sigma}}], \tag{16}$$

for each  $[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$ . We will show that each  $\mathcal{L}_k$  is indeed a continuous and coercive linear operator with continuity and coerciveness constants independent of  $t$ . Hence, (12) will be a straightforward consequence of (14) and (16). We proceed by splitting  $\mathcal{L}_k$  as

$$\mathcal{L}_k = \mathcal{L}_k^{(2)} \mathcal{L}_k^{(1)}, \tag{17}$$

where

$$\mathcal{L}_k^{(1)}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] := [\mathcal{C}_k\underline{\underline{\tau}}, \mathcal{C}_k\underline{\underline{\sigma}}] \quad \forall [\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)_s^4 \times L^2(\Omega)_s^4, \tag{18}$$

and

$$\mathcal{L}_k^{(2)}[\underline{\underline{\tau}}', \underline{\underline{\sigma}}'] := \left[ t^{-1}(z_k - z_{k-1})\underline{\underline{\tau}}' + \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\underline{\underline{\sigma}}', \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\underline{\underline{\tau}}' + \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3)\underline{\underline{\sigma}}' \right], \tag{19}$$

for each  $[\underline{\underline{\tau}}', \underline{\underline{\sigma}}'] \in L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$ . It is straightforward to check that both  $\mathcal{L}_k^{(1)}$  and  $\mathcal{L}_k^{(2)}$  are symmetric operators and they commute. Moreover,  $\mathcal{L}_k^{(1)}$  is clearly continuous and coercive with continuity and coerciveness constants independent of  $t$ . To study the features of  $\mathcal{L}_k^{(2)}$  we only need to consider the real matrix  $A = A(k)$  (cf. (19)), where  $A(k)$  is defined by

$$A(k) = \begin{bmatrix} t^{-1}(z_k - z_{k-1}) & \frac{t^{-2}(z_k^2 - z_{k-1}^2)}{2} \\ \frac{t^{-2}(z_k^2 - z_{k-1}^2)}{2} & \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3) \end{bmatrix}. \tag{20}$$

In fact, an easy computation shows that  $\mathcal{L}_k^{(2)}$  is coercive on  $L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$  if and only if  $A(k)$  is positive-definite. Thus, the determinant  $\Delta(k)$  of  $A(k)$  is

$$\Delta(k) = \frac{t^{-4}}{3}(z_k^3 - z_{k-1}^3)(z_k - z_{k-1}) - \frac{t^{-4}}{4}(z_k^2 - z_{k-1}^2)^2 = \frac{t^{-4}}{12}(z_k - z_{k-1})^4 = \frac{1}{12n^4} > 0, \tag{21}$$

since  $z_k - z_{k-1} = t/n$ . Moreover, its trace is

$$\text{Tr}(k) = \frac{1}{n} + \frac{t^{-2}}{3n} (z_k^2 + z_k z_{k-1} + z_{k-1}^2). \tag{22}$$

It follows that the eigenvalues are given by

$$\begin{aligned} \lambda_m(k) &= \frac{\text{Tr}(k)}{2} \left( 1 - \sqrt{1 - \frac{4\Delta(k)}{\text{Tr}(k)^2}} \right), \\ \lambda_M(k) &= \frac{\text{Tr}(k)}{2} \left( 1 + \sqrt{1 - \frac{4\Delta(k)}{\text{Tr}(k)^2}} \right). \end{aligned} \tag{23}$$

Noting that (cf. (22))

$$\frac{1}{n} \leq \text{Tr}(k) \leq \frac{5}{4n}, \tag{24}$$

and recalling (21), we get

$$\lambda_m(k) \geq C_1(k), \quad \lambda_M(k) \leq C_2(k), \tag{25}$$

for some positive constants  $C_1(k)$  and  $C_2(k)$ , independent of  $t$ .

Hence,  $\mathcal{L}_k^{(2)}$  is coercive and continuous on  $L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$ , with continuity and coerciveness constants independent of  $t$ . It follows that the composition  $\mathcal{L}_k = \mathcal{L}_k^{(2)} \mathcal{L}_k^{(1)}$  is coercive and continuous on  $L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$ , with continuity and coerciveness constants independent of  $t$ . Recalling also (14) and (16) we finally obtain

$$C_1 \left( \|\underline{\tau}\|_0^2 + \|\underline{\sigma}\|_0^2 \right) \leq (\mathcal{A}\underline{\tau}, \underline{\tau}) + 2(\mathcal{B}\underline{\tau}, \underline{\sigma}) + (\mathcal{D}\underline{\sigma}, \underline{\sigma}) \leq C_2 \left( \|\underline{\tau}\|_0^2 + \|\underline{\sigma}\|_0^2 \right), \tag{26}$$

for every  $\underline{\tau}, \underline{\sigma} \in L^2(\Omega)_s^4$ .  $\square$

**Remark 2.3.** We remark that the constants  $C_1$  and  $C_2$  above, even though they do not depend on  $t$ , do depend on the number  $n$  of layers in the laminated plate. The analysis of the model behaviour at fixed  $t$  and  $n \rightarrow +\infty$ , although interesting, will not be treated in this paper.

We are now ready to prove the following Proposition.

**Proposition 2.1.** For each  $t > 0$  fixed, variational Problem 2.1 has a unique solution  $(\underline{u}, \underline{\theta}, w; \underline{s})$  in  $U \times \Theta \times W \times S$ .

**Proof.** We proceed in two steps.

1. *Uniqueness.* Suppose  $f = 0$ . Choosing in (11)  $\underline{v} = \underline{u}$ ,  $\underline{\eta} = \underline{\theta}$ ,  $\zeta = w$  and  $\underline{r} = -\underline{s}$ , we get, summing the equations of (11)

$$(\mathcal{A}\underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{u})) + 2(\mathcal{B}\underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{\theta})) + (\mathcal{D}\underline{\varepsilon}(\underline{\theta}), \underline{\varepsilon}(\underline{\theta})) + t^2(\mathcal{H}^{-1}\underline{s}, \underline{s}) = 0. \tag{27}$$

We first remark that there exists  $C > 0$  such that (cf. (6))

$$t^2(\mathcal{H}^{-1}\underline{s}, \underline{s}) \geq Ct^2\|\underline{s}\|_0^2. \tag{28}$$

To continue, by Lemma 2.1 and Korn’s inequality, we have

$$(\mathcal{A}\underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{u})) + 2(\mathcal{B}\underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{\theta})) + (\mathcal{D}\underline{\varepsilon}(\underline{\theta}), \underline{\varepsilon}(\underline{\theta})) \geq C \left( \|\underline{u}\|_1^2 + \|\underline{\theta}\|_1^2 \right). \tag{29}$$

From (27)–(29) we infer

$$\underline{u} = 0, \quad \underline{\theta} = 0, \quad \underline{s} = 0. \tag{30}$$

Finally, using (30), from the third equation of (11) we get

$$(\underline{r}, \nabla w) = 0 \quad \forall \underline{r} \in S, \tag{31}$$

which means  $w = 0$ , due to Poincaré’s inequality. We conclude that for variational Problem 2.1 the solution, if it exists, is unique.

