eXtended Finite Element Method (XFEM)-
Modeling arbitrary discontinuities
and Failure analysis

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By

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The dissertation entitled "eXtended Finite Element Method(XFEM)-Modeling arbitrary discontinuities and Failure analysis”, by Awais Ahmed, has been approved in partial fulfillment of the requirements for the Master Degree in Earthquake Engineering.

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ABSTRACT

An eXtended Finite Element Method (XFEM) is implemented for modeling arbitrary discontinuities in 1D and 2D domains. XFEM is a local partition of unity based method where the key idea is to paste together special functions into the finite element approximation space to capture desired features in the solution.

In the events of severe seismic demands, earthquake induced stresses may exceed the elastic strength capacity of the material. This may cause the structural elements to respond in-elastically and may result in progressive failure of the structure and requires accurate and efficient methods to numerically model and simulate the structural behavior and damage pattern. All this necessitates a need to perform a failure analysis. Failure analysis is imperative not only to determine the ultimate capacity of the new materials and structures but also to predict the post-peak behavior correctly.

The Finite Element Method (FEM) has been used for decades to solve myriad of problems. However, there are number of instances where the usual FEM method poses restrictions in efficient application of the method, such problems involving interior boundaries, discontinuities or singularities, because of the need of remeshing and high mesh densities.

Extended finite element method and its coupling with level set method was studied and discussed in detail for geometric representation of discontinuities. The level set method allows for treatment of internal boundaries and interfaces without any explicit treatment of the interface geometry. This provides a convenient and an appealing means for tracking moving interfaces.

In this article XFEM is presented as a potential methodology for performing a failure analysis. An XFEM methodology is implemented to model flaws in the structures such as cracks, voids and inclusions, where their presence in a structure or in a structural component requires careful analysis to assess the true strength, durability and integrity of the structure/structural component. Problems involving static cracks in structures, evolving cracks, cracks emanating from voids were numerically
studied and the results were compared with the analytical and experimental results to demonstrate the robustness of the method. Exclusively, an analysis of multiple interacting cracks using an extended finite element method is presented, where complex stress distribution caused by interaction of many cracks is studied.
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Chapter 1

Introduction

1.1 Motivation

Finite element method (FEM) is one of the most common numerical tool for finding the approximate solutions of partial differential equations. It has been applied successfully in many areas of engineering sciences to study, model and predict the behavior of structures. The area ranges from aeronautical and aerospace engineering, automobile industry, mechanical engineering, civil engineering, biomechanics, geomechanics, material sciences and many more.

In order to predict not only the failure load but also the post-peak behavior correctly, robust and stable computational algorithms that are capable of dealing with the highly non-linear set of governing equations are an essential requirement. There are number of instances where the usual FEM method poses restrictions in an efficient application of the method. The FEM relies approximation properties of polynomials, hence they often require smooth solutions in order to obtain optimal accuracy. However, if the solution contains a non smooth behavior, like high gradients/singularities in stress and strain fields, strong discontinuities in the displacement field as in case of cracked bodies, then the FEM methodology becomes computationally expensive to get optimal convergence.

Engineering structures when subjected to high loading may result in stresses in the body exceeding the material strength and thus results in the progressive failure. These failures are often initiated by surface or near surface cracks. These cracks lowers the strength of the
1.1 Motivation

material. These material failure processes manifest themselves in quasi-brittle materials such as rocks and concrete as fracture process zones, shear (localization) bands in ductile metals, or discrete crack discontinuities in brittle materials. This requires accurate modeling and careful analysis of the structure to assess the true strength of the body. In addition to that, modeling holes and inclusions, modeling faults and landslides presents another form of problems where the usual FEM becomes an expensive choice to get optimal convergence of the solution.

Modeling of cracks in structures and specially evolving cracks requires the FEM mesh to conform the geometry of the crack and hence needs to be updated each time as the crack grows. This is not only computationally costly and cumbersome but also results in loss of accuracy as the data is mapped from old mesh to the new mesh.

