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The Modified Finite Particle Method in the context of meshless methods

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• Introduction

• Literature review on meshless methods

• Modified Finite Particle Method (MFPM)
  • Application to elasticity

• MFPM for incompressible elasticity
  • Discretization with collocation methods: numerical difficulties
  • Alternative formulations of incompressible elasticity equations
  • Applications

• MFPM in the framework of the Least Square Residual Method
  • Solution of linear and non linear problems
Numerical methods are computational algorithms used to find the approximated solution of (algebraic, ordinary differential, partial differential) equations for which an analytical solution is not available.

Here we focus on numerical algorithms for partial differential equations.

Numerical methods for partial differential equations can be classified in:

- **Collocation methods**: a solution is found by discretizing the partial differential equation in the strong form.

- **Variational methods**: a solution is found by discretizing a weak formulation of the partial differential equation.
Numerical methods for partial differential equations can be subdivided in:

- **Particle methods**: The domain is discretized into particles with physical characteristics (mass, velocity, energy, …)

- **Non-particle methods**: The domain is discretized into nodes or elements with no physical characteristics

and in:

- **Mesh-based methods**: The nodes of the domain have a predetermined connectivity

- **Meshless methods**: The connectivity among nodes is based on the current reciprocal positions
In the present work we focus on **meshless methods**.

**Interesting features** of meshless methods: since there is no rigid connectivity among nodes, such methods can easily model problems in which large deformations are involved, such as fast solid dynamics, explosions, fluid-dynamics.

In fluid dynamics, particularly, meshless methods permit the study of fluid flows using a **Lagrangian framework**, that is, the fluid motion is studied through the motion of each fluid particle. This approach is convenient when studying problems of fluid-structure interaction (FSI), or fluid-dynamics problems with moving boundaries.
The Smoothed Particle Hydrodynamics (SPH)

The first meshless method introduced in the literature is the **Smoothed Particle Hydrodynamics (SPH)**, introduced by **Lucy** (1977) and **Gingold and Monaghan** (1977) for the study of astrophysical problems.

The starting point of the approximation is

\[
f(x_i) = \int_{\Omega} f(x) \delta(x - x_i) dx
\]

\[
\delta(x - x_i)
\]

**Dirac Delta** distribution, centered in \( x_i \)

- Approximation of the Dirac Delta distribution:

\[
\delta(x - x_i) \approx W(x - x_i, h)
\]

- Function approximation

\[
f(x_i) = \int_{\Omega} f(x) W(x - x_i, h) dx
\]

\( W(x - x_i, h) \) **kernel function**

\( h \) **smoothing length**
Smoothed Particle Hydrodynamics (SPH)

- Properties of the kernel function

\[ \int_{\Omega} W(x - x_i, h) dx = 1 \]

\[
\begin{cases} 
W(x - x_i, h) \neq 0 & |x - x_i| < h \\
W(x - x_i, h) = 0 & |x - x_i| \geq h 
\end{cases}
\]

\[ \lim_{h \to 0} W(x - x_i, h) = \delta(x - x_i) \]

- Approximation of first order derivative

\[ f'(x_i) = \left[ f(x)W(x - x_i, h) \right]_{-\infty}^{+\infty} - \int_{\Omega} f(x)W'(x - x_i, h) dx \]

Here the first term is neglected due to the shape of \( W(x - x_i, h) \)
Discrete approximation for derivatives of order $n$

$$f^{(n)}(x_i) = (-1)^n \sum_j f(x_j) W^{(n)}(x_j - x_i) \Delta x_j$$

Integrals are discretized and replaced by summation

PROBLEM !!!

What about particles near to the boundary domain?

