eXtended Finite Element Method (XFEM) for material modelling. Application to cracks

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Introduction

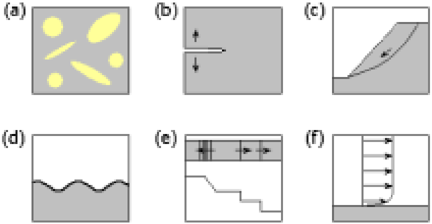
The eXtended Finite Element Method (XFEM) is a versatile tool for the analysis of problems characterized by discontinuities, singularities, localized deformations and complex geometries.

This method can simplify the solution of many problems in material modeling, such as

- the propagation of crack,
- the evolution of dislocations,
- the modeling of grain boundaries,
- the evolution of phase boundaries.

Non-smooth solutions properties: discontinuities and singularities

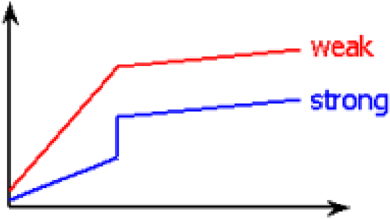
A discontinuity may be defined as a rapid change of a field quantity over a length which is negligable compared to the dimensions of the observed domain. In the real world, discontinuities are frequently found.



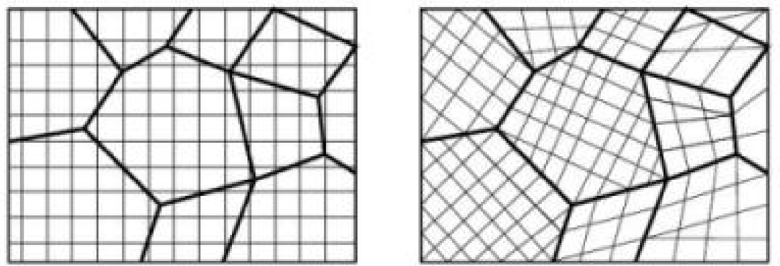
- (a) In solids stresses and strains are discontinuous across material interfaces.
- (b) In solids displacements are discontinuous at cracks.
- (c) Tangential displacements are discontinuous across shear bands.
- (d) In fluids, velocity and pressure fields may involve discontinuities at the interface of two fluids.
- (e)(f) Shocks and boundary layers can be interpreted as discontinuities.

Classification of discontinuities

- weak discontinuities (kink),
- strong discontinuities (jump).



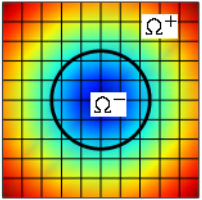
The advantages of this method is that the finite element mesh can be completely independent of the morphology of these entities.



The level-set method

The decription of discontinuities in the context of the XFEM is often realized by the level-set method. A level-set function is a scalar function f within the domain whose zero-level is interpreted as the discontinuity. As a consequence, the domain Ω is divided into two subdomains Ω^+ and Ω^- on either side of the discontinuity where the level-set function is positive or negative, respectively.

Example



Often, the signed distance function is used as a particular level-set function

$$f(x) = \pm \min_{x_\Gamma \in \Gamma} \|x - x_\Gamma\|$$

It is noted, that level-set functions are typically defined by discrete values at the nodes in the domain. They are then interpolated in the element interiors by standard finite element shape functions.

$$f^h(x) = \sum_{i \in I} f_i N_i(x)$$

$$\text{con } f_i = f(x_i)$$

Overview

XFEM is a numerical method that enables a local enrichment of approximation spaces.
The enrichment is realized through the partition of unity concept.
The method is useful for the approximation of solutions with pronounced non-smooth characteristics in small parts of the computational domain, for example near discontinuities and singularities.

General formulation

$$\text{XFEM} = \text{FEM} + \text{enrichment}$$

Finite element approximation: u^{FEM}

$$u^{FEM}(x) = \sum_{i \in I} N_i(x) u_i$$

with

- I set of all nodes in the domain,
- N_i standard FE function of node i,
- u_i unknown of the standard FE part at node i.