Extended finite element (XFEM) is a numerical technique that enables the incorporation of local enrichment of approximation spaces. The incorporation of any function, typically non-polynomials, is realized through the notion of partition of unity. Due to this it is then possible to incorporate any kind of function to locally approximate the field. These functions may include any analytical solution of the problem or any a priori knowledge of the solution from the experimental test results.

The enriched basis is formed by the combination of the nodal shape functions associated with the mesh and the product of nodal shape functions with discontinuous functions. This construction allows modeling of geometries that are independent of the mesh. Additionally the enrichment is added only locally i.e where the domain is required to be enriched. The resulting algebraic system of equations consists of two types of unknowns, i.e classical degrees of freedom and enriched degrees of freedom. Furthermore, the incorporation of enrichment functions using the notion of partition of unity ensures the maintenance of a measure of the sparsity in the system of equations. All of the above features provide the method with distinct advantages over standard finite element for modeling arbitrary discontinuities.
1.2 Literature review

Modeling discontinuities/localization zones has always remained a challenge in the field of computational mechanics. Cracks when modeled with the standard finite element method (FEM) requires the FEM mesh to conform the geometry of the crack. Additionally in order to capture the true stress and strain field around the crack tip, mesh refinement is a mandatory.

A re-meshing technique is traditionally used for modeling cracks within the framework of finite element method (see for example [Swenson and Ingraffea 1988]). Where a re-meshing is done near the crack to align the element edges with the crack faces. This becomes quite burdensome in case of static or quasi-static evolving cracks or dynamic crack propagation problems, where each time a new mesh is generated as the crack grows. This results in construction of totally new shape functions and all the calculations have to be repeated. Furthermore, the dynamic solution represents an evolving history because of inertia, and whenever the mesh is changed, this history must be preserved. This is accomplished by transferring the data from the old mesh to the new mesh. The process of mapping variables from the old mesh to the new mesh may also result in loss of accuracy.

Element deletion method is one of the simplest methods for simulation of crack growth problems. In the element deletion method, the discontinuities are not modeled explicitly, rather a constitutive relationship is modified in an element cut by the crack and is called as a failed element. For more details see for example [Beissel et al. 1998; Song et al. 2008].

In the inter-element separation method, the crack is allowed to form and propagate along the element boundaries. Hence the method depends upon the mesh, which should be so constructed that it provides a rich enough set of possible failure paths. In the formulation of Xu and Needleman [1994] all the elements are separated from the beginning and a proper cohesive law model is used to join the element’s boundaries, while in the approach of Camacho and Ortiz [1996] new surfaces are created adaptively along the previously coherent element’s boundaries, as the criteria is met according to the cohesive law model. This is done by duplicating the nodes along the element’s boundaries.
1.2 Literature review

Global-local methodologies introduced in some sense an idea of enriching the approximation field. The basic idea was to obtain a global solution using the coarse grid of finite elements and then detailed results were obtained by zooming to an area of interest (localization zones etc.), refining the mesh and using the displacements from the global analysis as an input for the refined mesh. The local (detailed) analysis were also carried out by incorporating known physical behaviors/analytical solutions (e.g. polar and/or edge functions for shells with cutouts [Pattibiran and 1974]) into the computational model of the structure to get a rapid convergence. A brief review and assessment of global local methodologies can be found in [Noor 1986]. For a recent application of global local methodologies for 3D crack growth problems and its coupling with GFEM see [Kim et al. 2008].

The idea of enriching the field with an analytical solution in the context of crack growth problems was utilized by Gifford and Hilton [1978], where the displacement approximation for an element was considered to be the combination of usual FEM polynomial displacement assumption and an enriched displacement i.e. $u = u_{std} + u_{env}$. Where the enriched part comes from singular displacement fields for cracks. However as a result of this enrichment, the sparsity of the matrix was lost. Additionally the method requires that the crack tip be located on the nodes of an element and not in the element interior.