When the smoothing length exceeds the boundary of the domain, the smoothing function is said not completely developed. In these cases the derivative approximation is not accurate.
SPH-based methods

- Reproducing Kernel Particle Method (Liu et al, 1995)
  \[ K_i(x) = C_i(x) W(x - x_i) \]

- Corrective Smoothed Particle Method (Chen et al, 1999)
  \[ f(x_i) \approx \frac{\int_{\Omega} f(x) W(x - x_i, h) \, dx}{\int_{\Omega} W(x - x_i, h) \, dx} \]

- Modified Smoothed Particle Method (Zhang and Batra, 2004)
  \[
  \begin{pmatrix}
  A_{11}^i & A_{12}^i \\
  A_{21}^i & A_{22}^i
  \end{pmatrix}
  \begin{pmatrix}
  f(x_i) \\
  f'(x_i)
  \end{pmatrix}
  =
  \begin{pmatrix}
  \int_{\Omega} f(x) W_i(x) \, dx \\
  \int_{\Omega} f(x) W'_i(x) \, dx
  \end{pmatrix} \]


Meshless methods based on shape functions

Radial Basis Collocation (Franke, 1998; Buhmann, 2003)

\[ f(x) = \sum_j a_j \phi_j(r_{ij}) \quad r_{ij} = \sqrt{(x_i - x_j) \cdot (x_i - x_j)} \]

Radial basis functions

\[ \phi(r) = e^{-r^2/c^2} \quad \phi(r) = (r^2 + c^2)^{n-3/2} \quad \phi(r) = r \log(r) \]

Local max-ent shape functions (Arroyo, 2006, 2007)

\[ f_\beta(x, p) \equiv \beta U(x, p) - H(p) \]

Shape function

\[ p_{\beta \alpha}(x) = \frac{1}{Z(x, \lambda^*(x))} \exp[-\beta |x - x_\alpha|^2 + \lambda^*(x) \cdot (x - x_\alpha)] \]

Partition function

\[ Z(x, \lambda) \equiv \sum_{\alpha=1}^{N} \exp[-\beta |x - x_\alpha|^2 + \lambda^*(x) \cdot (x - x_\alpha)] \]

Lagrangian multiplier

\[ \lambda^*(x) = \arg \min \log Z(x, \lambda) \]
References


Coadvisorship activity of the master thesis by **Alberto Cattenone**: Max-Ent approximants and applications with collocation methods. University of Pavia, October 2015.
The Modified Finite Particle Method is a *collocation, meshless, local* method based on projection functions.

**Collocation** method: we approximate partial differential equations in their strong form. This leads to avoid problems related to numerical integration of functions.

**Meshless** method: there is no *a priori* connectivity among nodes. The idea is to approximate functions and derivatives based only on the reciprocal position between nodes.

**Local** method: derivative approximations are based on nodes in a neighborhood of each collocation point.
Modified Finite Particle Method (MFPM)

Consider the **two dimensional** Taylor series of a function $u(x)$ centered in a point $x_i$

$$u(x) - u(x_i) = \frac{\partial u}{\partial x}(x_i)(x - x_i) + \frac{\partial u}{\partial y}(x_i)(y - y_i) +$$

$$\frac{1}{2} \frac{\partial^2 u}{\partial x^2}(x_i)(x - x_i)^2 + \frac{1}{2} \frac{\partial^2 u}{\partial y^2}(y - y_i)^2 + \frac{\partial^2 u}{\partial x \partial y}(x_i)(x - x_i)(y - y_i)$$

- 5 unknown derivatives have to be approximated
- 5 **projection functions** $W_\alpha^i = W_\alpha(x - x_i)$ for $\alpha = 1, \ldots, 5$ are introduced

$$W_1^i = x - x_i; \quad W_2^i = y - y_i; \quad W_3^i = (x - x_i)^2;$$

$$W_4^i = (y - y_i)^2; \quad W_5^i = (x - x_i)(y - y_i)$$

- **No unity** property is required
- **No compact support** property is required
- **No Dirac Delta property** is required
Modified Finite Particle Method (MFPM)

