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A UNIAXIAL MODEL FOR SHAPE-MEMORY ALLOYS

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Abstract—We present a uniaxial model for shape-memory alloys, cast within the *generalized plasticity* framework, previously developed. The model is based on two internal variables (the single-variant martensite fraction and the multiple-variant martensite fraction), for which evolution equations in rate form are proposed. The model reproduces the shape-memory effect and the superelastic behavior; moreover, for loading–unloading cycles, without completion of the phase transition, it presents a cyclic response with internal loops. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Materials undergoing phase transitions are receiving progressively more and more attention, mainly for their innovative use in practical applications. An important example is the family of *shape-memory alloys*, which have an intrinsic capacity of remembering their original configuration or shape.†

In terms of macroscopic quantities, such as stress, strain and temperature, the shape-memory alloys (SMA) present two main properties: the *superelasticity* and the *shape-memory effect*.

- Superelasticity (SE). At high temperature (characteristic of the specific alloy) a mechanical loading–unloading cycle induces highly-nonlinear large deformations. At the end of the cycle no permanent deformations are present. The stress–strain path usually presents a hysteresis loop.
- Shape-memory effect (SME). At low temperature (characteristic of the specific alloy) a mechanical loading–unloading cycle induces highly-nonlinear large deformations. At the end of the cycle residual deformations (up to 10–15%) may be present. Through a thermal cycle the material is able to recover such deformations, returning to its initial configuration (memory effect).

Both macroscopic behaviors are the consequence of martensitic transformations. From a metallurgic point of view (Khachaturyan, 1983; Wayman, 1964), a martensitic transformation is a solid–solid, diffusionless transition between a crystallographically more-ordered parent phase (*austenite*) and a crystallographically less-ordered product phase (*martensite*). For shape-memory alloys the transformation is reversible and, in many cases, rate-independent.

During the conversion of austenite into martensite, it is important to distinguish between two cases. If there is no preferred direction for the occurrence of the transformation, the martensite takes advantage of the existence of different possible habit planes (Wayman,

† In the literature it is possible to find many introductory papers describing shape-memory alloys, such as those by Wayman (1992, 1993), Wayman and Duerig (1990). Refer to these for more detailed, but still introductory, presentations. For a review of applications based on shape-memory alloys refer to Duerig (1990) and Pelton (1994).

1964; Khachaturyan, 1983), forming a series of crystallographically equivalent variants. The product phase is then termed *multiple-variant martensite* and is characterized by a *twinned* structure, which minimizes the misfit between the martensite and the surrounding austenite. On the other hand, if there is a preferred direction for the occurrence of the transformation (often associated with a state of stress), all the martensite crystals tend to be formed on the most favorable habit plane. The product phase is then termed *single-variant martensite* and is characterized by a *detwinned* structure, which again minimizes the misfit between the martensite and the surrounding austenite.

However, the macroscopic behavior is not simply the result of the crystallographic changes; in fact, several other mechanisms, each one with its own scale, play important roles (Otsuka, 1986; Funakubo, 1987). Accordingly, the development of macroscopic constitutive equations based on the phase transition micro-mechanics is a difficult task.

There have been many attempts to directly construct phenomenological models capable of representing the material macroscopic behavior. Some representative† works are by: Abeyaratne and Knowles (1993), Brandon and Rogers (1992) Brinson (1993), Cory and McNichols Jr (1985, 1987), Falk and Konopka (1990), Ivshin and Pence (1993, 1994), Liang and Rogers (1990, 1992), Muller and Xu (1991), Patoor *et al.* (1988), Raniecki and Lexcellent (1994), Sun and Hwang (1993a, 1993b), Tanaka *et al.* (1982, 1985, 1986, 1992), Tobushi *et al.* (1991), Wilmanski (1993).

Recently, based on the work of Lubliner (1984) and Lubliner *et al.* (Lubliner, 1991), Lubliner and Auricchio presented an inelastic theory, based on an internal variable formalism. To the authors' judgment, such a theory is well suited for the modeling of complex material behaviors, such as those occurring in materials undergoing solid–solid phase transitions (Auricchio, 1995; Lubliner, 1996).

In Lubliner (1995) a first application of this theory to the case of shape-memory alloys is presented. A one-dimensional model based on a single scalar internal variable is developed. The model describes the superelastic behavior with the internal loops. However, due to the presence of only one internal variable, the different behavior between the multiple-variant martensite and the single-variant martensite in terms of macroscopic effects is not taken into account and the shape-memory effect is only partially modeled.

We now propose a phenomenological model, based on two internal variables (the single-variant martensite fraction and the multiple-variant martensite fraction), for which evolution equations in rate form are presented. Brinson (1993) and Leclercq *et al.* (1994) already presented models for shape-memory alloys based on the use of the same two internal variables. The novelty of the paper is in the framework we use to develop the model; in fact, as addressed in the next Section, the framework is such to allow the uncoupling of the different phase transformation processes during the construction of the evolutionary equations. Appealing characteristics of the model herein discussed are:

- simplicity,
- complete soundness in a continuum mechanics framework,
- possibility of extending the model to describe three-dimensional and/or more complex behaviors,
- possibility of implementing the model (or its generalization) in a valid computational framework, such as finite elements.

2. A UNIAXIAL CONSTITUTIVE MODEL

In the following we present a uniaxial model able to reproduce both the shape-memory effect and the superelastic behavior. The model is a case within the *generalized plasticity* theory (Lubliner, 1996) and is based on the introduction of two scalar internal variables; accordingly, it can be considered as an extension of the work previously presented by the authors (Lubliner, 1996).

† But by no mean exhaustive of the literature richness on the subject.

2.1. *Control and internal variables*

As control variables we assume the uniaxial stress, σ, and the temperature, T. As internal variables, we might assume:

- the multiple-variant martensite fraction, ξ_M,
- the single-variant martensite fraction, ξ_S,
- the austenite fraction, ξ_A.†

Since the following relation must be fulfilled at any time:

$$\xi_M + \xi_S + \xi_A = 1 \tag{1}$$

there are only two independent fractions, chosen herein to be ξ_M and ξ_S. From eqn (1) we may also get a relation between the rates of the single fractions:

$$\dot{\xi}_M + \dot{\xi}_S + \dot{\xi}_A = 0 \tag{2}$$

where a superposed dot indicates the time derivative. For simplicity, during the construction of the model we prefer to deal with all three parameters, ensuring that the fraction evolutions always satisfy eqn (2); as a consequence, at any time eqn (1) is also satisfied. We also find it more convenient to establish first the evolutionary equations associated with those fractions that reduce, deriving the remaining evolutionary equations by implicit enforcement of eqn (2).

2.2. *Phase transitions and activation conditions*

Since we consider three fractions, we should consider the three corresponding production processes, that are:

- multiple-variant martensite production,
- single-variant martensite production,
- austenite production.

The regions in which such processes may occur are assumed to be delimited by straight lines (Funakubo, 1987, Melton, 1990) [Fig. 1].

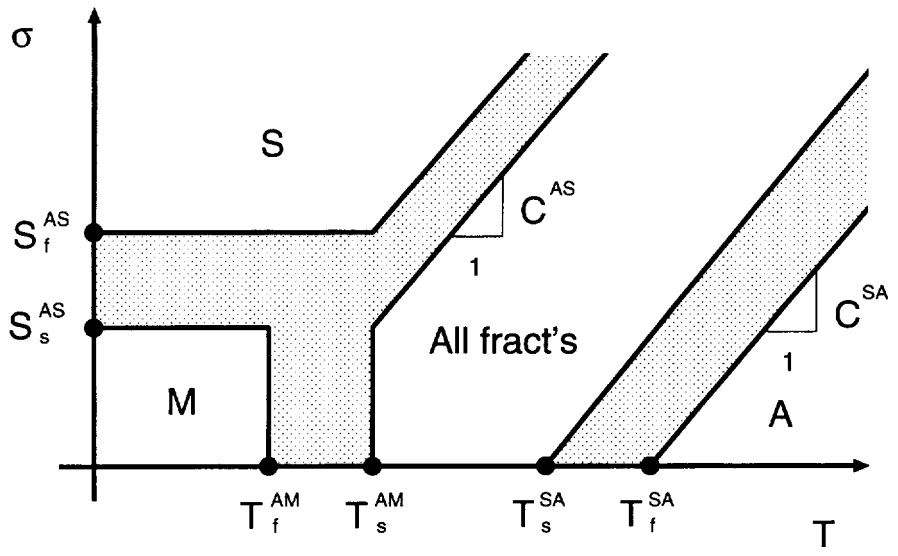

Fig. 1. Phase transformation zones. Experimentally it has been shown that in a uniaxial stress-temperature diagram and in the usual range of applications, the region in which phase transformations may occur are delimited with good approximation by straight lines (Funakubo, 1987; Melton, 1990). We also indicate the regions in which only single fractions are stable (S: single-variant martensite, M: multiple-variant martensite, A: austenite).