2. *Existence.* As far as existence is concerned, consider

**Problem 2.2.** Find  $(\underline{u}, \underline{\theta}, w) \in U \times \Theta \times W$ , solution of the following system:

$$\begin{aligned} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\underline{\theta})) &= 0 & \forall \underline{v} \in U, \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + t^{-2}(\mathcal{H}(\underline{\theta} - \nabla w), \underline{\eta} - \nabla \zeta) &= (g, \zeta) \quad \forall (\underline{\eta}, \zeta) \in \Theta \times W. \end{aligned} \tag{32}$$

By (29) and a little algebra, we can apply Lax–Milgram lemma to conclude that Problem 2.2 has a unique solution  $(\underline{u}, \underline{\theta}, w) \in U \times \Theta \times W$ . Setting

$$\underline{s} = t^{-2} \mathcal{H}(\underline{\theta} - \nabla w) \in S, \tag{33}$$

we easily see that  $(\underline{u}, \underline{\theta}, w, \underline{s}) \in U \times \Theta \times W \times S$  is a solution of Problem 2.1. The proof is now complete.  $\square$

Furthermore, for Problem 2.1 we have, using the technique of [12] and Lemma 2.1

**Proposition 2.2.** *There exist positive constants  $C_1$  and  $C_2$ , independent of  $t$ , such that*

$$C_1 \leq \|\underline{u}\|_1 + \|\underline{\theta}\|_1 + \|w\|_1 + \|\underline{s}\|_{H^{-1}(\text{div})} \leq C_2, \tag{34}$$

where

$$H^{-1}(\text{div}) = \left\{ \underline{r} : \underline{r} \in H^{-1}(\Omega)^2, \text{div } \underline{r} \in H^{-1}(\Omega) \right\},$$

equipped with the obvious norm.

Moreover, as  $t$  tends to zero,  $(\underline{u}, w)$  converges to  $(\underline{u}_0, w_0) \in H_0^1(\Omega)^2 \times H_0^2(\Omega)$ , solution of

$$\begin{aligned} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}_0), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\nabla w_0)) &= 0 & \forall \underline{v} \in H_0^1(\Omega)^2, \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}_0), \underline{\underline{\varepsilon}}(\nabla \zeta)) + (\mathcal{D}\underline{\underline{\varepsilon}}(\nabla w_0), \underline{\underline{\varepsilon}}(\nabla \zeta)) &= (g, \zeta) \quad \forall \zeta \in H_0^2(\Omega). \end{aligned} \tag{35}$$

**Remark 2.4.** Notice that problem (35) is a Kirchoff-type laminated composite plate problem (CLPT) – cf. [22]. In a distributional sense,  $(\underline{u}_0, w_0)$  is the solution of the system

$$\begin{aligned} \text{div} \left( \mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}_0) + \mathcal{B}\underline{\underline{\varepsilon}}(\nabla w_0) \right) &= 0 & \text{in } \Omega, \\ \text{div } \text{div} \left( \mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}_0) + \mathcal{D}\underline{\underline{\varepsilon}}(\nabla w_0) \right) &= g & \text{in } \Omega. \end{aligned} \tag{36}$$

**Remark 2.5.** In the sequel, we will use the variational formulation (11) as the starting point for our discretization procedure by means of finite elements.

### 3. The discretized problem and error analysis

The aim of this section is essentially to show that a performant discretization procedure can be obtained by coupling a good element for homogeneous plate problem with a good membrane element.

#### 3.1. The discretized scheme

To begin, we select finite element spaces  $U_h \subset U$ ,  $\Theta_h \subset \Theta$ ,  $W_h \subset W$  and  $S_h \subset S$  (cf. [13]),  $h > 0$  denoting the mesh size. We then consider the discretized problem.

**Problem 3.1.** Find  $(\underline{u}_h, \underline{\theta}_h, w_h; \underline{s}_h) \in U_h \times \Theta_h \times W_h \times S_h$ , solution of the following system:

$$\begin{aligned} (\mathcal{A}\underline{\varepsilon}(\underline{u}_h), \underline{\varepsilon}(\underline{v})) + (\mathcal{B}\underline{\varepsilon}(\underline{v}), \underline{\varepsilon}(\underline{\theta}_h)) &= 0 & \forall \underline{v} \in U_h, \\ (\mathcal{B}\underline{\varepsilon}(\underline{u}_h), \underline{\varepsilon}(\underline{\eta})) + (\mathcal{D}\underline{\varepsilon}(\underline{\theta}_h), \underline{\varepsilon}(\underline{\eta})) + (\underline{s}_h, R_h \underline{\eta} - \underline{\nabla} \zeta) &= G_h(\underline{\eta}, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta_h \times W_h, \\ (\underline{r}, R_h \underline{\theta}_h - \underline{\nabla} w_h) - t^2(\mathcal{H}^{-1} \underline{s}_h, \underline{r}) &= 0 & \forall \underline{r} \in S_h. \end{aligned} \tag{37}$$

Above,  $R_h$  is an interpolation or projection operator acting on a suitable space and valued in  $S_h$ .

**Remark 3.1.** The loading term  $G_h(\underline{\eta}, \zeta)$  appearing in (37) should be a proper approximation of  $G(\underline{\eta}, \zeta) =: (g, \zeta)$  (cf. (10)). In many cases, but not always, it holds  $G_h(\underline{\eta}, \zeta) = G(\underline{\eta}, \zeta) \forall (\underline{\eta}, \zeta) \in \Theta_h \times W_h$ . Moreover, we will suppose that  $(G - G_h)(\underline{\eta}, \zeta) = (G - G_h)(\underline{\eta})$ , i.e., the consistency error depends only on the rotational field; this is the case of schemes based on the linked interpolation technique.

In the sequel, we suppose that  $U_h \subset U$  is a good finite element space for standard second-order elliptic problems, i.e., for every  $\underline{u} \in U$  it holds

$$\inf_{\underline{v} \in U_h} \|\underline{u} - \underline{v}\|_1 = O(h). \tag{38}$$

Moreover, we assume that  $\Theta_h$ ,  $W_h$  and  $S_h$ , together with the reduction operator  $R_h$ , provide a good scheme for homogeneous plate problem. This means that we assume the following hypothesis.