- Taylor series is projected on the projection functions $W_{\alpha}^i$

$$\int_{\Omega} [u(x) - u(x_i)] W_{\alpha}^i \, d\Omega = \frac{\partial u}{\partial x}(x_i) \int_{\Omega} (x - x_i) W_{\alpha}^i \, d\Omega + \frac{\partial u}{\partial y}(x_i) \int_{\Omega} (y - y_i) W_{\alpha}^i \, d\Omega$$

$$+ \frac{\partial^2 u}{\partial x^2}(x_i) \int_{\Omega} \frac{1}{2} (x - x_i)^2 W_{\alpha}^i \, d\Omega + \frac{\partial^2 u}{\partial y^2}(x_i) \int_{\Omega} \frac{1}{2} (y - y_i)^2 W_{\alpha}^i \, d\Omega$$

$$+ \frac{\partial^2 u}{\partial x \partial y}(x_i) \int_{\Omega} (x - x_i)(y - y_i) W_{\alpha}^i \, d\Omega \quad \alpha = 1, \ldots, 5$$

- In matrix form

$$A_i \begin{pmatrix} \frac{\partial u(x_i)}{\partial x} \\ \frac{\partial u(x_i)}{\partial y} \\ \frac{\partial^2 u(x_i)}{\partial x^2} \\ \frac{\partial^2 u(x_i)}{\partial y^2} \\ \frac{\partial^2 u(x_i)}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} \int_{\Omega} [u(x) - u(x_i)] W_1^i \, d\Omega \\ \int_{\Omega} [u(x) - u(x_i)] W_2^i \, d\Omega \\ \int_{\Omega} [u(x) - u(x_i)] W_3^i \, d\Omega \\ \int_{\Omega} [u(x) - u(x_i)] W_4^i \, d\Omega \\ \int_{\Omega} [u(x) - u(x_i)] W_5^i \, d\Omega \end{pmatrix}$$
Modified Finite Particle Method (MFPM)

• Integrals have to be discretized.

• Voronoi procedure is used to divide the domain in subdomains.

• Each particle $x_j$ is assigned a subdomain $\Delta A_j$.

• Integrals are replaced by summation

$$A_i \begin{pmatrix} \partial u(x_i)/\partial x \\ \partial u(x_i)/\partial y \\ \partial^2 u(x_i)/\partial x^2 \\ \partial^2 u(x_i)/\partial y^2 \\ \partial^2 u(x_i)/\partial x \partial y \end{pmatrix} = \begin{pmatrix} \sum_j [u(x_j) - u(x_i)]W_{1ij} \Delta A_j \\ \sum_j [u(x_j) - u(x_i)]W_{2ij} \Delta A_j \\ \sum_j [u(x_j) - u(x_i)]W_{3ij} \Delta A_j \\ \sum_j [u(x_j) - u(x_i)]W_{4ij} \Delta A_j \\ \sum_j [u(x_j) - u(x_i)]W_{5ij} \Delta A_j \end{pmatrix}$$

• Inverting this system, we get derivative approximations

PROBLEMS

• The Voronoi tessellation procedure is time consuming

• Numerical errors introduced in the approximation of integrals
Consider the Taylor series expansion of $u(x)$ centered at $x_i$.

Evaluate $u(x_j) - u(x_i)$ in a set of supporting nodes $x_j$.

$$u(x_j) - u(x_i) = \frac{\partial u}{\partial x}(x_i)(x_j - x_i) + \frac{\partial u}{\partial y}(x_i)(y_j - y_i)$$

$$+ \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(x_i)(x_j - x_i)^2 + \frac{1}{2} \frac{\partial^2 u}{\partial y^2}(y_j - y_i)^2$$

$$+ \frac{\partial^2 u}{\partial x \partial y}(x_i)(x_j - x_i)(y_j - y_i)$$

Collect the evaluations in a vector $q^i = \{u(x_j) - u(x_i)\}, j = 1, \ldots, N_{supp}$ where $N_{supp}$ is the number of the nodes supporting $x_i$.

Evaluate five projection functions $W^i_\alpha = W_\alpha (x - x_i)$ in the same supporting nodes $x_j$.