† By convention, the capital letters M, S and A used as indices refer to specific fractions (M = multiple-variant martensite, S = single-variant martensite, A = austenite). Moreover, $\xi_i = 0$ ($i = M, S, A$) indicates the absence of the corresponding phase in the material, while $\xi_i = 1$ indicates that the material is completely in such a phase.

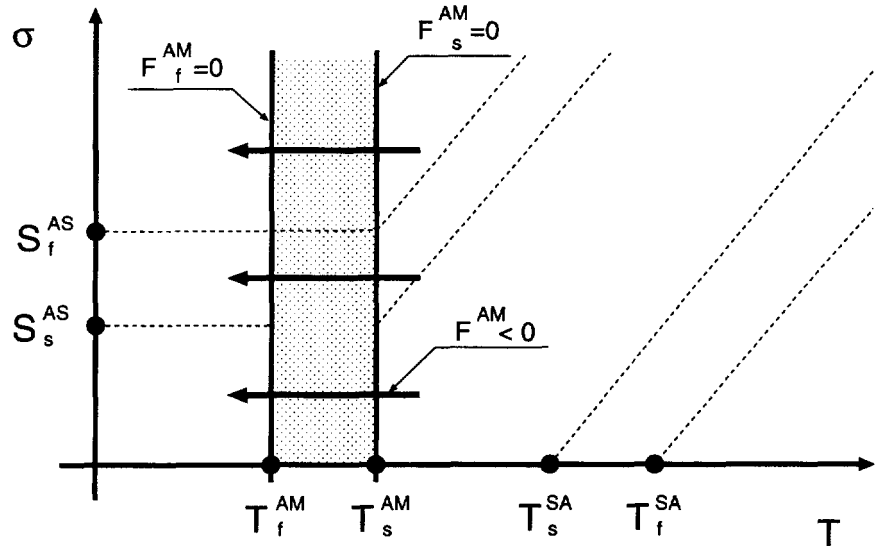

Fig. 2. Production of multiple-variant martensite. The phase transformation may occur within the shaded area. The arrows indicate the direction for activation of the transformation.

Multiple-variant martensite production. The production of multiple-variant martensite can occur only as conversion (reduction) of austenite ($A \to M$). We set:

$$\begin{aligned} F^{AM} &= T \\ F_s^{AM} &= T - T_s^{AM} \\ F_f^{AM} &= T - T_f^{AM} \end{aligned} \tag{3}$$

where T_s^{AM} and T_f^{AM} are the starting and final temperatures at which the transformation may occur at zero stress (Fig. 2).† The region in which the transformation may take place is described by:

$$F_s^{AM} < 0, \quad F_f^{AM} > 0 \;\Rightarrow\; F_s^{AM} F_f^{AM} < 0. \tag{4}$$

Moreover, for inducing the transformation a temperature decrease should occur:

$$\dot{F}^{AM} < 0. \tag{5}$$

As already mentioned, we first construct the rate equations associated with the reducing fraction (in this case the austenite), deriving the evolution of the remaining fractions by implicit enforcement of eqn (2). Accordingly, we set:

$$\begin{aligned} \dot{\xi}_A^{AM} &= K^{AM} \langle -F_s^{AM} F_f^{AM} \rangle \langle -F^{AM} \rangle \\ \dot{\xi}_M^{AM} &= -\dot{\xi}_A^{AM} \\ \dot{\xi}_S^{AM} &= 0 \end{aligned} \tag{6}$$

where K^{AM} is a function of the state variables and $\langle \cdot \rangle$ is the Macaulay bracket, defined as: $\langle x \rangle = (x + |x|)/2$. For K^{AM} we choose a simple form such that:

† By convention, the superscripts refer to specific evolution processes; accordingly, the superscript AM refers to the conversion of austenite into multiple-variant martensite.

The temperature at which the conversion of austenite into multiple-variant martensite starts at zero stress is often indicated in the literature as M_s. For the same quantity, we prefer to introduce the notation T_s^{AM}, which indicates also from which material fraction the multiple-variant martensite is produced. This is an important information when dealing with multiple phase transformation resulting in the same product phase.

$$\dot{\xi}_A^{AM} = -\beta^{AM}\xi_A \frac{\langle -F_s^{AM}F_f^{AM}\rangle}{|F_s^{AM}F_f^{AM}|}\frac{\langle -\dot{F}^{AM}\rangle}{(F_f^{AM})^2}$$

$$\dot{\xi}_M^{AM} = +\beta^{AM}\xi_A \frac{\langle -F_s^{AM}F_f^{AM}\rangle}{|F_s^{AM}F_f^{AM}|}\frac{\langle -\dot{F}^{AM}\rangle}{(F_f^{AM})^2}$$

$$\dot{\xi}_S^{AM} = 0. \tag{7}$$

The β^{AM} parameter measures the rates at which the transformation proceeds. A brief discussion of the evolutionary equation of the type here adopted is presented in Appendix A.

Single-variant martensite production. There are two different evolution processes which can result in a production of single-variant martensite: a conversion of austenite into single-variant martensite ($A \rightarrow S$) and a conversion of multiple-variant martensite into single-variant martensite ($M \rightarrow S$). We distinguish between the two conversion processes, also if the zones in which they may occur are assumed to be identical (Fig. 3). We set:

$$\begin{aligned}
F_1^{AS} &= \sigma \\
F_{1,s}^{AS} &= \sigma - S_s^{AS} \\
F_{1,f}^{AS} &= \sigma - S_f^{AS} \\
F_2^{AS} &= \sigma - C^{AS}T \\
F_{2,s}^{AS} &= (\sigma - S_s^{AS}) - C^{AS}(T - T_s^{AM}) \\
F_{2,f}^{AS} &= (\sigma - S_f^{AS}) - C^{AS}(T - T_s^{AM})
\end{aligned} \tag{8}$$

where C^{AS} is a material parameter and S_s^{AS} and S_f^{AS} are the stress values at which the

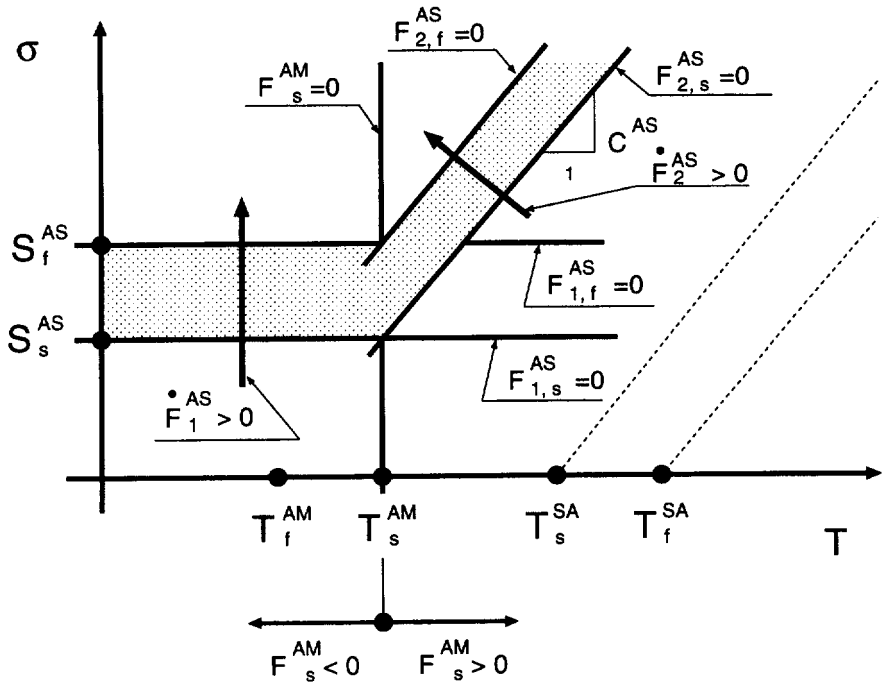

Fig. 3. Production of single-variant martensite. The phase transformation may occur within the shaded area. The arrows indicate the direction for activation of the transformation.

transformation starts and finishes if $T = T_s^{AM}$. Due to the specific form of the transformation zone, we need to distinguish on the sign of F_s^{AM}.