Consider the problem

**Problem 3.2.** Find  $(\underline{\theta}^1, w^1; \underline{s}^1) \in \Theta \times W \times S$ , solution of the following system:

$$\begin{aligned} (\mathcal{D}_1 \underline{\varepsilon}(\underline{\theta}^1), \underline{\varepsilon}(\underline{\eta})) + (\underline{s}^1, \underline{\eta} - \underline{\nabla} \zeta) &= (\underline{m}, \underline{\eta}) + (g, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta \times W, \\ (\underline{r}, \underline{\theta}^1 - \underline{\nabla} w^1) - t^2(\mathcal{H}_1^{-1} \underline{s}^1, \underline{r}) &= 0 & \forall \underline{r} \in S, \end{aligned} \tag{39}$$

where  $\mathcal{D}_1$  and  $\mathcal{H}_1$  are defined by (4) and (5).

We suppose that for the discretized problem.

**Problem 3.3.** Find  $(\underline{\theta}_h^1, w_h^1; \underline{s}_h^1) \in \Theta_h \times W_h \times S_h$ , solution of the following system:

$$\begin{aligned} (\mathcal{D}_1 \underline{\varepsilon}(\underline{\theta}_h^1), \underline{\varepsilon}(\underline{\eta})) + (\underline{s}_h^1, R_h \underline{\eta} - \underline{\nabla} \zeta) &= (\underline{m}, \underline{\eta}) + G_h(\underline{\eta}, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta_h \times W_h, \\ (\underline{r}, R_h \underline{\theta}_h^1 - \underline{\nabla} w_h^1) - t^2(\mathcal{H}_1^{-1} \underline{s}_h^1, \underline{r}) &= 0 & \forall \underline{r} \in S_h, \end{aligned} \tag{40}$$

we have the error estimate

$$\|\underline{\theta}^1 - \underline{\theta}_h^1\|_1 + \|w^1 - w_h^1\|_1 + t\|\underline{s}^1 - \underline{s}_h^1\|_0 \leq Ch. \tag{41}$$

3.2. Error estimates

We begin by setting, for notational simplicity,

$$\mathcal{E}(\underline{u}, \underline{\theta}; \underline{v}, \underline{\eta}) =: (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\underline{\theta})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})). \tag{42}$$

We remark that, by Lemma 2.1 and Korn’s inequality, it holds

$$\mathcal{E}(\underline{v}, \underline{\eta}; \underline{v}, \underline{\eta}) \geq C \left( \|\underline{v}\|_1^2 + \|\underline{\eta}\|_1^2 \right) \quad \forall (\underline{v}, \underline{\eta}) \in U \times \Theta, \tag{43}$$

with  $C$  independent of  $t$ .

We have the following error estimate, whose proof is very similar to that developed in [14].

**Proposition 3.1.** *Suppose that  $\nabla W_h \subset S_h$  and that*

$$\|R_h \underline{\eta} - \underline{\eta}\|_0 \leq Ch \|\underline{\eta}\|_1 \quad \forall \underline{\eta} \in \Theta. \tag{44}$$

Let  $\underline{u}_I \in U_h, \underline{\theta}_I \in \Theta_h$  and  $w_I \in W_h$ . Set

$$\underline{s}_I = t^{-2} \mathcal{H}(R_h \underline{\theta}_I - \nabla w_I). \tag{45}$$

Then it holds

$$\begin{aligned} \|\underline{u} - \underline{u}_h\|_1 + \|\underline{\theta} - \underline{\theta}_h\|_1 + t \|\underline{s} - \underline{s}_h\|_0 &\leq C (\|\underline{u}_I - \underline{u}\|_1 + \|\underline{\theta}_I - \underline{\theta}\|_1 + t \|\underline{s}_I - \underline{s}\|_0 \\ &+ h \|\underline{s}\|_0 + \|G - G_h\|_{-1}). \end{aligned} \tag{46}$$

**Proof.** Subtracting the first two equations of (37) from the first two of (11) we have the error equation

$$\mathcal{E}(\underline{u} - \underline{u}_h, \underline{\theta} - \underline{\theta}_h; \underline{v}, \underline{\eta}) + (\underline{s} - \underline{s}_h, R_h \underline{\eta} - \nabla \zeta) = (G - G_h)(\underline{\eta}, \zeta) + (\underline{s}, R_h \underline{\eta} - \underline{\eta}), \tag{47}$$

for every  $\underline{v} \in U_h, \underline{\eta} \in \Theta_h$  and  $\zeta \in W_h$ .

Hence we get, recalling also that  $(G - G_h)(\underline{\eta}, \zeta) = (G - G_h)(\underline{\eta})$  (cf. Remark 3.1)

$$\begin{aligned} \mathcal{E}(\underline{u}_I - \underline{u}_h, \underline{\theta}_I - \underline{\theta}_h; \underline{v}, \underline{\eta}) + (\underline{s}_I - \underline{s}_h, R_h \underline{\eta} - \nabla \zeta) &= \mathcal{E}(\underline{u}_I - \underline{u}, \underline{\theta}_I - \underline{\theta}; \underline{v}, \underline{\eta}) + (\underline{s}_I - \underline{s}, R_h \underline{\eta} - \nabla \zeta) \\ &+ (G - G_h)(\underline{\eta}) + (\underline{s}, R_h \underline{\eta} - \underline{\eta}). \end{aligned} \tag{48}$$

Choosing  $\underline{v} = \underline{u}_I - \underline{u}_h, \underline{\eta} = \underline{\theta}_I - \underline{\theta}_h$  and  $\zeta = w_I - w_h$ , we have

$$R_h \underline{\eta} - \nabla \zeta = t^2 \mathcal{H}^{-1}(\underline{s}_I - \underline{s}_h). \tag{49}$$

Inserting (49) in (48), using coercivity (cf. (43)) we get

$$\begin{aligned} \|\underline{u}_I - \underline{u}_h\|_1^2 + \|\underline{\theta}_I - \underline{\theta}_h\|_1^2 + t^2 \|\underline{s}_I - \underline{s}_h\|_0^2 &\leq C (\mathcal{E}(\underline{u}_I - \underline{u}, \underline{\theta}_I - \underline{\theta}; \underline{u}_I - \underline{u}_h, \underline{\theta}_I - \underline{\theta}_h) + t^2 (\underline{s}_I - \underline{s}, \mathcal{H}^{-1}(\underline{s}_I - \underline{s}_h)) \\ &+ (G - G_h)(\underline{\theta}_I - \underline{\theta}_h) + (\underline{s}, R_h [\underline{\theta}_I - \underline{\theta}_h] - [\underline{\theta}_I - \underline{\theta}_h])). \end{aligned} \tag{50}$$

We thus easily obtain

$$\|\underline{u}_I - \underline{u}_h\|_1 + \|\underline{\theta}_I - \underline{\theta}_h\|_1 + t \|\underline{s}_I - \underline{s}_h\|_0 \leq C (\|\underline{u}_I - \underline{u}\|_1 + \|\underline{\theta}_I - \underline{\theta}\|_1 + t \|\underline{s}_I - \underline{s}\|_0 + h \|\underline{s}\|_0 + \|G - G_h\|_{-1}). \tag{51}$$