Collect the evaluations of $W^i_\alpha$ in five vectors $W^i_\alpha = \{W^{ij}_\alpha\}, j = 1, \ldots, N_{supp}$ where $W^{ij}_\alpha = W_\alpha (x_j - x_i)$.
Modified Finite Particle Method: novel formulation

- Scalarly multiply $W^i_{\alpha} \cdot q^i$

$$ \sum_{j}^{N_{\text{supp}}} [u(x) - u(x_i)]W^{ij}_{\alpha} = \frac{\partial u}{\partial x}(x_i) \sum_{j}^{N_{\text{supp}}} (x_j - x_i)W^{ij}_{\alpha} + \frac{\partial u}{\partial y}(x_i) \sum_{j}^{N_{\text{supp}}} (y_j - y_i)W^{ij}_{\alpha} $$

\[ + \frac{\partial^2 u}{\partial x^2}(x_i) \sum_{j}^{N_{\text{supp}}} \frac{1}{2} (x_j - x_i)^2 W^{ij}_{\alpha} + \frac{\partial^2 u}{\partial y^2}(x_i) \sum_{j}^{N_{\text{supp}}} \frac{1}{2} (y_j - y_i)^2 W^{ij}_{\alpha} \]

\[ + \frac{\partial^2 u}{\partial x \partial y}(x_i) \sum_{j}^{N_{\text{supp}}} (x_j - x_i)(y_j - y_i)W^{ij}_{\alpha} \]

\[ \alpha = 1, \ldots, 5 \]

- In matrix form

$$ A_i = \begin{pmatrix} \frac{\partial u(x_i)}{\partial x} \\ \frac{\partial u(x_i)}{\partial y} \\ \frac{\partial^2 u(x_i)}{\partial x^2} \\ \frac{\partial^2 u(x_i)}{\partial y^2} \\ \frac{\partial^2 u(x_i)}{\partial x \partial y} \end{pmatrix} \begin{pmatrix} \sum_{j}^{N_{\text{supp}}} [u(x_j) - u(x_i)]W_{1}^{ij} \\ \sum_{j}^{N_{\text{supp}}} [u(x_j) - u(x_i)]W_{2}^{ij} \\ \sum_{j}^{N_{\text{supp}}} [u(x_j) - u(x_i)]W_{3}^{ij} \\ \sum_{j}^{N_{\text{supp}}} [u(x_j) - u(x_i)]W_{4}^{ij} \\ \sum_{j}^{N_{\text{supp}}} [u(x_j) - u(x_i)]W_{5}^{ij} \end{pmatrix} $$

- Inverting the system, derivative approximations are obtained

ADVANTAGES

- No integral discretization is needed
- No Voronoi tessellation is required
Consider a linear elastic body $\Omega$ subjected to internal forces $b = b(x)$, constrained displacements $\bar{u} = \bar{u}(x)$ on the Dirichlet boundary $\Gamma_D$ and tractions $\bar{t} = \bar{t}(x)$ on the Neumann boundary $\Gamma_N$.

Dynamic equilibrium equations

\[
\begin{cases}
\rho \frac{\partial^2 s}{\partial t^2} = \nabla \cdot \sigma + b & x \in \Omega \\
\sigma n = \bar{t}(t) & x \in \Gamma_N \\
s = \bar{s}(t) & x \in \Gamma_D \\
s \big|_{t=0} = s_0(x) & x \in \Omega \\
\frac{\partial s}{\partial t} \bigg|_{t=0} = \dot{s}_0(x) & x \in \Omega
\end{cases}
\]

Spatial discretization of the problem

\[
\begin{pmatrix}
\rho \ddot{u} \\
\rho \ddot{v} \\
\rho \ddot{w}
\end{pmatrix}
= \begin{pmatrix}
\hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} \\
\hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} \\
\hat{K}_{31} & \hat{K}_{32} & \hat{K}_{33}
\end{pmatrix}
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix}
+ \begin{pmatrix}
b_x \\
b_y \\
b_z
\end{pmatrix}
\]