Partition of unity

The foundation of this method is the partition of unity concept for enriching finite element approximation.
A global partition of unity in a domain Ω is a set of functions $\{\varphi_i\}$ such that

$$\sum_i \varphi_i(x) = 1, \quad \text{per ogni } x \in \Omega$$

General formulation

$$u^{XFEM}(x) = u^{FEM} + u^{\text{enriched}}$$

$$u^{XFEM}(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I^*} \varphi_i(x) \Psi(x) q_i$$

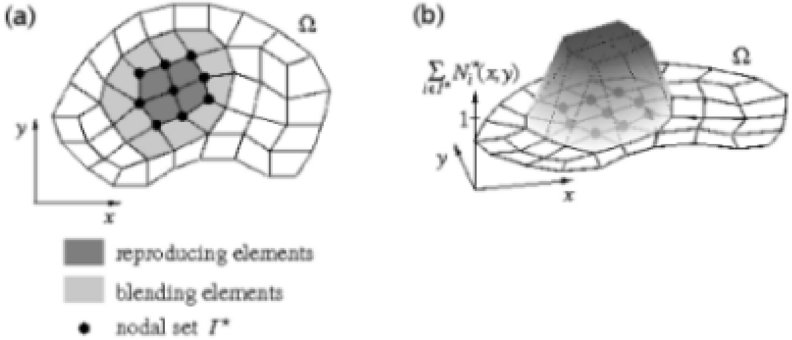
with

- $I^* \subset I$ nodal subset of the enrichment,
- φ_i partition of unity function of node i,
- Ψ global enrichment functions,
- q_i unknown parameter at node i.

$M_i(x) = \varphi_i(x)\Psi(x)$ local enrichment function of node i

Classification of elements

- reproducing elements,
- blending elements (!).



Choice of enriched nodes

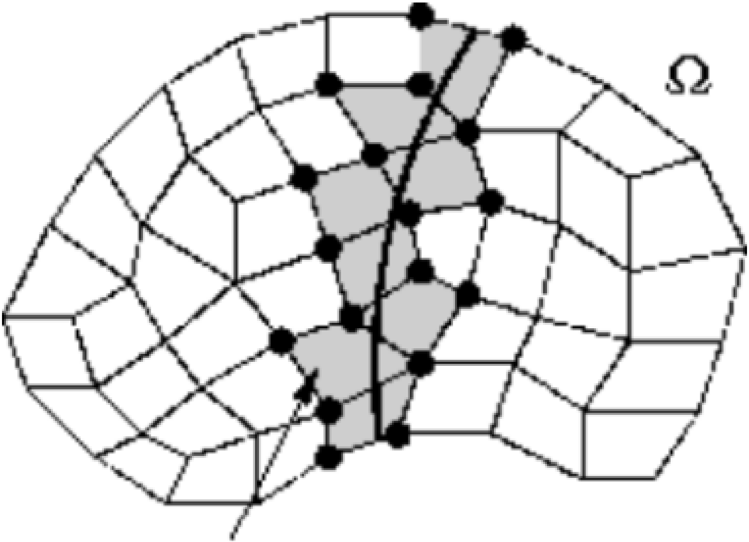
For weak and strong discontinuities, the nodal subset I^* is built from all nodes of elements that are cut by the discontinuity.

$$\text{cut elements:} \quad \min_{i \in I^{el}} (f_i) \cdot \max_{i \in I^{el}} (f_i) < 0$$

$$\text{uncut elements:} \quad \min_{i \in I^{el}} (f_i) \cdot \max_{i \in I^{el}} (f_i) > 0$$

with

- f level-set function,
- I^{el} set of element nodes.