The triangle inequality now leads to the following estimate

$$\|\underline{u} - \underline{u}_h\|_1 + \|\underline{\theta} - \underline{\theta}_h\|_1 + t \|\underline{s} - \underline{s}_h\|_0 \leq C (\|\underline{u}_I - \underline{u}\|_1 + \|\underline{\theta}_I - \underline{\theta}\|_1 + t \|\underline{s}_I - \underline{s}\|_0 + h \|\underline{s}\|_0 + \|G - G_h\|_{-1}). \tag{52}$$

The proof is complete.  $\square$



We now choose suitable interpolants  $\underline{u}_I \in U_h$ ,  $\underline{\theta}_I \in \Theta_h$  and  $w_I \in W_h$ . For  $\underline{u}_I$  we take the usual Clément interpolant of  $\underline{u}$ . For the other variables, we take advantage from the fact that we have selected a performant scheme for homogeneous plate problems. More precisely, fix  $(\underline{\theta}, w; \underline{s}) \in \Theta \times W \times S$ , part of the solution of Problem 2.1. In particular, we have

$$\underline{s} = t^{-2} \mathcal{H}(\underline{\theta} - \nabla w). \tag{53}$$

Consider now the following homogeneous-type problem.

**Problem 3.4.** Find  $(\hat{\underline{\theta}}, \hat{w}; \hat{\underline{s}}) \in \Theta \times W \times S$ , solution of the following system:

$$\begin{aligned} (\mathcal{D}\underline{\underline{\varepsilon}}(\hat{\underline{\theta}}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\hat{\underline{s}}, \underline{\eta} - \nabla \underline{\zeta}) &= (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{s}, \underline{\eta} - \nabla \underline{\zeta}) \quad \forall (\underline{\eta}, \underline{\zeta}) \in \Theta \times W, \\ (\underline{r}, \hat{\underline{\theta}} - \nabla \hat{w}) - t^2(\mathcal{H}^{-1}\hat{\underline{s}}, \underline{r}) &= 0 \quad \forall \underline{r} \in S. \end{aligned} \tag{54}$$

Clearly, Problem 3.4 has the unique solution  $(\hat{\underline{\theta}}, \hat{w}; \hat{\underline{s}}) = (\underline{\theta}, w; \underline{s})$ . For the discretized problem.

**Problem 3.5.** Find  $(\hat{\underline{\theta}}_I, \hat{w}_I; \hat{\underline{s}}_I) \in \Theta_h \times W_h \times S_h$ , solution of the following system:

$$\begin{aligned} (\mathcal{D}\underline{\underline{\varepsilon}}(\hat{\underline{\theta}}_I), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\hat{\underline{s}}_I, R_h \underline{\eta} - \nabla \underline{\zeta}) &= (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{s}, \underline{\eta} - \nabla \underline{\zeta}) \quad \forall (\underline{\eta}, \underline{\zeta}) \in \Theta_h \times W_h, \\ (\underline{r}, R_h \hat{\underline{\theta}}_I - \nabla \hat{w}_I) - t^2(\mathcal{H}^{-1}\hat{\underline{s}}_I, \underline{r}) &= 0 \quad \forall \underline{r} \in S_h, \end{aligned} \tag{55}$$

we thus have the error estimate (cf. (41))

$$\|\underline{\theta} - \hat{\underline{\theta}}_I\|_1 + \|w - \hat{w}_I\|_1 + t\|\underline{s} - \hat{\underline{s}}_I\|_0 \leq Ch. \tag{56}$$

If we now choose  $\underline{\theta}_I = \hat{\underline{\theta}}_I$  and  $w_I = \hat{w}_I$ , we see that (cf. (45))

$$\underline{s}_I = t^{-2} \mathcal{H}(R_h \underline{\theta}_I - \nabla w_I) = \hat{\underline{s}}_I. \tag{57}$$

Hence, from Proposition 3.1, (56), (57) and standard interpolation estimates (cf. [13]), we get

$$\|\underline{u} - \underline{u}_h\|_1 + \|\underline{\theta} - \underline{\theta}_h\|_1 + t\|\underline{s} - \underline{s}_h\|_0 \leq C(h + \|G - G_h\|_{-1,h}). \tag{58}$$

If the consistency error  $\|G - G_h\|_{-1}$  is  $O(h)$ , we obtain

$$\|\underline{u} - \underline{u}_h\|_1 + \|\underline{\theta} - \underline{\theta}_h\|_1 + t\|\underline{s} - \underline{s}_h\|_0 \leq Ch. \tag{59}$$

The estimate above provides an error bound for all the variables, but the vertical displacement fields. We notice that

$$\underline{s} = t^{-2} \mathcal{H}(\underline{\theta} - \nabla w), \quad \underline{s}_h = t^{-2} \mathcal{H}(R_h \underline{\theta}_h - \nabla w_h), \tag{60}$$

so that

$$\nabla(w - w_h) = t^2 \mathcal{H}^{-1}(\underline{s}_h - \underline{s}) + (\underline{\theta} - R_h \underline{\theta}_h). \tag{61}$$

Hence we obtain

$$\|\nabla(w - w_h)\|_0 \leq C(t^2 \|\underline{s}_h - \underline{s}\|_0 + \|\underline{\theta} - R_h \underline{\theta}_h\|_0 + \|R_h(\underline{\theta} - \underline{\theta}_h)\|_0). \tag{62}$$

By Poincaré’s inequality, Proposition 3.1 and (59), we get

$$\|w - w_h\|_1 \leq Ch. \tag{63}$$

We can summarize what we have developed so far in the following:

**Proposition 3.2.** *Let  $(\underline{u}, \underline{\varrho}, w; \underline{s})$  be the solution of Problem 2.1, and  $(\underline{u}_h, \underline{\varrho}_h, w_h; \underline{s}_h)$  be the solution of Problem 3.1. If*

- $U_h$  is chosen so that

$$\inf_{\underline{v} \in U_h} \|\underline{u} - \underline{v}\|_1 = O(h); \tag{64}$$

- $\Theta_h, W_h, S_h$  and  $R_h$  provide a good approximation scheme for homogeneous plate bending problem (in particular  $\underline{\nabla} W_h \subset S_h$ , (41) and (44) hold);
- the consistency error  $\|G - G_h\|_{-1}$  is  $O(h)$ , then the following error bound holds true:

$$\|\underline{u} - \underline{u}_h\|_1 + \|\underline{\varrho} - \underline{\varrho}_h\|_1 + \|w - w_h\|_1 + t\|\underline{s} - \underline{s}_h\|_0 \leq Ch. \tag{65}$$