\[
\begin{align*}
\hat{K}_{11} &= (\lambda + 2\mu)D_{xx} + \mu(D_{yy} + D_{zz}) \\
\hat{K}_{22} &= (\lambda + 2\mu)D_{yy} + \mu(D_{xx} + D_{zz}) \\
\hat{K}_{33} &= (\lambda + 2\mu)D_{zz} + \mu(D_{xx} + D_{yy}) \\
\hat{K}_{12} &= \hat{K}_{21} = (\lambda + \mu)D_{xy} \\
\hat{K}_{13} &= \hat{K}_{31} = (\lambda + \mu)D_{xz} \\
\hat{K}_{23} &= \hat{K}_{32} = (\lambda + \mu)D_{yz}
\end{align*}
\]
Traction of an infinitely extended plate with a circular hole

\[ \nabla \cdot \sigma + b = 0 \quad x \in \Omega \]
\[ \sigma n = 0 \quad \text{on } \Gamma_1 \text{ and } \Gamma_4 \]
\[ \sigma n \cdot t = 0 \quad \text{and} \quad s \cdot n = 0 \quad \text{on } \Gamma_2 \text{ and } \Gamma_5 \]
\[ \sigma n = [\sigma_0 \ 0]^T \quad \text{on } \Gamma_3 \]

Geometry and boundary conditions for a bar under impulsive traction

Convergence diagram of the error in stress for the original and novel formulation

- Second order convergence of the error in both the original and novel formulation
- The error constant is reduced in the novel formulation

Stress component \( \sigma_{xx} \) obtained with MFPM
3D problem: parallelepiped with spherical hole under traction

The parallelepiped is under traction in the $x$ direction.

$x = 0$, $y = 0$, $z = 0$ are symmetry planes

• The novel formulation is easily extensible to 3D, while the original formulation implies 3D Voronoi tessellation that is difficult to code and extremely time consuming.
Dynamics: 2d bar under impulsive load

Geometry and boundary conditions for a bar under impulsive traction

Stress component $\sigma_{xx}$ in three time instant. The stress wave propagation can be appreciated.

Also in the case of dynamics, second order spatial accuracy is achieved both for original and novel formulation.


Incompressibility

The equation that models the dynamics of an incompressible (solid or fluid) body is

\[ \rho \, a = -\nabla p + \mu \Delta u + b \]

\( \rho \) density \quad \( a \) material acceleration
\( p \) pressure \quad \( \mu \) elastic shear modulus
\( b \) body loads

The variable \( u \) is the body displacement field in the solid case, and is the body velocity field in the fluid case.

The incompressibility condition is \( \nabla \cdot u = 0 \)

Lagrangian point of view: the observer follows body particles during their motion.

Eulerian point of view: the observer is in a fixed position during the fluid motion.

\[ a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} \]

\[ a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + c \cdot \nabla u \]

\( c = u - u_{ref} \) convective velocity
Stokes and Navier-Stokes Equations

Stokes equations
\[ \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{b} \]
\[ \nabla \cdot \mathbf{u} = 0 \]
- Describe highly-viscous flows in an Eulerian framework
- Describe all flows in a Lagrangian framework

Navier-Stokes equations
\[ \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{b} \]
\[ \nabla \cdot \mathbf{u} = 0 \]
- Describe all flows in an Eulerian framework
- Are non linear equations

Stokes and Navier-Stokes Equations are parabolic problems with a constraint equation. For its solution, one initial condition is required on the whole domain; two boundary conditions are required at each boundary.

Possible boundary conditions
- Dirichlet boundary conditions (known velocity at the boundary)
- Neumann boundary conditions (known traction at the boundary)
- No boundary conditions are required for the constraint equation
The same interpolation of velocity and pressure leads to a well known instability of the pressure field.

This instability has been first studied by Brezzi (1974), which established a condition that has to be respected in order to avoid pressure instability. This condition is known as LBB condition or inf-sup condition.

In the field of the Finite Element Method, usually velocity is discretized using quadratic interpolation, and pressure using linear interpolation.

In the field of Finite Difference Method, the difficulty is overcome using staggered grids, that is, velocity and pressure are discretized in different nodes, and also the equilibrium equations and incompressibility constrains are collocated in different points.