Ω

Global enrichment functions

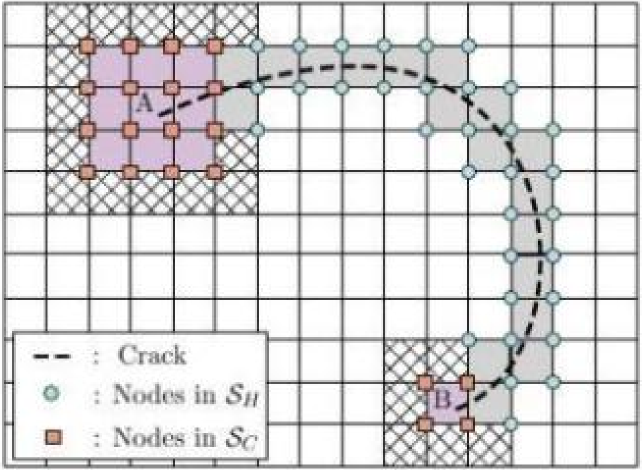
- weak discontinuities: abs-functions of the level set-function

$$\Psi(x) = |f(x)|$$

- strong discontinuities: Heaviside-function of the level set-function

$$\Psi(x) = H(f(x))$$

Application to cracks modelling



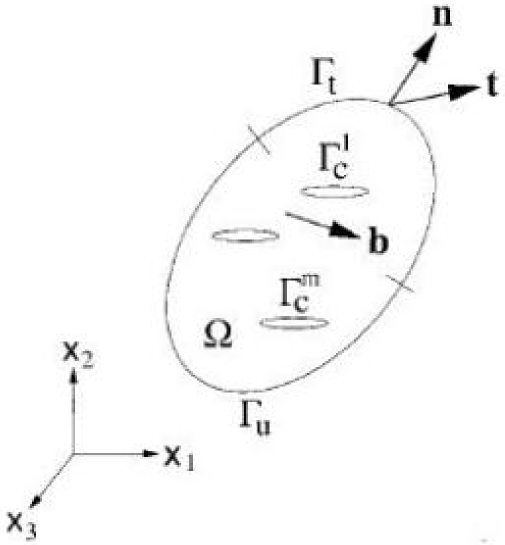
$$u^h(x) = \sum_{i \in S} N_i(x) u_i + \sum_{i \in S_H} N_i(x) \left[H(f(x)) - H(f(x_i)) \right] q_i^0 + \sum_{j=1}^{n} \sum_{i \in S_C} N_i(x) \left[\Psi^j(x) - \Psi^j(x_i) \right] q_i^j$$

- S set of all nodes of finite element mesh,
- S_C set of nodes of elements around the crack tip,
- S_H set of nodes of elements cut by the crack but not in S_C.
- $\left\{ \Psi^j \right\}_{j=1,\dots,n}$ set of enrichment functions (approximation of near tip beheviour)

Example: cracks in elastic materials

$$\left\{\Psi^j\right\}_{j=1}^4 = \sqrt{r}\left\{\cos(\theta/2), \sin(\theta/2), \sin(\theta/2) sin(\theta), \cos(\theta/2)\sin(\theta)\right\}$$

- based on asymptotic solution of Williams
- r, θ polar coordinates


n
t
Γ_t
Γ_c^l
b
Γ_c^m
Ω
Γ_u
x_2
x_1
x_3

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in } \Omega$$

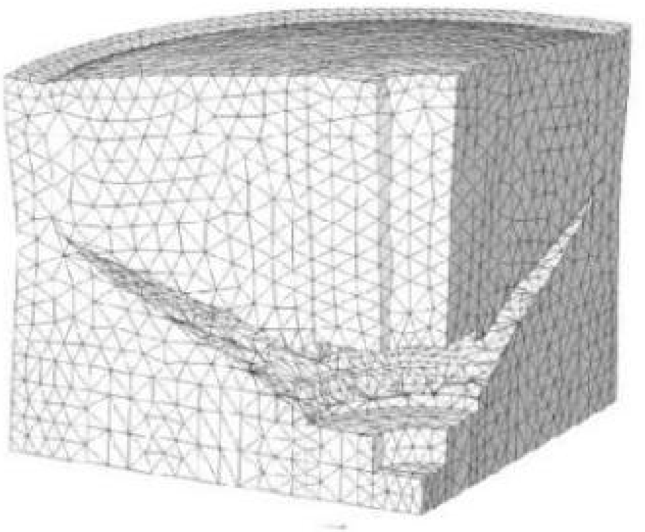
$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c^i \quad (i = 1, 2, \ldots, m)$$



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