#### 4. Applications

In this section we will present two examples of laminated composite plate element, showing that they meet the analysis developed above. We will suppose that our computational domain  $\Omega$  is a rectangle, and we will partition it into uniform meshes  $\mathcal{T}_h$  of rectangles  $K$ , whose diameters are bounded by  $h$ . For the in-plane displacement field, we choose standard bilinear and continuous functions for both the schemes, i.e., we set

$$U_h = \left\{ \underline{u}_h \in \Theta : \underline{u}_{h|K} \in Q_1(K)^2 \ \forall K \in \mathcal{T}_h \right\}, \tag{66}$$

where  $Q_1(T)$  is the space of the functions defined on  $K$  and bilinear with respect to the usual local coordinates  $\xi$  and  $\eta$ . Hence, from standard approximation theory we have

$$\inf_{\underline{v} \in U_h} \|\underline{u} - \underline{v}\|_1 = O(h). \tag{67}$$

##### 4.1. A linked interpolation scheme

Our first example arises from an element based on the linked interpolation technique, (cf. [5,23–25,27]) which has already been analysed in the framework of homogeneous plate problems (cf. [6]). We first set

$$S_h = \left\{ \underline{s}_h \in S : \underline{s}_{h|K} = (a + b\eta, c + d\xi) \ \forall K \in \mathcal{T}_h \right\}. \tag{68}$$

We then select

$$\Theta_h = \left\{ \underline{\varrho}_h \in \Theta : \underline{\varrho}_{h|K} \in Q_1(K)^2 \oplus S_h b_4 \ \forall K \in \mathcal{T}_h \right\}, \tag{69}$$

where  $b_4 = (1 - \xi^2)(1 - \eta^2)$ . Finally, we take

$$W_h = \left\{ w_h \in W : w_{h|K} \in Q_1(K) \ \forall K \in \mathcal{T}_h \right\}. \tag{70}$$

Notice that in this case  $\underline{\nabla} W_h \subset S_h$ . The reduction operator  $R_h$  is defined through the relation

$$R_h \underline{\eta} = P_h (Id - \underline{\nabla} L) \underline{\eta}, \tag{71}$$

where  $P_h$  is the  $L^2$ -projection operator on  $S_h$ ,  $Id$  the identity operator and  $L$  is the so-called Linking operator, which we define below. Let us first introduce for each  $K \in \mathcal{T}_h$  the functions

$$\varphi_i = \lambda_j \lambda_k \lambda_m. \tag{72}$$

In (72)  $\{\lambda_i\}_{1 \leq i \leq 4}$  are the equations of the sides of  $K$  and the indices  $(i, j, k, m)$  form a permutation of the set  $(1, 2, 3, 4)$ . The function  $\varphi_i$  is an edge bubble relatively to the edge  $e_i$  of  $K$ . Let us now set

$$EB(K) = \text{Span}\{\varphi_i\}_{1 \leq i \leq 4}. \tag{73}$$

The operator  $L$  is locally defined as

$$L|_K \underline{\eta}_h = \sum_{i=1}^4 \alpha_i \varphi_i \in EB(K), \tag{74}$$

by requiring that

$$(\underline{\eta}_h - \nabla L \underline{\eta}_h) \cdot \underline{t}_i = \text{constant along each } e_i, \tag{75}$$

where  $\underline{t}_i$  is the tangent vector along the side  $e_i$ . Notice that with the definition above,  $L$  is defined only on  $\Theta_h$ , but a natural extension to  $\Theta$  is provided by setting

$$L \underline{\eta} := L \underline{\eta}_p \quad \forall \underline{\eta} \in \Theta, \tag{76}$$

where  $\underline{\eta}_p$  is the  $H^1$ -projection of  $\underline{\eta}$  over  $\Theta_h$ . Thus, also  $R_h$  is defined on the whole  $\Theta$ . Next, we define  $G_h$  by

$$G_h(\underline{\eta}, \zeta) = (g, \zeta + L \underline{\eta}), \tag{77}$$

so that

$$(G - G_h)(\underline{\eta}, \zeta) = -(g, L \underline{\eta}). \tag{78}$$

It has been proved that for the operator  $L$  it holds (cf. [6])

$$\|L \underline{\eta}\|_1 \leq Ch \|\underline{\eta}\|_1 \quad \forall \underline{\eta} \in \Theta_h, \tag{79}$$

so that (cf. (76))

$$\|L \underline{\eta}\|_1 = \|L \underline{\eta}_p\|_1 \leq Ch \|\underline{\eta}_p\|_1 \leq Ch \|\underline{\eta}\|_1 \quad \forall \underline{\eta} \in \Theta. \tag{80}$$

It follows that

- $\|G - G_h\|_{-1} \leq Ch$  (cf. (78));
- for the operator  $R_h$  we have (cf. (71))

$$\|\underline{\eta} - R_h \underline{\eta}\|_0 = \|\underline{\eta} - P_h \underline{\eta} + P_h \nabla L \underline{\eta}\|_0 \leq \|\underline{\eta} - P_h \underline{\eta}\|_0 + \|P_h \nabla L \underline{\eta}\|_0 \leq Ch \|\underline{\eta}\|_1 \quad \forall \underline{\eta} \in \Theta. \tag{81}$$

Moreover, the scheme under consideration is a robust and first-order convergent when applied to homogeneous plate bending problems (cf. [6]). We thus conclude that our method is first-order convergent also when used in the case of a laminated composite structure (cf. Proposition 3.2).

#### 4.2. A mixed interpolated with tensorial components scheme

The second example we provide comes from the well-known MITC4 plate element (cf. [8,10]), which has been successfully used for plate bending problems in the case of homogeneous materials. For this scheme we set

$$S_h = \left\{ \underline{s}_h \in H_0(\text{rot}; \Omega) : \underline{s}_{h|K} = (a + b \eta, c + d \xi) \quad \forall K \in \mathcal{T}_h \right\}, \tag{82}$$

where (cf. [9], for instance)

$$H_0(\text{rot}; \Omega) = \left\{ \underline{r} \in L^2(\Omega)^2 : \text{rot } \underline{r} \in L^2(\Omega), \underline{r} \cdot \underline{t} = 0 \text{ on } \partial\Omega \right\}.$$

For the approximated space of rotations we select

$$\Theta_h = \left\{ \underline{\varrho}_h \in \Theta : \underline{\varrho}_{h|K} \in \mathcal{Q}_1(K)^2 \quad \forall K \in \mathcal{T}_h \right\}. \tag{83}$$

Finally, we take

$$W_h = \{w_h \in W : w_{hK} \in Q_1(K) \forall K \in \mathcal{T}_h\}. \quad (84)$$

Notice that also in this case  $\nabla W_h \subset S_h$ . The reduction operator  $R_h$  is defined through the relation (cf. [10])

$$R_h \underline{\eta} \in S_h, \quad \int_e R_h \underline{\eta} \cdot \underline{t}_e = \int_e \underline{\eta} \cdot \underline{t}_e, \quad (85)$$

for  $\underline{\eta}$  smooth enough, and for each side  $e$  of the rectangles in the mesh.