Staggered grids cannot be used in meshless methods, where nodes could be randomly distributed.
Stokes Equations – alternative formulations

Different formulations should be used in order to solve incompressibility equations on non staggered grids.

\[-\nabla p + \mu \Delta u = -b\]
\[\nabla \cdot u = 0\]

\[\Delta p = \nabla \cdot b\]
\[(-\nabla p + \mu \Delta u) \cdot n = -b \cdot n \quad \text{on } \partial \Omega\]

**S1 (Sani, 2006)**
\[-\nabla p + \mu \Delta u = -b\]
\[\Delta p = \nabla \cdot b\]
\[(-\nabla p + \mu \Delta u) \cdot n = -b \cdot n \quad \text{on } \partial \Omega\]

**S2 (Sani, 2006)**
\[-\nabla p + \mu \Delta u = -b\]
\[\Delta p - \mu \nabla \cdot (\Delta u) = \nabla \cdot b\]

No BC for the constraint equation

**S3 (Sani, 2006)**
\[-\nabla p + \mu \Delta u = -b\]
\[\Delta p = \nabla \cdot b\]
\[\nabla \cdot u = 0 \quad \text{on } \partial \Omega\]

**S4 (Brezzi&Douglas, 1988)**
\[-\nabla p + \mu \Delta u = -b\]
\[\nabla \cdot u + \varepsilon \Delta p = 0\]
\[\nabla \cdot u = 0 \quad \text{on } \partial \Omega\]

**S5 (Wang&Liu, 2000)**
\[u = a - \nabla \phi\]
\[p = -\Delta \phi\]
\[\mu \Delta a = -b\]
\[\nabla \cdot a - \Delta \phi = 0\]
\[\partial \phi / \partial n = 0 \quad \text{on } \partial \Omega\]

In formulations S1, S2 and S3 the incompressibility is not explicitly imposed!
Formulations S1, S2 and S3 do not impose directly the incompressibility constrain, but a derived one.

- Lid-driven cavity flow (Stokes case)

From the above figures we see that only in formulation S3 the constraint is respected. For this reason, formulations S1 and S2 are not further investigated.

**Problem of a square with polynomial solution**

\[ u(x, y) = 20xy^3 \]
\[ v(x, y) = 5(x^4 - y^4) \]
\[ p(x, y) = (60x^2y - 20y^3 + C') \]

Convergence diagrams for displacements and pressure


• Asprone, D., F. Auricchio, A. Montanino, and A. Reali Modified Finite Particle Method for Stokes problems. *Submitted to Computer & fluids*
Modified Finite Particle Method: extended formulation

Two different sets of points

- **Collocation points** \( x = [x, y]^T \): the points where function and derivatives are computed

- **Control points** \( \xi = [\xi, \eta]^T \): the auxiliary points used for approximating function and derivatives in the collocation points

The derivative approximation technique starts from the Taylor series of \( u(x) \), centered in \( x_i \), evaluated in \( \xi_j \):

\[
\begin{align*}
u(\xi_j) &= u(x_i) + D_x u(x_i)(\xi_j - x_i) + D_y u(x_i)(\eta_j - y_i) + \frac{1}{2}D_{xx}^2 u(x_i)(\xi_j - x_i)^2 \\
&\quad+ \frac{1}{2}D_{yy}^2 u(x_i)(\eta_j - y_i)^2 + D_{xy} u(x_i)(\xi_j - x_i)(\eta_j - y_i)
\end{align*}
\]

6 unknowns: \( u(x_i), D_x u(x_i), D_y u(x_i), D_{xx} u(x_i), D_{yy} u(x_i), D_{xy} u(x_i) \)
Introduce 6 known projection functions  \( W_\alpha^i = W_\alpha (\xi - x_i) \) and evaluate them in the same points \( \xi_j \).