- If $F_s^{AM} < 0$, then the region in which the transformation may take place is:

$$F_{1,s}^{AS} > 0, \quad F_{1,f}^{AS} < 0 \Rightarrow F_{1,s}^{AS} F_{1,f}^{AS} < 0. \tag{9}$$

Moreover, for the activation of the transformation we require a stress increase:

$$\dot{F}_1^{AS} > 0. \tag{10}$$

- If $F_s^{AM} > 0$, then the region in which the transformation may take place is:

$$F_{2,s}^{AS} > 0, \quad F_{2,f}^{AS} < 0 \Rightarrow F_{2,s}^{AS} F_{2,f}^{AS} < 0. \tag{11}$$

Moreover, for the activation of the transformation we require a stress increase, a temperature decrease or a proper combination of those actions, that is:

$$\dot{F}_2^{AS} > 0. \tag{12}$$

Hence, for the conversion of austenite into single-variant martensite we have the following evolutionary equations:†

$$\begin{aligned}
\dot{\xi}_A^{AS} &= K_1^{AS} \langle -F_s^{AM} \rangle \langle -F_{1,s}^{AS} F_{1,f}^{AS} \rangle \langle \dot{F}_1^{AS} \rangle \\
&\quad + K_2^{AS} \langle F_s^{AM} \rangle \langle -F_{2,s}^{AS} F_{2,f}^{AS} \rangle \langle \dot{F}_2^{AS} \rangle \\
\dot{\xi}_S^{AS} &= -\dot{\xi}_A^{AS} \\
\dot{\xi}_M^{AS} &= 0
\end{aligned} \tag{13}$$

where K_1^{AS} and K_2^{AS} are scalar functions of the state variables. We choose K_1^{AS} and K_2^{AS} such that:

$$\begin{aligned}
\dot{\xi}_A^{AS} &= -\beta^{AS} \xi_A \left[\frac{\langle -F_s^{AM} \rangle}{|F_s^{AM}|} \frac{\langle -F_{1,s}^{AS} F_{1,f}^{AS} \rangle}{|F_{1,s}^{AS} F_{1,f}^{AS}|} \frac{\langle \dot{F}_1^{AS} \rangle}{(F_{1,f}^{AS})^2} \right. \\
&\quad \left. + \frac{\langle F_s^{AM} \rangle}{|F_s^{AM}|} \frac{\langle -F_{2,s}^{AS} F_{2,f}^{AS} \rangle}{|F_{2,s}^{AS} F_{2,f}^{AS}|} \frac{\langle \dot{F}_2^{AS} \rangle}{(F_{2,f}^{AS})^2} \right] \\
\dot{\xi}_S^{AS} &= \beta^{AS} \xi_A \left[\frac{\langle -F_s^{AM} \rangle}{|F_s^{AM}|} \frac{\langle -F_{1,s}^{AS} F_{1,f}^{AS} \rangle}{|F_{1,s}^{AS} F_{1,f}^{AS}|} \frac{\langle \dot{F}_1^{AS} \rangle}{(F_{1,f}^{AS})^2} \right. \\
&\quad \left. + \frac{\langle F_s^{AM} \rangle}{|F_s^{AM}|} \frac{\langle -F_{2,s}^{AS} F_{2,f}^{AS} \rangle}{|F_{2,s}^{AS} F_{2,f}^{AS}|} \frac{\langle \dot{F}_2^{AS} \rangle}{(F_{2,f}^{AS})^2} \right] \\
\dot{\xi}_M^{AS} &= 0.
\end{aligned} \tag{14}$$

In a similar way, for the conversion of multiple-variant martensite into single-variant martensite ($M \to S$) we have the following evolutionary equations:

† The segment $\{(\sigma, T) | S_s^{MS} \leqslant \sigma \leqslant S_f^{MS} \text{ and } F_s^{AM}(T) = 0\}$ can be considered as belonging to the transformation area with $F_s^{AM} < 0$ as well as to the one with $F_s^{AM} > 0$. In the following, we consider it as included in the first region, also if not explicitly stated in the flow rule expression.

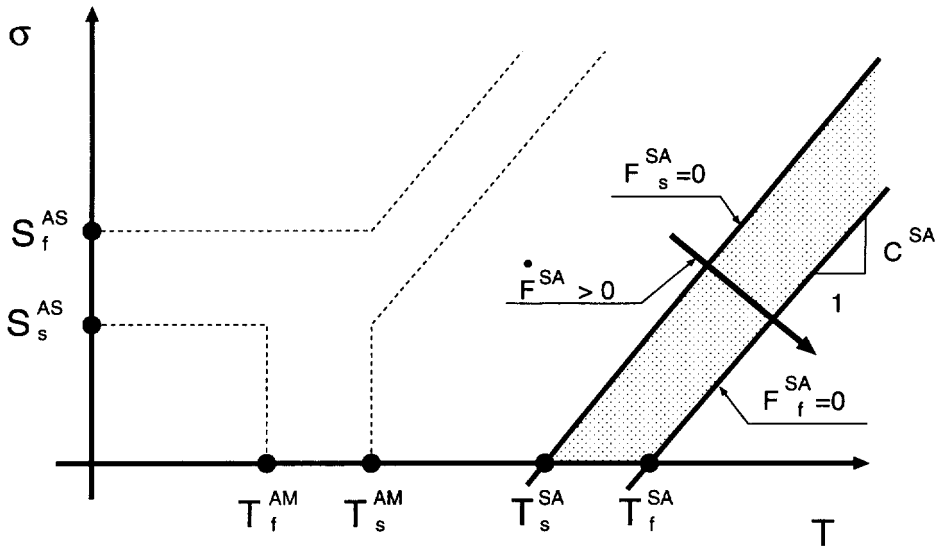

Fig. 4. Production of austenite. The phase transformation may occur within the shaded area. The arrows indicate the direction for activation of the transformation.

$$\begin{aligned}
\dot{\xi}_M^{MS} &= K_1^{MS}\langle -F_s^{AM}\rangle\langle -F_{1,s}^{AS}F_{1,f}^{AS}\rangle\langle \dot{F}_1^{AS}\rangle \\
&\quad + K_2^{MS}\langle F_s^{AM}\rangle\langle -F_{2,s}^{AS}F_{2,f}^{AS}\rangle\langle \dot{F}_2^{AS}\rangle \\
\dot{\xi}_S^{MS} &= -\dot{\xi}_M^{MS} \\
\dot{\xi}_A^{MS} &= 0
\end{aligned} \tag{15}$$

where K_1^{MS} and K_2^{MS} are again scalar functions of the state variable. We choose K_1^{MS} and K_2^{MS} such that:

$$\begin{aligned}
\dot{\xi}_M^{MS} &= -\beta^{MS}\xi_M\left[\frac{\langle -F_s^{AM}\rangle}{|F_s^{AM}|}\frac{\langle -F_{1,s}^{AS}F_{1,f}^{AS}\rangle}{|F_{1,s}^{AS}F_{1,f}^{AS}|}\frac{\langle \dot{F}_1^{AS}\rangle}{(F_{1,f}^{AS})^2}\right. \\
&\quad \left. + \frac{\langle F_s^{AM}\rangle}{|F_s^{AM}|}\frac{\langle -F_{2,s}^{AS}F_{2,f}^{AS}\rangle}{|F_{2,s}^{AS}F_{2,f}^{AS}|}\frac{\langle \dot{F}_2^{AS}\rangle}{(F_{2,f}^{AS})^2}\right] \\
\dot{\xi}_S^{MS} &= \beta^{MS}\xi_M\left[\frac{\langle -F_s^{AM}\rangle}{|F_s^{AM}|}\frac{\langle -F_{1,s}^{AS}F_{1,f}^{AS}\rangle}{|F_{1,s}^{AS}F_{1,f}^{AS}|}\frac{\langle \dot{F}_1^{AS}\rangle}{(F_{1,f}^{AS})^2}\right. \\
&\quad \left. + \frac{\langle F_s^{AM}\rangle}{|F_s^{AM}|}\frac{\langle -F_{2,s}^{AS}F_{2,f}^{AS}\rangle}{|F_{2,s}^{AS}F_{2,f}^{AS}|}\frac{\langle \dot{F}_2^{AS}\rangle}{(F_{2,f}^{AS})^2}\right] \\
\dot{\xi}_A^{MS} &= 0.
\end{aligned} \tag{16}$$