The work of Belytschko et al. [1988] is one of the pioneering work towards the local enrichment of the approximation field at an element level for the localization problems. Where the strain field is modified to get the required jumps in the strain field within the framework of three-field variational principle. Embedded finite element method (EFEM) uses an element enrichment scheme, where the field is modified/enriched within the framework of three-field variational principle. The three fields are the displacement field $u$, the strain field $\epsilon$ and the stress field $\sigma$. The enriched approximation to the field in generic form can be expressed as $u \approx N_d + N_c d_c$ and $\epsilon \approx B_d + G e$. Where $N$ and $B$ are the standard FEM displacement interpolation and strain interpolation matrices and $d$ is the FEM standard degrees of freedom. $N_c$ and $G$ are the matrices containing enrichment terms for the displacement and strain fields. $d_c$ and $e$ are the enriched degrees of freedoms and are unknown. These unknowns are found by imposing traction continuity and compatibility within the element. The prominent feature in this method is that, the enrichment is localized to an element level. However these methods requires the
1.2 Literature review

continuity of the crack path. Extended finite element method (XFEM) on the contrary is also a local enrichment scheme but uses a notion of partition of unity to incorporate an enrichment to the approximating field. In XFEM, in contrast to element enrichment scheme a nodal enrichment scheme is practiced. A prominent feature of using the notion of partition of unity in XFEM in particular or in any partition of unity method in general is that, it automatically enforces the conformity of the global approximation space. For a reference on EFEM see for example [Oliver et al. 1999; Jirasek 2000].

Extended finite element method (XFEM) developed by Belytschko and Black [1999], is able to incorporate the local enrichment into the approximation space within the framework of finite elements. The resulting enriched space is then capable of capturing the non-smooth solutions with optimal convergence rate. This becomes possible due to the notion of partition of unity as identified by Melenk and Babuska [1996] and Duarte and Oden [1996].

Modeling complicated domains was a bit difficult and cumbersome with standard finite element method as the finite element mesh was required to be aligned with the domain boundaries, such as modeling re-entrant corners. In this view efforts were made to develop methods which are mesh independent. Element Free Galerkin method (EFG) is one of the results of such efforts. For a few applications on the EFG, see [Belytchko et al. 1996; Phu et al. 2008; krysl and Belytschko 1999]. The approach was intuitive, in a sense that the method relies on defining arbitrary nodes/particles in an irregular domain and then constructing a cloud over each node/particle such that it forms a covering to the whole domain. The field is then approximated using shape functions which may be weighting functions or moving least squares functions or else, see for instance [Belytchko et al. 1996; Phu et al. 2008; Dolbow and Beytchko 1998]. Detail theory and application on meshless methods can be found in [Liu 2003].

The notion of partition of unity (PoU) was first identified and exploited by Duarte and Oden [1996] and Melenk and Babuska [1996]. The idea was to define a set of functions over a certain domain $\Omega^{PoU}$, such that they form partition of unity subordinate to the cover PoU, or in other words they sums up to 1. This property was a crucial as it corresponds to the ability of the partition of unity shape functions to reproduce a constant, and this is essential for convergence. The hp-cloud method by Duarte and Oden [1996] used the extrinsic basis function to
1.2 Literature review

increase the order of approximation analogous to p-refinement using the concept of partition of unity. Melenk and Babuska [1996] realized the same and applied it in the framework of finite element method (FEM), a method called partition of unity finite element method (PUFEM). The method was similar to hp-cloud method, in spite the fact that PUFEM uses a lagrangian basis function and where the FEM elements sharing the same nodes forms the support or cloud for nodal shape functions. The main idea in both the methods was to incorporate a non-smooth enrichment function, typically non-polynomial into the approximation space using partition of unity. This generates an enriched basis function which could be non-smooth, non-polynomial depending upon the type of enrichment used. Hence it was possible to locally approximate the field with a non-smooth approximation function. Such as used in crack propagation problems.

Using the idea of PoU to paste together non-polynomial functions into the approximation space, successful efforts were made to incorporate discontinuities in the approximation spaces or incorporating discontinuities in the derivatives of the approximations in the framework of meshless methods, for example enriched element free galerkin method (EEFG). For a few applications in the above spirit see [Flemming et al. 1997; Krongauz and Beytchko 1998; Belytchko and Flemming 1999].