Multiply left- and right-hand sides of Equation (1) in each \( \xi_j \) by the evaluations \( W_\alpha^{ij} = W_\alpha (\xi_j - x_i) \), and sum all terms, obtaining 6 equations of the type

\[
\begin{align*}
& u_i \sum_j W_\alpha^{ij} + D_x u_i \sum_j (\xi_j - x_i) W_\alpha^{ij} + D_y u_i \sum_j (\eta_j - y_i) W_\alpha^{ij} + \\
& + \frac{1}{2} D_{xx}^2 u_i \sum_j (\xi_j - x_i)^2 W_\alpha^{ij} + \frac{1}{2} D_{yy}^2 u_i \sum_j (\eta_j - y_i)^2 W_\alpha^{ij} + \\
& + D_{xy}^2 u_i \sum_j (\xi_j - x_i)(\eta_j - y_i) W_\alpha^{ij} = \sum_j u(\xi_j) W_\alpha^{ij} \quad \alpha = 1, \ldots, 6
\end{align*}
\]

**Matrix form**

\[
A^i \begin{pmatrix} u(x_i) \\ D_x u(x_i) \\ D_y u(x_i) \\ D_{xx}^2 u(x_i) \\ D_{yy}^2 u(x_i) \\ D_{xy}^2 u(x_i) \end{pmatrix} = \begin{pmatrix} \sum_j u(\xi_j) W_1^{ij} \\ \sum_j u(\xi_j) W_2^{ij} \\ \sum_j u(\xi_j) W_3^{ij} \\ \sum_j u(\xi_j) W_4^{ij} \\ \sum_j u(\xi_j) W_5^{ij} \\ \sum_j u(\xi_j) W_6^{ij} \end{pmatrix}
\]

**Compact form**

\[
A^i D(u_i) = W^i u
\]

If we assume to know the values of \( u(\xi) \) collected in \( u \) we can retrieve the approximations of function and derivatives collected in \( D(u_i) \) in \( x_i \)
Stokes Equations: discretization

Stokes problem

\[ \nabla p = \mu \Delta u + b \]

\[ \nabla \cdot u = 0 \]

MFPM discretization of the Stokes problem

\[
\begin{bmatrix}
\mu L & 0 & -D_x \\
0 & \mu L & -D_y \\
D_x & D_y & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{p}
\end{bmatrix}
=
\begin{bmatrix}
\hat{f}_x \\
\hat{f}_y \\
0
\end{bmatrix}
\]

\[ K \mathbf{d} = \mathbf{f} \]

\( L, D_x, D_y \): MFPM discrete differential operators

The above algebraic system is completed with suitable discrete boundary condition on the velocity or on the boundary outward stress.

When collocation points \( x \) and control points \( \xi \) coincide, the linear system is a square system, that can be solved through direct inversion.

Unfortunately, if collocation points coincide with control nodes, the solution of linear system exhibits unphysical oscillations of the pressure, known in the literature as pressure checkerboard instability.
The problem is discretized using a number of collocation points higher than the number of control points. The corresponding linear system is overdetermined. Its solution is approximated through an error minimization.

Global error

\[ E = (Kd - f)^T (Kd - f) \]

Minimization

\[ \frac{\partial E}{\partial d} = 0 \quad \Rightarrow \quad K^T Kd = K^T f \]

The present error definition is the sum of the squared errors coming from the equilibrium equations, from the incompressibility constraint, and from Dirichlet and Neumann boundary conditions. They in general contribute differently to the total squared error.

Hu et al (2007) and Chi et al. (2014) proposed a weighted squared error, based on the balance of different components of the error.

Weighted error

\[ E_w = (Kd - f)^T A (Kd - f) \]

\( A \): diagonal matrix composed by the squared weights associated to the different error components
Choice of the weights

**Weighted error minimization**

\[ K^T A K d = K^T A f \]

The present linear system is a well-posed square system, which gives the control nodal values \( \hat{u}, \hat{v}, \) and \( \hat{p} \) from which it is possible, using the MFPM approximation procedure, to retrieve the evaluations of \( u(x), v(x), \) and \( p(x) \) in the collocation points.