Austenite production. There are two different evolution process also for the production of austenite, the first relative to conversion of single-variant martensite into austenite ($S \rightarrow A$), the second relative to conversion of multiple-variant martensite into austenite ($M \rightarrow A$). We distinguish between the two conversion processes, also if the zones in which they may occur are identical (Fig. 4). We set:

$$\begin{aligned}
F^{SA} &= \sigma - C^{SA}T \\
F_s^{SA} &= \sigma - C^{SA}(T - T_s^{SA}) \\
F_f^{SA} &= \sigma - C^{SA}(T - T_f^{SA})
\end{aligned} \tag{17}$$

where C^{SA}, T_s^{SA} and T_f^{SA} are material parameters, with the latter two presenting starting and final temperatures at which the transformations may occur at zero stress. The region in which the transformation may take place is described by:

$$F_s^{SA} < 0, \quad F_f^{SA} > 0 \Rightarrow F_s^{SA} F_f^{SA} < 0. \tag{18}$$

Moreover, for the activation of the transformation we require a stress decrease, a temperature increase or a proper combination of those actions, that is:

$$\dot{F}^{SA} < 0. \tag{19}$$

Hence, for the conversion of single-variant martensite into austenite we have the following evolutionary equations:

$$\begin{aligned} \dot{\xi}_S^{SA} &= K^{SA} \langle -F_s^{SA} F_f^{SA} \rangle \langle -\dot{F}^{SA} \rangle \\ \dot{\xi}_A^{SA} &= -\dot{\xi}_S^{SA} \\ \dot{\xi}_M^{SA} &= 0 \end{aligned} \tag{20}$$

where K^{SA} is chosen such that:

$$\begin{aligned} \dot{\xi}_S^{SA} &= -\beta^{SA} \xi_S \frac{\langle -F_s^{SA} F_f^{SA} \rangle}{|F_s^{SA} F_f^{SA}|} \frac{\langle -\dot{F}^{SA} \rangle}{(F_f^{SA})^2} \\ \dot{\xi}_A^{SA} &= +\beta^{SA} \xi_S \frac{\langle -F_s^{SA} F_f^{SA} \rangle}{|F_s^{SA} F_f^{SA}|} \frac{\langle -\dot{F}^{SA} \rangle}{(F_f^{SA})^2} \\ \dot{\xi}_M^{SA} &= 0. \end{aligned} \tag{21}$$

In a similar fashion, for the conversion of multiple-variant martensite into austenite we have the following evolutionary equations:

$$\begin{aligned} \dot{\xi}_M^{MA} &= K^{MA} \langle -F_s^{SA} F_f^{SA} \rangle \langle -\dot{F}^{SA} \rangle \\ \dot{\xi}_A^{MA} &= -\dot{\xi}_M^{MA} \\ \dot{\xi}_S^{MA} &= 0 \end{aligned} \tag{22}$$

where K^{MA} is chosen such that:

$$\begin{aligned} \dot{\xi}_M^{MA} &= -\beta^{MA} \xi_M \frac{\langle -F_s^{SA} F_f^{SA} \rangle}{|F_s^{SA} F_f^{SA}|} \frac{\langle -\dot{F}^{SA} \rangle}{(F_f^{SA})^2} \\ \dot{\xi}_A^{MA} &= +\beta^{MA} \xi_M \frac{\langle -F_s^{SA} F_f^{SA} \rangle}{|F_s^{SA} \dot{F}_f^{SA}|} \frac{\langle -F^{SA} \rangle}{(F_f^{SA})^2} \\ \dot{\xi}_S^{MA} &= 0. \end{aligned} \tag{23}$$

Remark 1. Because of the general framework in which the model is developed (Lubliner, 1995), there is no limitation on the relative position of the phase-transition zones; hence they may intersect or they may be disjoint, since neither case would be problematic for the constitutive model.

Remark 2. Also, if we are dealing with three different fractions, the material may present only five (and not six!) solid–solid phase transformations. In fact, it is not possible to transform directly the single-variant martensite into the multiple-variant martensite

$(S \nrightarrow M)$; the only way to perform such transformation is to convert first the single-martensite fraction into austenite and then the austenite into multiple-martensite fraction. This is one of the key aspects for understanding the peculiar SMA macroscopic behavior.

2.3. *Flow rule*

Since during the construction of the evolutionary equations, we paid attention to guarantee the satisfaction of eqns (2), to obtain the final form of the flow rule, we may simply sum up all the evolutionary equations. The Macaulay brackets manage the choice of the active evolution process.

$$\begin{aligned}
\dot{\xi}_S &= \dot{\xi}_S^{AM} + \dot{\xi}_S^{AS} + \dot{\xi}_S^{MS} + \dot{\xi}_S^{MA} + \dot{\xi}_S^{SA} \\
\dot{\xi}_M &= \dot{\xi}_M^{AM} + \dot{\xi}_M^{AS} + \dot{\xi}_M^{MS} + \dot{\xi}_M^{MA} + \dot{\xi}_M^{SA} \\
\dot{\xi}_A &= \dot{\xi}_A^{AM} + \dot{\xi}_A^{AS} + \dot{\xi}_A^{MS} + \dot{\xi}_A^{MA} + \dot{\xi}_A^{SA}.
\end{aligned} \tag{24}$$

Then, recalling that the model has only two independent internal variables, chosen to be ξ_S and ξ_M, we may:

- neglect the evolutionary equation for ξ_A, since the value of this fraction can always be evaluated using eqn (1);
- express the evolutional equation for ξ_S and ξ_M all in terms of the independent internal variables.

Accordingly, the final version of the evolutionary equations are:

$$\begin{aligned}
\dot{\xi}_S = [\beta^{AS}(1-\xi_S-\xi_M)+\beta^{MS}\xi_M]\Bigg[&\langle -F_s^{AM}\rangle \frac{\langle -F_{1,s}^{AS}F_{1,f}^{AS}\rangle}{|F_{1,s}^{AS}F_{1,f}^{AS}|}\frac{\langle \dot{F}_1^{AS}\rangle}{(F_{1,f}^{AS})^2} \\
&+\langle -F_s^{AM}\rangle \frac{\langle -F_{2,s}^{AS}F_{2,f}^{AS}\rangle}{|F_{2,s}^{AS}F_{2,f}^{AS}|}\frac{\langle \dot{F}_2^{AS}\rangle}{(F_{2,f}^{AS})^2}\Bigg] - \beta^{SA}\xi_S \frac{\langle -F_s^{SA}F_f^{SA}\rangle}{|F_s^{SA}F_f^{SA}|}\frac{\langle -\dot{F}^{SA}\rangle}{(F_f^{SA})^2}
\end{aligned} \tag{25}$$

$$\begin{aligned}
\dot{\xi}_M = +\beta^{AM}(1-\xi_S-\xi_M)\frac{\langle -F_s^{AM}F_f^{AM}\rangle}{|F_s^{AM}F_f^{AM}|}\frac{\langle -\dot{F}^{AM}\rangle}{(F_f^{AM})^2} - \beta^{MS}\xi_M\Bigg[&\langle -F_s^{AM}\rangle \frac{\langle -F_{1,s}^{AS}F_{1,f}^{AS}\rangle}{|F_{1,s}^{AS}F_{1,f}^{AS}|}\frac{\langle \dot{F}_1^{AS}\rangle}{(F_{1,f}^{AS})^2} \\
&+\langle F_s^{AM}\rangle \frac{\langle -F_{2,s}^{AS}F_{2,f}^{AS}\rangle}{|F_{2,s}^{AS}F_{2,f}^{AS}|}\frac{\langle \dot{F}_2^{AS}\rangle}{(F_{2,f}^{AS})^2}\Bigg] - \beta^{MA}\xi_M \frac{\langle -F_s^{SA}F_f^{SA}\rangle}{|F_s^{SA}F_f^{SA}|}\frac{\langle -\dot{F}^{SA}\rangle}{(F_f^{SA})^2}.
\end{aligned} \tag{26}$$

If we set:

$$\mathscr{H}_1^{AS} = \begin{cases} 1 & \text{if } F_s^{AM} < 0 \quad \text{and} \quad F_{1,s}^{AS}F_{1,f}^{AS} < 0 \quad \text{and} \quad \dot{F}_1^{AS} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathscr{H}_2^{AS} = \begin{cases} 1 & \text{if } F_s^{AM} < 0 \quad \text{and} \quad F_{2,s}^{AS}F_{2,f}^{AS} < 0 \quad \text{and} \quad \dot{F}_2^{AS} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathscr{H}^{SA} = \begin{cases} 1 & \text{if } F_s^{SA}F_f^{SA} < 0 \quad \text{and} \quad \dot{F}^{SA} < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathscr{H}^{AM} = \begin{cases} 1 & \text{if } F_s^{AM}F_f^{AM} < 0 \quad \text{and} \quad \dot{F}^{AM} < 0 \\ 0 & \text{otherwise} \end{cases}$$

noting that $\langle -F_s^{AM}F_f^{AM}\rangle = 0$ when $F_s^{AM} > 0$, we may rewrite the flow rules in the following more compact form:

• If $F_s^{AM} \leqslant 0$ then

$$\dot{\xi}_S = +\mathscr{H}_1^{AS}[\beta^{AS}(1-\xi_S-\xi_M)+\beta^{MS}\xi_M]\frac{\dot{F}_1^{AS}}{(F_{1,f}^{AS})^2} + \mathscr{H}^{SA}\beta^{SA}\xi_S\frac{\dot{F}^{SA}}{(F_f^{SA})^2}$$

$$\dot{\xi}_M = -\mathscr{H}_1^{AS}\beta^{MS}\xi_M\frac{\dot{F}_1^{AS}}{(F_{1,f}^{AS})^2} + \mathscr{H}^{SA}\beta^{MA}\xi_M\frac{\dot{F}^{SA}}{(F_f^{SA})^2} - \mathscr{H}^{AM}\beta^{AM}(1-\xi_S-\xi_M)\frac{\dot{F}^{AM}}{(F_f^{AM})^2}. \tag{27}$$

• If $F_s^{AM} > 0$ then

$$\dot{\xi}_S = \mathscr{H}^{SA}\beta^{SA}\xi_S\frac{\dot{F}^{SA}}{(F_f^{SA})^2} + \mathscr{H}_2^{AS}[\beta^{AS}(1-\xi_S-\xi_M)+\beta^{MS}\xi_M]\frac{\dot{F}_2^{AS}}{(F_{2,f}^{AS})^2}$$

$$\dot{\xi}_M = \mathscr{H}^{SA}\beta^{MA}\xi_M\frac{\dot{F}^{SA}}{(F_f^{SA})^2} - \mathscr{H}_2^{SA}\beta^{MS}\xi_M\frac{\dot{F}_2^{AS}}{(F_{2,f}^{AS})^2}. \tag{28}$$

2.4. *Strain decomposition and elastic equation*

Limiting the discussion to a small deformation regime, we assume an additive decomposition of the total strain ε:

$$\varepsilon = \varepsilon^e + \varepsilon_L\xi_S \tag{29}$$

where ε^e is the elastic strain and ε_L is the maximum recoverable strain.† The elastic strain is assumed to be linearly related to the stress:

$$\sigma = E\varepsilon^e \tag{30}$$

with E the elastic modulus.

Remark 3. The kinematic assumption—small deformation and strain decomposition—as well as the elastic constitutive equation are fundamentally different and unrelated from the inelastic constitutive equation governing the evolution of the martensite fractions. The formers are here introduced mainly to perform some simple example tests. The inelastic constitutive equations can be used also within a nonlinear kinematic framework.

Remark 4. For some simple loading case, the flow rules can be integrated in closed form as discussed in Appendix B.

3. TEST EXAMPLES

We now discuss the response of the proposed constitutive model under isothermal mechanical loading and under combined thermal and mechanical loading.

† The maximum recoverable strain ε_L, regarded as a material constant, is a measure of the maximum deformation obtainable only by multiple-variant martensite detwinning, hence, a measure of the maximum deformation obtainable aligning all the single-variant martensites in one direction.

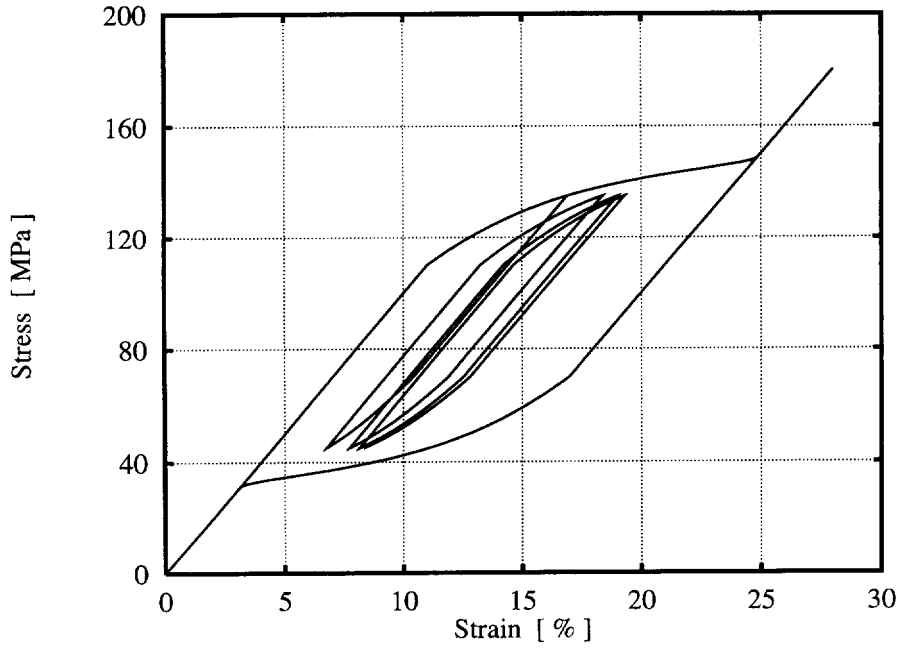

Fig. 5. Superelasticity: stress vs strain. A complete transformation path is followed by partial loading and partial unloading.

We choose to work with an hypothetical alloy, for which the material parameters are chosen as:

$$E = 1000\,\text{MPa} \quad C^{AS} = C^{SA} = 1\,\text{MPa}/^{\circ}\text{C} \quad \varepsilon_L = 10\%$$

$$T_f^{AM} = 10^{\circ}\text{C} \quad T_s^{AM} = 50^{\circ}\text{C} \quad T_s^{SA} = 80^{\circ}\text{C} \quad T_f^{SA} = 120^{\circ}\text{C}$$

$$S_s^{MS} = 10\,\text{MPa} \quad S_f^{MS} = 50\,\text{MPa}.$$

The β-parameters are set equal to 10. The material is always assumed to start from a fully austenitic phase ($\xi_M = \xi_S = 0$).

3.1. *Superelastic behavior*

We test the model's ability to reproduce the superelastic behavior under multiple stress cycles, while keeping the temperature constant ($T = 160 > T_f^{SA}$). We start always from a specimen in the parent phase ($\xi_S = 0, \xi_A = 1$). In the first simulation (Figs 5–6) we first induce a complete set of transformations (that is, a complete $A \to S$ and a complete $S \to A$ transformation), followed by partial unloading–reloading cycles. A *partial reloading* implies an incomplete direct transformation ($A \to S$), while a *partial unloading* implies an incomplete reverse transformation ($S \to A$). Note that the model describes a series of loops, which are internal to the complete loading–unloading cycle; such internal loops present ratcheting, which stabilizes after a few cycles. We also consider the case of partial loading with complete unloading, the case of partial unloading with complete loading and the case of partial loading and partial unloading (Fig. 7–9). Comparing with experimental results presented in the literature, it is possible to conclude that the model has the appropriate qualitative behavior (McNichols, 1987, Muller, 1991, Lim, 1994).

3.2. *Shape-memory effect*

We test the behavior of the model under a thermal-stress–thermal cycle. Starting from $T = 60$ and zero stress we cool the specimen (inducing a conversion of austenite into multiple-variant martensite). Thereafter, we first bring the temperature back to

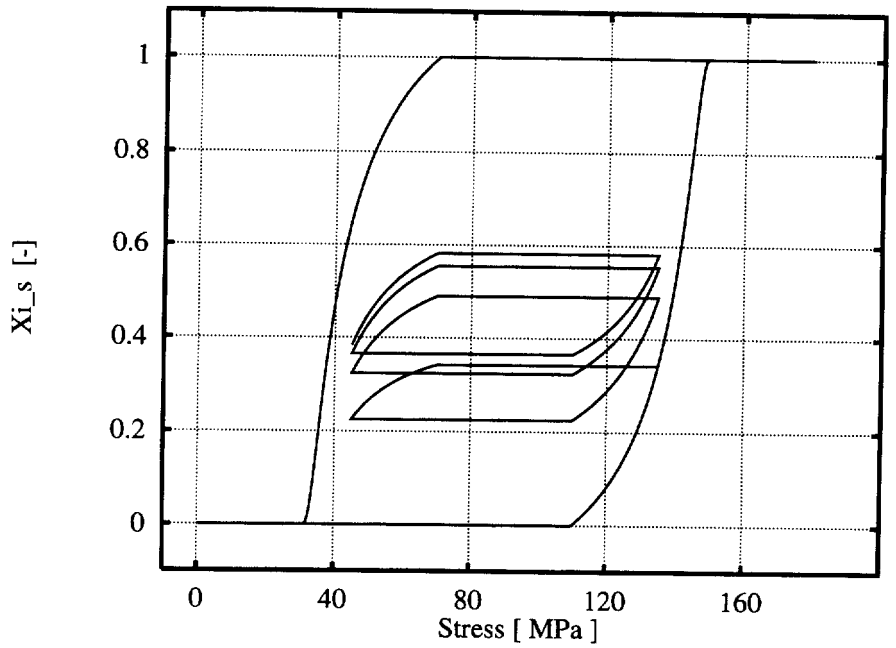

Fig. 6. Superelasticity: single variant martensite fraction versus stress. A complete transformation path is followed by partial loading and partial unloading.