Later on Strouboulis et al. [2000] used the same concept of partition of unity and showed that different partition of unity functions can be embedded into the finite element approximation to locally enrich the field. The method was called as Generalized Finite Element Method (GFEM). The generalized finite element method relies on incorporating analytical solution to locally approximate the field using the partition of unity. For more details on GFEM see [Oden et al. 1998; Strouboulis et al. 2000; Strouboulis et al. 2000; Duarte et al. 2000; Kim et al. 2008].

Belytschko and Black [1999] developed another finite element based method (later on developed into extended finite element method, XFEM) to locally enrich the field using the partition of unity. One of the differences with GFEM was that, any kind of generic function can be incorporated in XFEM to construct the enriched basis function, however the current form of GFEM has no such differences with XFEM, in spite the fact that XFEM is coined with Northwestern university and GFEM name was adopted by the Texas school. In its first attempt
1.2 Literature review

towards the extended finite element method, a local enrichment of the domain for crack propagation problem was proposed by Belytschko and Black [1999] using the partition of unity. The enriched basis function was constructed by simple multiplication of the enrichment function with the standard finite element basis functions. The analytical solution for the displacement and stress field near the crack tip were known from the theory of linear elastic fracture mechanics (LEFM). So they used near tip enrichment functions to enrich the field near the crack throughout the crack length. By this method no remeshing was required as the crack grows, however for severely curved cracks a remeshing was required near the crack root. In addition to that for curved or kink cracks, it was required to align the discontinuity in the enriching functions with the crack by a sequence of mapping that rotates each segment of the crack onto the crack model. However a noticeable thing was that, the method was able to model the crack arbitrarily aligned with finite element mesh with minimal amount of remeshing.

Next a modification in the method was proposed by Moes et al. [1999]. The modified version what is now called as extended finite element method (XFEM) removed the need for minimal mesh refinement. They showed, that any type of generic function that best describes the field can be incorporated into the approximation space. This emphasizes less dependence on the analytical/closed form solution as opposed to the earlier version of GFEM, where analytical solution or accurate numerical solutions were incorporated as an enrichment functions. This capability of XFEM makes it more flexible to a variety of problems. In the methodology for crack propagation problems, two types of enrichment functions were proposed. Due to the fact that partition of unity property allows one to incorporate any kind of non-smooth, non-polynomial enrichment function into the approximation space, a Haar/Discontinuous function is used to enrich the field throughout the length of the crack, thus giving the required discontinuity along the crack length. The exact solutions for the stress and displacement fields near the crack tip were already known in the world of LEFM. So Near tip enrichment functions derived from analytical solutions were used to enrich the field near the crack tip. This helps in approximating the high strain/stress gradient fields near the crack tip with optimal convergence. The enrichment is applied at the nodes. Thus increasing the number of degrees of freedom equal to the number of enrichment functions assigned to that nodal, in addition to standard degrees of freedom.

The main idea of XFEM (and any partition of unity based method) lies in applying
the appropriate enrichment function locally in the domain of interest using the partition of unity. The whole beauty of XFEM lies in subdividing the problem into two parts A) generating mesh without cracks/inclusions etc. B) enriching the FEM approximation with additional/enrichment functions that models the discontinuities. This alleviates the need for remeshing or explicit geometric modeling of the discontinuity. Using the same methodology, the XFEM is successfully applied to model number of arbitrary moving and intersecting discontinuities [Duax et al. 2000]. For a few applications in the above spirit see also [Dolbow et al. 2000a; Dolbow et al. 2000b; Dolbow 1999; Sukumar and Prevost 2003; Huag et al. 2003; Bechet et al. 2005; Moes et al. 2006; Rozycki et al. 2008].

In reference [Sukumar et al. 2000] XFEM was applied for modeling 3D crack propagation problems, however issues regarding the accurate crack modeling, determination of correct crack surfaces and crack path in 3D is still under debate. For more details, see for example [Areias and Belytschko 2005; Jager et al. 2008; Rabczuk et al. 2008].