The weights are selected on the basis of the **scale** of each equation

- **Equilibrium equations:** \( o(\mu \partial u/\partial x^2) \approx o(\mu/h^2) \)
- **Incompressibility constraints:** \( o(\partial u/\partial x) \approx o(1/h) \)
- **Dirichlet boundary conditions:** \( o(u) \approx o(1) \)
- **Neumann boundary conditions:** \( o(\mu \partial u/\partial x) \approx o(\mu/h) \)

To restore the same order of magnitude of equations, the weights are

\[ \alpha_{eq} = 1; \quad \alpha_{inc} = \mu/h; \quad \alpha_{Dir} = \mu/h^2; \quad \alpha_{Neum} = 1/h \]
Quarter of annulus under polynomial body loads

Stokes flow in a quarter of annulus with exact solution

\[
\begin{align*}
    u &= 10^{-6} x^2 y^4 (x^2 + y^2 - 16)(x^2 + y^2 - 1)(5x^4 + 18x^2 y^2 - 85x^2 + 13y^4 + 80 - 153y^2) \\
    v &= -2 \cdot 10^{-6} xy^5 (x^2 + y^2 - 16)(x^2 + y^2 - 1)(5x^4 - 51x^2 + 6x^2 y^2 - 17y^2 + 16 + y^4)
\end{align*}
\]

Collocation and control nodes distributions

Convergence diagram of the error
Navier-Stokes Equations

Stationary Navier-Stokes equations:

\[ \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mu \Delta \mathbf{u} + \mathbf{b} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

The MFPM/LSRM procedure is extended to the non-linear case, and the solution of a problem is found through the Newton Raphson algorithm.

Iterative linearization process

\[ K^k(d) \Delta d^k = R^k \]

Since the present system is overdetermined \( (N_c > N_s) \), a LSRM procedure is required to approximate the solution increment at each step.

Iteration weighted squared error

\[ E^k_w = (K^k \Delta d^k - R^k)^T A^k (K^k \Delta d^k - R^k) \]

Minimization

\[ (K^k)^T A^k K^k \Delta d^k = (K^k)^T A^k R^k \]

Solution until convergence

\[ \alpha_i = A^k_{ii} = \left( \frac{3N_i}{\sum_{j=1}^{N_i} K^k_{ij}} \right)^2 \]

Selected weights
Application: lid-driven cavity

Domain and boundary condition for the lid-driven cavity problem

Pressure contour plot

Vorticity contour plot

Horizontal velocity profile at $x=0$ and comparison with Ghia et al. (1982)

$Re = 400$


• **Asprone, D., F. Auricchio, A. Montanino, and A. Reali.** Solution of the stationary Stokes and Navier-Stokes equations using the Modified Finite Particle Method in the framework of a Least Square Residual Method. *To be submitted*
• In this thesis we study meshless method, and apply and extend the Modified Finite Particle Method to compressible linear elasticity, incompressible elasticity, fluid dynamics.

• For linear elastic problems, in the tests where an analytical solution is available, MFPM shows correct second-order accuracy.

• When approaching incompressible elasticity, MFPM, similarly to other numerical methods, suffers from spurious oscillations of the pressure, unless the so-called inf-sup condition is respected, or alternative formulations are used.

• An extended formulation of the MFPM has been introduced and used in combination with a Least Square Residual Method for the solution of Stokes and Navier-Stokes problem. The method is more robust with respect to the original one when approaching problems with complicated geometries and/or randomly distributed set of nodes, both in the linear and in the non-linear case.
Additional activities

**International conferences**

- Particle-Based Methods III – Stuttgart, September 18-20, 2013
- SPH and Particular Methods for Fluids and Fluid Structure Interaction – Lille, January 21-22, 2015 - coauthor
- SPHeric workshop 2015 – Parma, June 16-18, 2015
- Particle-Based Methods IV – Barcelona, September 28-30, 2015

**Teaching activities**


**Collaboration with companies**

- Design and verification of steel doors under blast load, according to the American standard UFC-3-340
THANK YOU

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