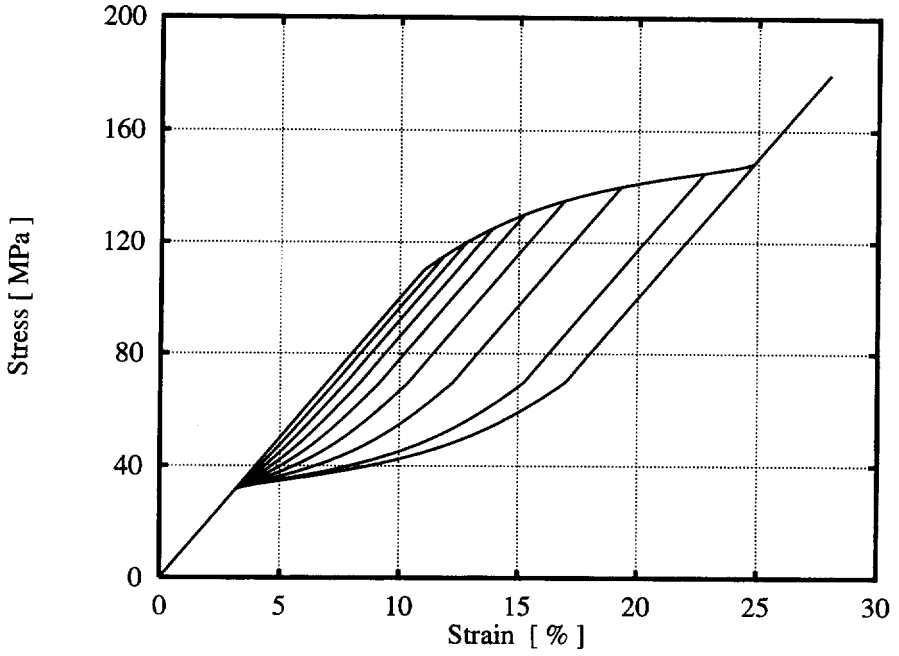

Fig. 7. Superelasticity: stress vs strain. A complete transformation path is followed by partial loading and partial unloading.

$T = 60 < T_s^{SA}$ and then load the specimen (including a conversion of multiple-variant martensite into single-variant martensite). Finally, we unload the specimen and at zero stress we increase the temperature (inducing a conversion of single-variant martensite into austenite). The stress–strain–temperature response and the martensite fraction evolution are presented in Figs 10 and 11. Note that the permanent deformation obtained after the production of single-variant martensite is recovered with the thermal cycle. Accordingly the model reproduces the shape-memory effect.

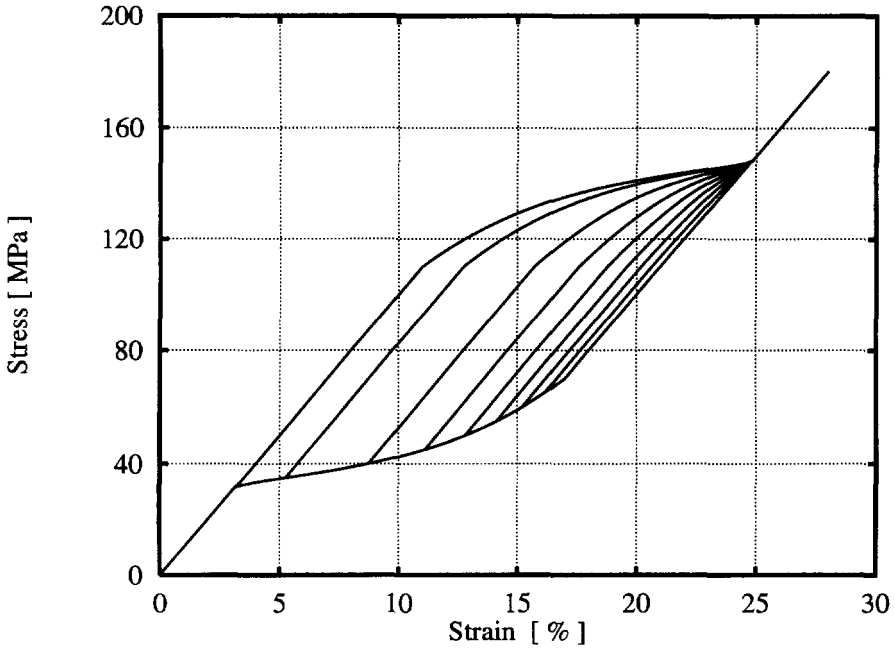

Fig. 8. Superelasticity: stress vs strain. A complete transformation path is followed by complete loading and partial unloading.

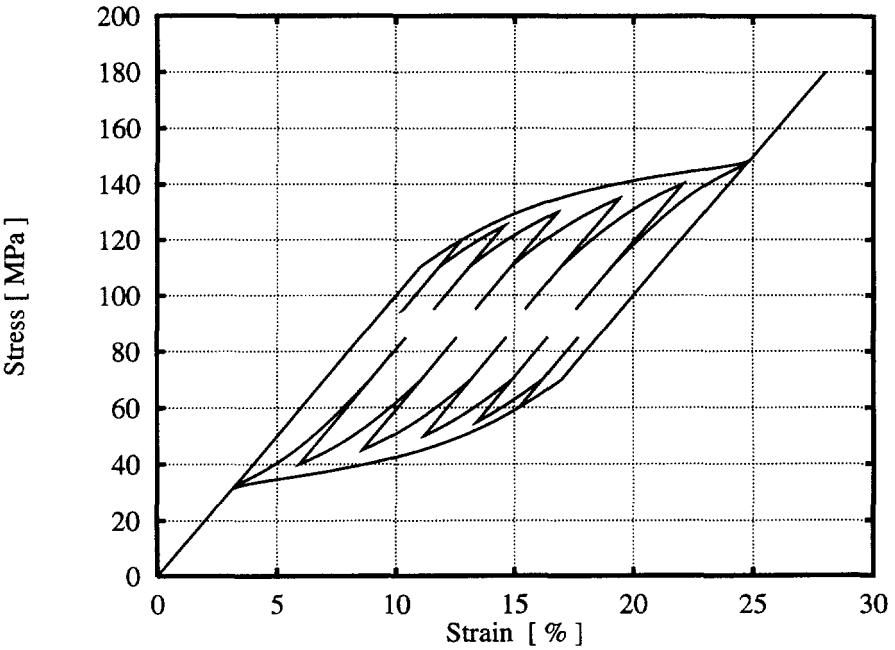

Fig. 9. Superelasticity: stress vs strain. A complete transformation path is followed by a series of partial loading and a series of partial unloading.

4. CLOSURE AND FUTURE DIRECTIONS OF RESEARCH

In the present work we presented a uniaxial model for shape-memory alloys, cast within the framework of a general and flexible inelastic theory previously developed (Lubliner, 1995). The model is based on two internal variables (the single-variant martensite fraction and the multiple-variant martensite fraction), for which evolution equations in rate form are proposed. Despite its simplicity, the model predicts the shape-memory effect and the superelastic behavior; moreover, for stress loading–unloading cycles, without completion of the phase transition, the model presents the correct cyclic response.