As for  $G_h$ , we simply choose

$$G_h(\underline{\eta}, \zeta) = (g, \zeta), \quad (86)$$

so that

$$(G - G_h)(\underline{\eta}, \zeta) = 0. \quad (87)$$

It is well known (cf. [8,14]) that if the refinement of the mesh is obtained by splitting each rectangle into 16 equal subrectangles, such a scheme meets all the features required by Proposition 3.2 (in particular, estimate (41) holds). Hence, we can conclude that in this case our method is first-order convergent when used for a laminated composite structure. We remark that some numerical tests relative to this element have been already presented in [1].

**Remark 4.1.** The analysis of [8,14] can be easily extended to the case of a refinement obtained by splitting each rectangle into four equal subrectangles (cf. also the results in [16,20], concerning the analysis of the  $Q_1 - P_0$  element for the Stokes problem, to which the MITC4 element is strictly linked).

## 5. Numerical examples

The aim of this section is to investigate the numerical performances of the interpolating schemes previously described. To reach this goal the schemes have been implemented into the Finite Element Analysis Program (FEAP) [26] and the convergence rate of the proposed schemes is checked on a model problem, for which the exact solution can be computed.

We consider a composite made with two layers of a high modulus graphite/epoxy material, whose properties are set as follows:

$$E_L = 25 \text{ MPa}, \quad E_T = 1 \text{ MPa}, \quad \nu_{TT} = 0.25, \quad G_{LT} = 0.5 \text{ MPa}, \quad G_{TT} = 0.2 \text{ MPa},$$

where the indexes L and T indicate the longitudinal and the transversal directions, while  $E$  indicates a Young's modulus,  $\nu$  a Poisson ratio,  $G$  a shear modulus. In particular, we observe that

$$\frac{E_L}{E_T} = 25, \quad \frac{G_{LT}}{E_T} = 0.5, \quad \frac{G_{TT}}{E_T} = 0.2.$$

We study the case of an unsymmetric 0/90 cross-ply laminate. Accordingly, recalling Eqs. (2)–(5) and expressing all the entries in MPa, we have

$$\mathcal{C}_1 = \begin{bmatrix} 250.627 & 0.250627 & 0 \\ 0.250627 & 1.00251 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \mathcal{C}_2 = \begin{bmatrix} 1.00251 & 0.250627 & 0 \\ 0.250627 & 250.627 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

The plate midplane is assumed to be the square  $\Omega = [0, a] \times [0, a]$  and subjected to the following sinusoidal load

$$f(x, y) = t^3 g(x, y) = t^3 q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{88}$$

with  $q_0$  measuring the maximum load intensity. Finally, we consider the following simply supported boundary conditions

$$\begin{aligned} \text{at } x = 0 \quad \text{and} \quad x = a &\Rightarrow u_2 = 0, \quad w = 0, \quad \theta_1 = 0, \\ \text{at } y = 0 \quad \text{and} \quad y = a &\Rightarrow u_1 = 0, \quad w = 0, \quad \theta_2 = 0, \end{aligned}$$

and the shear factor is set equal to  $5/6$ .

Following Refs. [21,22], the exact solution has form:

$$u_1(x, y) = u_{1,0} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right), \tag{89}$$

$$u_2(x, y) = u_{2,0} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right), \tag{90}$$

$$w(x, y) = w_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right), \tag{91}$$

$$\theta_1(x, y) = \theta_{1,0} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right), \tag{92}$$

$$\theta_2(x, y) = \theta_{2,0} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right), \tag{93}$$

where  $u_{1,0}$ ,  $u_{2,0}$ ,  $w_0$ ,  $\theta_{1,0}$  and  $\theta_{2,0}$  are factors that can be easily computed. To do so, we start from functional (10), we take the variation with respect to  $\underline{g}$  and we require a strong satisfaction of the corresponding equation. It follows that  $(\underline{u}, \underline{\theta}, w)$  minimizes the potential energy functional

$$\begin{aligned} \tilde{\Pi}_t = & \frac{1}{2} \left( \int_{\Omega} \mathcal{A} \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{u}) \, d\Omega + 2 \int_{\Omega} \mathcal{B} \underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{\theta}) \, d\Omega + \int_{\Omega} \mathcal{D} \underline{\underline{\varepsilon}}(\underline{\theta}) : \underline{\underline{\varepsilon}}(\underline{\theta}) \, d\Omega \right) \\ & + \frac{t^{-2}}{2} \int_{\Omega} \mathcal{H}(\underline{\theta} - \underline{\nabla}w) \cdot (\underline{\theta} - \underline{\nabla}w) \, d\Omega - \int_{\Omega} gw \, d\Omega. \end{aligned}$$

Hence, using positions (88)–(93) and requiring the potential energy stationarity, we obtain a system of five equations, which can be solved in terms of the five quantities  $u_{1,0}$ ,  $u_{2,0}$ ,  $w_0$ ,  $\theta_{1,0}$  and  $\theta_{2,0}$ . The solution clearly depends on the specific form of the fourth-order tensors  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{D}$ ,  $\mathcal{H}$ , hence on the specific lamination sequence as well as on the plate geometry.

Although the numerical computations have been performed for a simply supported boundary condition (while the theoretical analysis has been developed for the case of a clamped plate), we nonetheless believe that the presented tests are significant of the actual performances of the schemes.

The error of a discrete solution is measured through the relative errors  $E_u$ ,  $E_w$ ,  $E_{\theta}$  and  $E_{\text{tot}}$ , defined as

$$E_u^2 = \frac{\sum_{N_i} \left[ (u_{h1}(N_i) - u_1(N_i))^2 + (u_{h2}(N_i) - u_2(N_i))^2 \right]}{\sum_{N_i} \left[ (u_1(N_i))^2 + (u_2(N_i))^2 \right]}, \tag{95}$$

$$E_w^2 = \frac{\sum_{N_i} (w_h(N_i) - w(N_i))^2}{\sum_{N_i} w(N_i)^2}, \tag{96}$$