XFEM experienced another improvement in its implementation, when the XFEM was coupled with Level set method [Stolarska et al. 2001]. Level set method is a numerical technique to track the discontinuities, and was devised by Osher and Sethian [1988]. For details on level set methods see also [Osher and Fedkiw 2001]. The basic idea of level set method is to define a level set function such that the discontinuity is represented as a zero level set function. Level set function on one hand not only helps in tracking discontinuities arbitrarily aligned with the finite element mesh but on the other hand also helps in defining the position of a point in crack tip polar coordinate system and evaluation of commonly used enrichment functions such as step function and a distance function for modeling strong and weak discontinuities respectively. Duflot [2007] has presented an overview of the representation and an update techniques of the level set functions for 2D and 3D crack propagation problems.

For evolving cracks a fast marching method by Sethian [1996] was used, where only level set functions within the narrow band around an existing discontinuity is updated. The narrow band is marched forward, freezing the values of existing points and bringing new ones in the narrow band to update. The method was then extended to three dimensions in [Gravouil et al. 2002a; Gravouil et al. 2002b]. However for modeling open discontinuities using standard
1.2 Literature review

form of level set function rendered complexities in the algorithm by the need to freeze the level set describing the existing crack/discontinuity. Ventura et al. [2003] proposed vector level sets for modeling crack growth problems in 2D. Sukumar et al. [2008] couples the fast marching method (FMM) [Sethian 1996] to a three dimensional implementation of the extended finite element method. Furthermore, they used distinct meshes for the mechanical model (extended finite element analysis) and the FMM. As an application of the XFEM coupled with level set method see also [Bordas 2003].

Due to the possibility of defining the discontinuities arbitrarily aligned, independent of the mesh, XFEM is also able to be applied successfully for modeling holes and inclusions, which on the other hand using the standard finite element method requires the mesh to conform(align) the geometry or the material interfaces [Sukumar et al. 2001]. Material interfaces in composites can also be modeled to predict the mechanical behaviors using XFEM. Similar kind of approach is also applied in the framework of GFEM, Where [Strouboulis et al. 2000] used local enrichment functions in the GFEM for modeling re-entrant corners and in [Strouboulis et al. 2000] enrichment functions for holes were proposed. For Some other applications of XFEM in modeling holes and cracks emanating from holes, see [Yan 2006; Belytschko et al. 2001; Belytcschko and Gracie 2007].

XFEM was initially developed for crack growth problems in brittle materials. The theory of linear elastic fracture mechanics (LEFM) is valid only when the fracture process zone behind the crack tip is small compare to the size of the crack and size of the specimen. In other cases fracture process zone needs to be taken into account for analysis. In cohesive crack growth the crack propagation is governed by the traction-separation law at the crack faces. This kind of models were first presented in sixties for metals, like one by Dugdale [1960]. The cohesive crack growth simulations were first incorporated into XFEM by Wells and Sullays [2001]. This was accomplished by modifying the variational form where a traction separation law was incorporated to make the energy balance. Later on, Moes and Belytschko [2002] improved their earlier method [Dolbow et al. 2001] and provided a more comprehensive model for cohesive crack growth within the framework of XFEM, that addressed the issue of extent of cohesive zone. They also proposed a partly cracked element which is enriched with the set of non-singular branch functions to model the displacement field around the tip of the crack.
In Zi and Belytschko [2003] they proposed a new crack tip element where the entire crack is enriched with one type of enrichment function including the elements containing the crack tip so that the partition of unity holds in the entire enriched sub domain by using shifted enrichment. In their approach they used a sign function to enrich the nodes whose support is cut by the crack. In Asferg et al. [2007], they showed that the new crack tip element proposed in Zi and Belytschko [2003] cannot model equal stresses on both sides of the crack and proposed a new partly cracked XFEM element for cohesive crack growth with extra enrichment to cracked elements. The extra enrichment is constructed as a superposition of the standard nodal shape functions and standard nodal shape functions created for a sub-area of the cracked element. For some of the applications of XFEM in modeling cohesive cracks see also [Khoei and Nikbakht 2006; Unger et al. 2007].