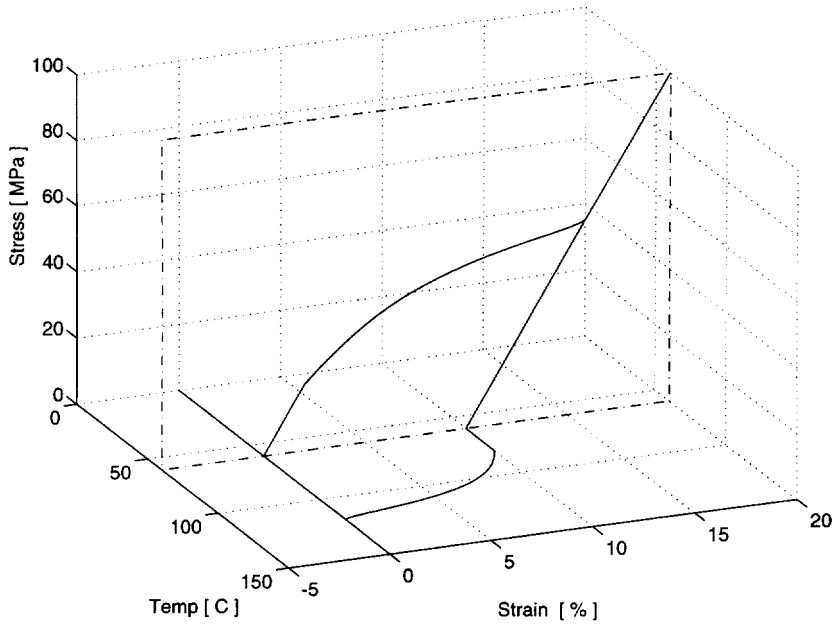

Fig. 10. Shape-memory effect: strain vs stress-temperature. Initially the material is a fully austenitic state. The specimen is first thermally cycled (cooled) to induce a conversion of austenite into multiple-variant martensite. It is then mechanically cycled (loading–unloading), inducing a conversion of multiple-variant martensite into single-variant martensite (showing a residual strain). Finally, the initial configuration is recovered through a thermal cycle (heating).

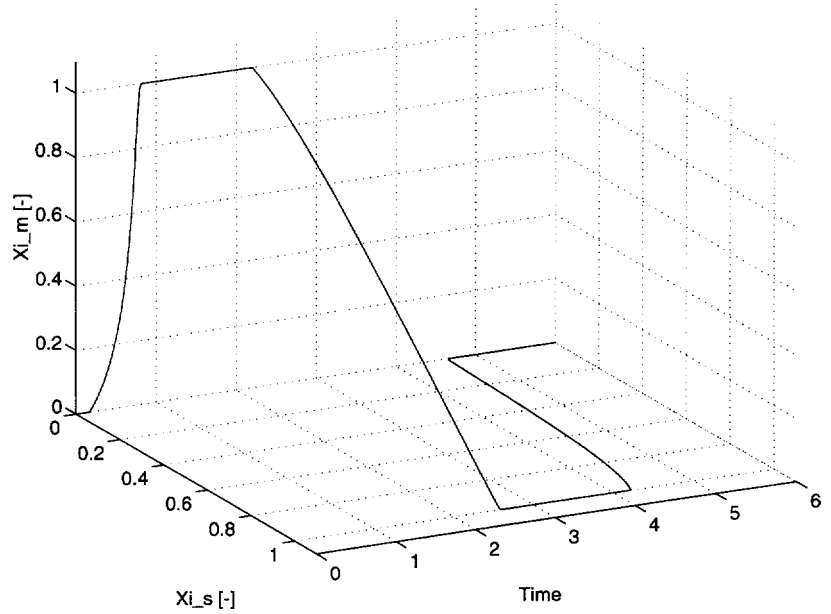

Fig. 11. Shape-memory effect: evolution of the martensite fractions. Initially the material is a fully austenitic state. The sequence of phase transformations due to the thermo-mechanical loading pattern can be observed: conversion of austenite into multiple-variant martensite (cooling), conversion of multiple-variant martensite into single-variant martensite (mechanical loading), conversion of single-variant martensite into austenite (heating).

Though the model is developed in a uniaxial context, it may be extended to include three-dimensional as well as more complex behaviors. Such extensions as well as the implementation in a real computational and design environment, such as a finite-element code, will be presented in forthcoming papers.

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APPENDIX A. FLOW RULE

We now want to highlight the flexibility of the flow rules proposed in the present work. To do so, it is sufficient to consider only one internal variable, ξ, whose evolutionary process is governed by the rate-independent equations:

$$\dot{\xi} = \beta\xi \frac{\dot{\sigma}}{(\sigma-\sigma_f)^2} \quad \text{if } \sigma_s < \sigma < \sigma_f, \quad \dot{\sigma} < 0 \tag{A1}$$

$$\dot{\xi} = \beta(1-\xi) \frac{\dot{\sigma}}{(\sigma-\sigma_f)^2} \quad \text{if } \sigma_s < \sigma < \sigma_f, \quad \dot{\sigma} < 0 \tag{A2}$$

where β, σ_s and σ_f are material constants. Integration of the equations leads to:

$$\xi = \xi_o \exp\left[\frac{\beta}{\Delta\sigma}\left(\frac{\sigma-\sigma_s}{\sigma-\sigma_f}\right)\right] \tag{A3}$$

$$\xi = 1-\xi_o \exp\left[\frac{\beta}{\Delta\sigma}\left(\frac{\sigma-\sigma_s}{\sigma-\sigma_f}\right)\right] \tag{A4}$$

with $\Delta\sigma = \sigma_f - \sigma_s$ and ξ_o being the initial condition. Since the two solutions differ only by a constant, we may limit the discussion only to the first equation. Assuming $\xi_o = 1$, in Fig. A1 we plot the value of the fraction ξ vs $(\sigma-\sigma_i)/(\Delta\sigma)$ for different values of $\beta/(\Delta\sigma)$. It is interesting to observe how varying the ratio $\beta/(\Delta\sigma)$ very different evolution processes can be obtained; the inspection of the figure clearly show the flexibility of the model in terms of the β-parameter.

APPENDIX B. CLOSED FORM SOLUTIONS

In many real applications the material is subjected to processes in which only one quantity between temperature and stress vary at the time. Accordingly, it is of practical interest to consider how the evolution processes specialize for the case in which one control variable is kept fixed, while the other varies. To simplify the equation we are dealing with, we assume $T_s^{SA} > T_s^{AM}$ and $C^{AS} = C^{SA} = \bar{C}$; therefore, the transformation zone relative to the production of austenite does not intersect any region in which production of martensite may occur. Accordingly, we may distinguish between the following three cases:

- $F_s^{AM} \leqslant 0$
- $F_s^{AM} > 0$ and $F_s^{SA}F_f^{SA} < 0$
- $F_s^{AM} > 0$ and $F_s^{SA}F_f^{SA} > 0$.

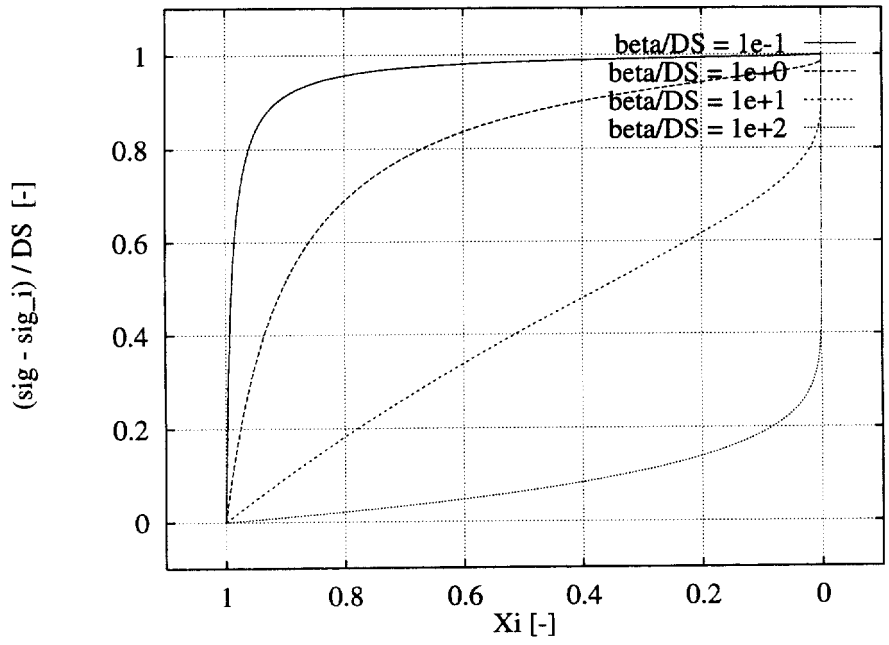

Fig. A1. Martensite fraction vs $\sigma/\Delta\sigma$ for different values of $\beta/(\Delta\sigma)$.