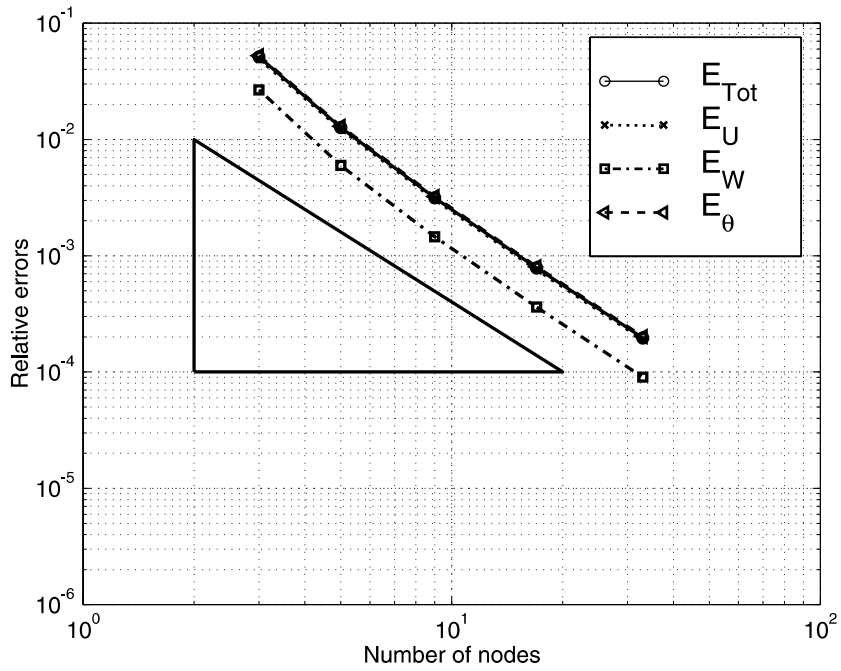


Fig. 1. Linked interpolation scheme: thick case ( $t = 10^{-1}$ ). Relative errors ( $E_{tot}$ ,  $E_u$ ,  $E_w$ ,  $E_\theta$ ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

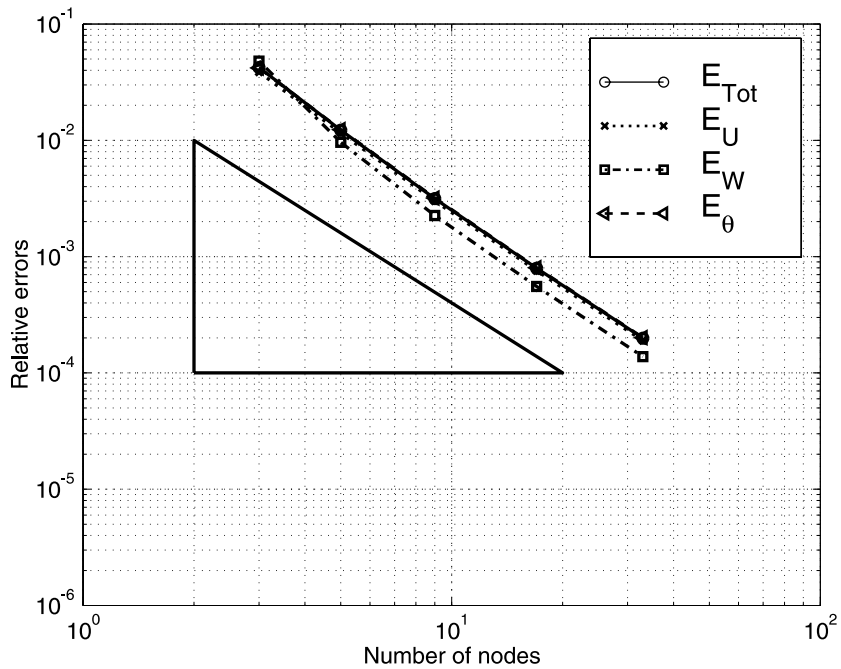


Fig. 2. Linked interpolation scheme: thin case ( $t = 10^{-3}$ ). Relative errors ( $E_{tot}$ ,  $E_u$ ,  $E_w$ ,  $E_\theta$ ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

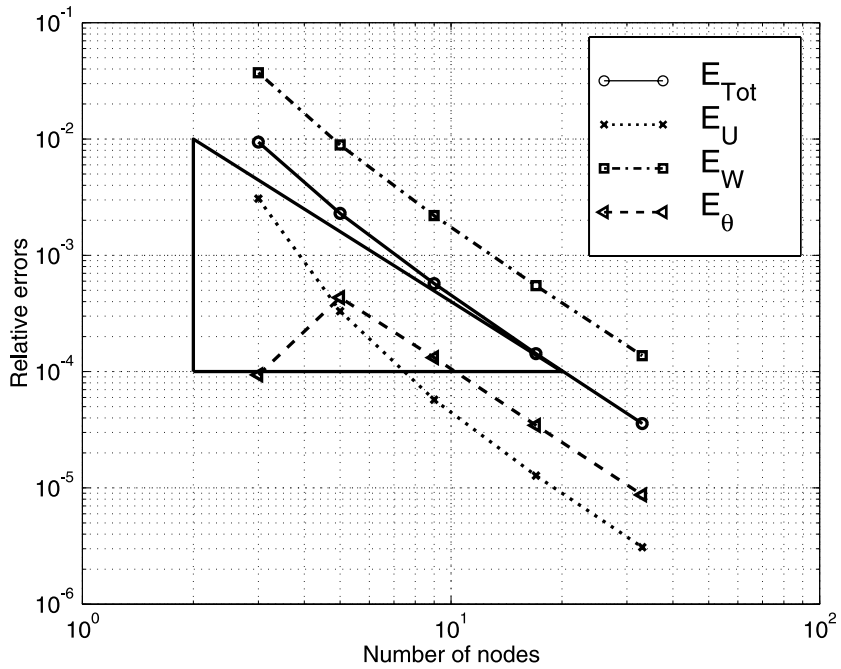


Fig. 3. MITC4 interpolation scheme: thick case ( $t = 10^{-1}$ ). Relative errors ( $E_{tot}$ ,  $E_u$ ,  $E_w$ ,  $E_\theta$ ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

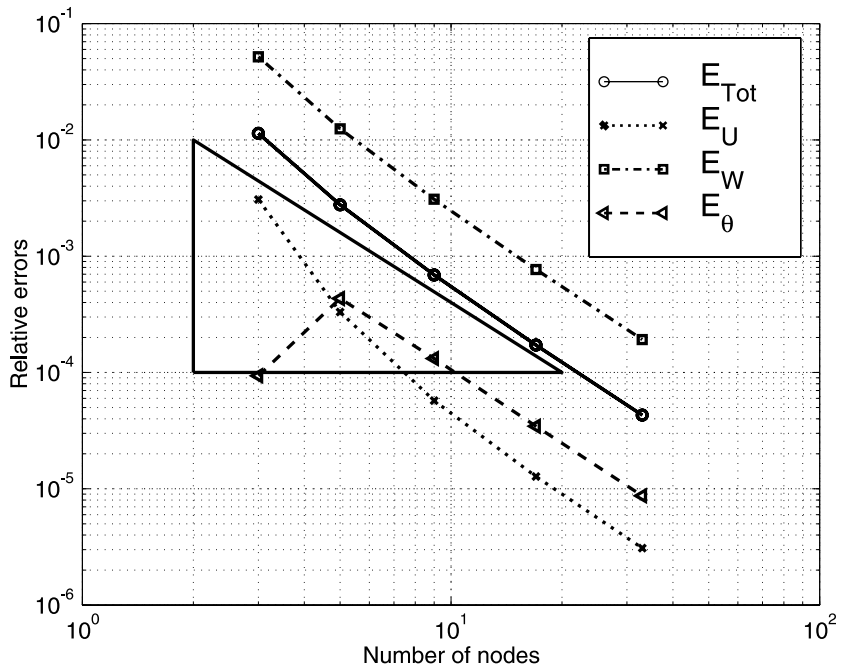


Fig. 4. MITC4 interpolation scheme: thin case ( $t = 10^{-3}$ ). Relative errors ( $E_{tot}$ ,  $E_u$ ,  $E_w$ ,  $E_\theta$ ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

$$E_{\theta}^2 = \frac{\sum_{N_i} [(\theta_{h1}(N_i) - \theta_1(N_i))^2 + (\theta_{h2}(N_i) - \theta_2(N_i))^2]}{\sum_{N_i} [(\theta_1(N_i))^2 + (\theta_2(N_i))^2]}, \quad (97)$$

$$E_{\text{tot}}^2 = E_u^2 + E_w^2 + E_{\theta}^2. \quad (98)$$

For simplicity, the summations are performed on all the nodes  $N_i$  relative to global interpolation parameters (that is, the internal parameters associated with bubble functions are neglected). The above error measures can also be seen as discrete  $L^2$ -type errors and a  $h^2$  convergence rate in  $L^2$  norm actually means a  $h$  convergence rate in the  $H^1$  energy-type norm.

The analyses are performed using uniform meshes and discretizing only one quarter of the plate, due to symmetry considerations. Moreover, the refinement procedure is obtained by splitting each square of a mesh into four equal subsquares. The plate side length is set equal to 1, while two different values of thickness are considered, i.e.,  $t \in \{10^{-1}, 10^{-3}\}$ , indicated in the following as thick and thin case, respectively.

Figs. 1–4 show the relative errors versus the number of nodes per side for the interpolation schemes considered.

It is interesting to observe that:

- Both the methods show the appropriate convergence rate for both rotations and vertical displacements.
- Both the methods are almost insensitive to the thickness, in such a way that the error graphs for two different choices of  $t$  are very close to each other. As a consequence, all the proposed elements are actually locking-free and they can be used for both thick and thin plate problems.

## References

- [1] G. Alfano, F. Auricchio, L. Rosati, E. Sacco, MITC finite elements for laminated composite plates, *Int. J. Numer. Methods Engrg.* 50 (2001) 707–738.
- [2] D.N. Arnold, Innovative finite element methods for plates, *Math. Appli. Comp.* V.10 (1991) 77–88.
- [3] D.N. Arnold, R.S. Falk, A uniformly accurate finite element method for the Reissner–Mindlin plate model, *SIAM J. Numer. Anal.* 26 (6) (1989) 1276–1290.
- [4] F. Auricchio, E. Sacco, A mixed-enhanced finite element for the analysis of laminated composite plates, *Int. J. Numer. Methods Engrg.* 44 (1999) 1481–1504.
- [5] F. Auricchio, R.L. Taylor, A shear deformable plate element with an exact thin limit, *Comput. Methods Appl. Mech. Engrg.* 118 (1994) 393–412.
- [6] F. Auricchio, C. Lovadina, Analysis of kinematic linked interpolation methods for Reissner–Mindlin plate problems, *Comput. Methods Appl. Mech. Engrg.* 19 (2001) 2465–2482.
- [7] K.-J. Bathe, *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, NJ, 1995.
- [8] K.-J. Bathe, F. Brezzi, On the convergence of a four-node plate bending element based on Mindlin–Reissner plate theory and a mixed interpolation, in: J.R. Whiteman (Ed.), *MAFELAP V*, London, 1985, pp. 491–503.
- [9] F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer, Berlin, 1991.
- [10] F. Brezzi, K.-J. Bathe, M. Fortin, Mixed-Interpolated elements for Reissner–Mindlin plates, *Int. J. Numer. Methods Engrg.* 28 (1989) 1787–1801.
- [11] F. Brezzi, M. Fortin, R. Stenberg, Error analysis of mixed-interpolated elements for Reissner–Mindlin plates, *Math. Models Methods Appl. Sci.* 1 (1991) 125–151.
- [12] D. Chenais, J.C. Paumier, On the locking phenomenon for a class of elliptic problems, *Numer. Math.* 67 (1994) 427–440.
- [13] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
- [14] R.G. Duran, E. Liberman, On mixed finite element methods for the Reissner–Mindlin plate model, *Math. Comput.* 58 (198) (1992) 561–573.
- [15] T.J.R. Hughes, *The Finite Element Method*, Prentice-Hall, Englewood Cliffs, NY, 1987.
- [16] C. Johnson, J. Pitkäranta, Analysis of some mixed finite element methods related to reduced integration, *Math. Comput.* 38 (158) (1982) 375–400.
- [17] C. Lovadina, Analysis of a mixed finite element method for Reissner–Mindlin plate problem, *Comput. Methods Appl. Mech. Engrg.* 163 (1998) 71–85.
- [18] T. Massard (Ed.), in: *Proceedings of the 12th International Conference on Composite Materials*, Paris, 5–9 July 1999.
- [19] J. Pitkäranta, Analysis of some low-order finite element schemes for Mindlin–Reissner and Kirchhoff plates, *Numer. Math.* 53 (1988) 237–254.
- [20] J. Pitkäranta, R. Stenberg, Error bounds for the approximation of the Stokes problem using bilinear/constant elements on irregular quadrilateral meshes, in: J.R. Whiteman (Ed.), *MAFELAP V*, London, 1985, pp. 325–334.



- [21] J.N. Reddy, *Energy and Variational Methods in Applied Mechanics*, Wiley, New York, 1984.
- [22] J.N. Reddy, *Mechanics of Laminated Composite Plates – Theory and Analysis*, CRC Press, Boca Raton, 1997.
- [23] R.L. Taylor, F. Auricchio, Linked interpolation for Reissner–Mindlin plate elements. Part II – A simple triangle, *Int. J. Numer. Methods Engrg.* 36 (1993) 3057–3066.
- [24] Z. Xu, A simple and efficient triangular finite element for plate bending, *Acta Mech. Sin.* 2 (1986) 185–192.
- [25] Z. Xu, A thick–thin triangular plate element, *Int. J. Numer. Methods Engrg.* 33 (1992) 963–973.
- [26] O.C. Zienkiewicz, R.L. Taylor, *The Finite Element Methods*, McGraw-Hill, New York, 1989.
- [27] O.C. Zienkiewicz, Z. Xu, L.F. Zeng, A. Samuelsson, N.-E. Wiberg, Linked interpolation for Reissner–Mindlin plate elements. Part I – A simple quadrilateral, *Int. J. Numer. Methods Engrg.* 36 (1993) 3043–3056.