Constant stress, variable temperature

We have:

$$\dot{F}^{AM} = \dot{T}, \quad \dot{F}_1^{AS} = 0, \quad \dot{F}_2^{AS} = -\bar{C}\dot{T}, \quad \dot{F}^{SA} = -\bar{C}\dot{T}.$$

If we indicate with a superposed $\mathring{}$ the derivative with respect to the independent control variable T (that is, $\mathring{\xi}_S = \mathrm{d}\xi_S/\mathrm{d}T$ and $\mathring{\xi}_M = \mathrm{d}\xi_M/\mathrm{d}T$), the flow rules reduce as follows:

T1 $F_s^{AM} \leqslant 0$, $F_s^{AM}F_f^{AM} < 0$ and $\dot{F}^{AM} = \dot{T} < 0$:

$$\mathring{\xi}_S = 0 \tag{A5}$$

$$\mathring{\xi}_M = \left[\beta^{AM}\frac{1}{(F_f^{AM})^2}\right]\xi_S + \left[\beta^{AM}\frac{1}{(F_f^{AM})^2}\right]\xi_M - \beta^{AM}\frac{1}{(F_f^{AM})^2} \tag{A6}$$

T2 $F_s^{AM} > 0$, $F_{2,s}^{AS}F_{2,f}^{AS} < 0$ and $\dot{F}_2^{AS} = -\bar{C}\dot{T} > 0$:

$$\mathring{\xi}_S = \left[\beta^{AS}\frac{\bar{C}}{(F_{2,f}^{AS})^2}\right]\xi_S + \left[(\beta^{AS}-\beta^{MS})\frac{\bar{C}}{(F_{2,f}^{AS})^2}\right]\xi_M - \beta^{AS}\frac{\bar{C}}{(F_{2,f}^{AS})^2} \tag{A7}$$

$$\mathring{\xi}_M = \left[\beta^{MS}\frac{\bar{C}}{(F_{2,f}^{AS})^2}\right]\xi_M \tag{A8}$$

T3 $F_s^{AM} > 0$, $F_s^{SA}F_f^{SA} < 0$ and $\dot{F}^{SA} = -\bar{C}\dot{T} < 0$:

$$\mathring{\xi}_S = \left[-\beta^{SA}\frac{\bar{C}}{(F_f^{SA})^2}\right]\xi_S \tag{A9}$$

$$\mathring{\xi}_M = \left[-\beta^{MA}\frac{\bar{C}}{(F_f^{SA})^2}\right]\xi_M. \tag{A10}$$

The equations can be integrated in closed form for the three cases:

T1 $F_s^{AM} \leqslant 0$, $F_s^{AM}F_f^{AM} < 0$ and $\dot{F}^{AM} = \dot{T} < 0$:

$$\xi_S = \xi_S^0 \tag{A11}$$

$$\xi_M = (1-\xi_S^0) - (1-\xi_S^0-\xi_M^0)\exp\left[\beta^{AM}\frac{F_s^{AM}}{\Delta M F_f^{AM}}\right] \tag{A12}$$

T2 $F_s^{AM} > 0$, $F_{2,s}^{AS}F_{2,f}^{AS} < 0$ and $\dot{F}_2^{AS} = -\bar{C}\dot{T} > 0$:

$$\xi_S = 1-\xi_M^0\exp\left[\beta^{MS}\frac{F_{2,s}^{AS}}{\bar{C}\Delta M F_{2,f}^{AS}}\right] - (1-\xi_S^0-\xi_M^0)\exp\left[\beta^{AS}\frac{F_{2,s}^{AS}}{\bar{C}\Delta M F_{2,f}^{AS}}\right] \tag{A13}$$

$$\xi_M = \xi_M^0\exp\left[\beta^{MS}\frac{F_{2,s}^{AS}}{\bar{C}\Delta M F_{2,f}^{AS}}\right] \tag{A14}$$

T3 $F_s^{AM} > 0$, $F_s^{SA}F_f^{SA} < 0$ and $\dot{F}^{SA} = -\bar{C}\dot{T} < 0$:

$$\xi_S = \xi_S^0\exp\left[\beta^{SA}\frac{F_s^{SA}}{\bar{C}\Delta A F_f^{SA}}\right] \tag{A15}$$

$$\xi_M = \xi_M^0\exp\left[\beta^{MA}\frac{F_s^{SA}}{\bar{C}\Delta A F_f^{SA}}\right] \tag{A16}$$

where ξ_S^0 and ξ_M^0 are the initial values, $\Delta M = T_s^{AM} - T_f^{AM} > 0$ and $\Delta A = T_f^{SA} - T_s^{SA} > 0$.

Constant temperature, variable stress

We have:

$$\dot{F}^{AM} = 0, \quad \dot{F}_1^{AS} = \dot{F}_2^{AS} = \dot{F}^{SA} = \dot{\sigma}.$$

Indicating with a superposed $\mathring{}$ the derivative with respect the independent control variable σ (that is, $\mathring{\xi}_S = \mathrm{d}\xi_S/\mathrm{d}\sigma$ and $\mathring{\xi}_M = \mathrm{d}\xi_M/\mathrm{d}\sigma$), the flow rules reduce as follows:

S1 $F_s^{AM} \leqslant 0$, $F_{1,s}^{AS} F_{1,f}^{AS} < 0$ and $\dot{F}_1^{AS} = \dot{\sigma} > 0$:

$$\xi^{\circ}_S = \left[-\beta^{AS} \frac{1}{(F_{1,f}^{AS})^2} \right] \xi_S + \left[(-\beta^{AS} + \beta^{MS}) \frac{1}{(F_{1,f}^{AS})^2} \right] \xi_M + \beta^{AS} \frac{1}{(F_{1,f}^{AS})^2} \tag{A17}$$

$$\xi^{\circ}_M = \left[-\frac{1\beta^{MS}}{(F_{1,f}^{AS})^2} \right] \xi_M \tag{A18}$$

S2 $F_s^{AM} > 0$, $F_{2,s}^{AS} F_{2,f}^{AS} < 0$ and $\dot{F}_2^{AS} = \dot{\sigma} > 0$:

$$\xi^{\circ}_S = \left[-\beta^{AS} \frac{1}{(F_{2,f}^{AS})^2} \right] \xi_S + \left[(-\beta^{AS} + \beta^{MS}) \frac{1}{(F_{2,f}^{AS})^2} \right] \xi_M + \beta^{AS} \frac{1}{(F_{2,f}^{AS})^2} \tag{A19}$$

$$\xi^{\circ}_M = \left[-\beta^{MS} \frac{1}{(F_{2,f}^{AS})^2} \right] \xi_M \tag{A20}$$

S3 $F_s^{AM} > 0$, $F_s^{SA} F_f^{SA} < 0$ and $\dot{F}^{SA} = \dot{\sigma} < 0$:

$$\xi^{\circ}_S = \left[\beta^{SA} \frac{1}{(F_f^{SA})^2} \right] \xi_S \tag{A21}$$

$$\xi^{\circ}_M = \left[\beta^{MA} \frac{1}{(F_f^{SA})^2} \right] \xi_M. \tag{A22}$$

The equations can be integrated in closed form for the three cases:

S1 $F_s^{AM} \leqslant 0$, $F_s^{AS} F_f^{AS} < 0$ and $\dot{F}_1^{AS} = \dot{\sigma} > 0$:

$$\xi_S = 1 - \xi_M^0 \exp\left[\beta^{MS} \frac{F_{1,s}^{AS}}{\bar{C} \Delta S F_{1,f}^{AS}} \right] - (1 - \xi_S^0 - \xi_M^0) \exp\left[\beta^{AS} \frac{F_{1,s}^{AS}}{\bar{C} \Delta S F_{1,s}^{AS}} \right] \tag{A23}$$

$$\xi_M = \xi_M^0 \exp\left[\beta^{MS} \frac{F_{1,s}^{AS}}{\bar{C} \Delta S F_{1,s}^{AS}} \right] \tag{A24}$$

S2 $F_s^{AM} > 0$, $F_{2,s}^{AS} F_{2,f}^{AS} < 0$ and $\dot{F}_2^{AS} = \dot{\sigma} > 0$:

$$\xi_S = 1 - \xi_M^0 \exp\left[\beta^{MS} \frac{F_{2,s}^{AS}}{\bar{C} \Delta M F_{2,f}^{AS}} \right] - (1 - \xi_S^0 - \xi_M^0) \exp\left[\beta^{AS} \frac{F_{2,s}^{AS}}{\bar{C} \Delta M F_{2,f}^{AS}} \right] \tag{A25}$$

$$\xi_M = \xi_M^0 \exp\left[\beta^{MS} \frac{F_{2,s}^{AS}}{\bar{C} \Delta M F_{2,f}^{AS}} \right] \tag{A26}$$

S3 $F_s^{AM} > 0$, $F_s^{SA} F_f^{SA} < 0$ and $\dot{F}^{SA} = \dot{\sigma} < 0$:

$$\xi_S = \xi_S^0 \exp\left[\beta^{SA} \frac{F_s^{SA}}{\bar{C} \Delta A F_f^{SA}} \right] \tag{A27}$$

$$\xi_M = \xi_M^0 \exp\left[\beta^{MA} \frac{F_s^{SA}}{\bar{C} \Delta A F_f^{SA}} \right] \tag{A28}$$

where $\Delta S = \sigma_f - \sigma_s > 0$.