In Meschke and Dumstorff [2007, Dumstorff and Meschke [2007] proposed a global energy based method within the frame work of XFEM for modeling cohesive as well as cohesion less cracks in brittle and quasi brittle materials. The prominent feature of the work was that, the crack propagation angle and length of the new crack segment was introduced into the variational principle as an additional unknowns and have to to solved for. The basic idea is to use the minimization of the total potential of the body to get the crack direction and length. As a result of this the crack propagation direction and length of the new crack segment are the direct outcomes of the analysis.

1.3 Outline

The document is organized as follows. Chapter 2 gives a brief introduction on the fracture mechanics, basic theories of fracture and some recent developments as regard to the numerical analysis of cracked bodies. Chapter 3 gives a comparison among the finite element, partition of unity and extended finite element method to have a better understanding of the basic philosophy involved in any partition of unity methods in general and XFEM in particular, using simple 1D example. Chapter 4 discusses in detail the level set methodology and its coupling with the extended finite element method. A common form of level set function usually employed with XFEM is studied and the advantages and disadvantages of using that form of level
1.3 Outline

set function is discussed. Chapter 5 gives a comprehensive insight on extended finite element
method for modeling arbitrary discontinuities. For the sake of completeness and comprehen-
siveness of the document and to give a reader an overall and understanding of the partition of
unity and specially the extended finite element method, some basic theories have been revised
using self explanatory figures and arguments to grasp the idea well. Chapter 6 discusses the im-
plementation issues regarding extended finite element method. In Chapter 7 numerical results
are presented to show th efficiency and accuracy of the extended finite element methodology
and chapter 8 briefly reviews and summarizes the numerical results and possible lines of future
work.
Chapter 2

Fracture Mechanics

2.1 Introduction

Strength of the materials were evaluated in the past based on two possible hypotheses [Griffith 1921]. A material is said to fracture if maximum tensile stress or maximum extension in a body exceeds a certain threshold value. Hence the strength of the material was basically considered to be dependent on the material properties. Effect of fracture on the strength was not taken into account or not understood properly. This sometimes resulted in a very high theoretical strength values, but practically the strength of the material was lower than the actual. One of earliest recorded incidents of brittle fracture failure was the Montrose bridges 1830 [Erdogan 2000]. There have been many incidents due to fracture failure after that e.g the event of Tay Rail Bridge failure in 1879. All this led people to think about the fracture strength of the material. During the years of 1930 to 1950, fracture failure of commercial jet airplanes and welded ships further aggreviated the mechanicians. Up to that time Griffith’s and Irwin’s work has led the foundations for a new engineering branch “Engineering Fracture Mechanics” to flourish, and soon after that Fracture mechanics evolved as an important engineering branch and a lot of research work was started, which made fracture mechanics to what we see today. A very good review on fracture mechanics can be found in Erdogan [2000]. More details on engineering fracture mechanics can also be found in [Wang 1996]
2.2 Griffith’s Work

The early strength theories were based on maximum tensile stress and in this connection uni-
axial tensile strength were used to find the material fracture strength. The fracture strength of
the material is considered to be size independent. It was after Griffith’s [Griffith 1921] work
that the concept of size dependence on material strength was explicitly understood. The key
points that motivated Griffith were

- The measured fracture stress of a bulk glass is around 100Mpa
- The theoretical fracture strength to break the atomic bond is much higher, 10GPA (approx,
ten times higher).

Griffith himself performed experiments on glass fibers and observed that the frac-
ture strength increases with a decrease in thickness of the fiber and vice versa. The observa-
tions were in agreement with the known fact, that strength of material is one-tenth the strength
deduced from physical data. He attributed this behavior due to the presence of microscopic
cracks/flaws in the bulk material.

To support his argument Griffith performed an experiment on a thin glass plate and
introduced in it a large crack. He found that the breaking load of a thin plate of glass having
in it sufficiently long crack normal to the applied stress, is inversely proportional to the square
root of the flaw length.

\[ \sigma \propto \sqrt{\frac{1}{a}} \]  
(2.1)

or we can also state

\[ \sigma \sqrt{a} = C \]  
(2.2)

where \( a \) is the flaw length.

The answer to such a behavior is not available in linear elasticity as it predicts the
stress to be infinite in linear elastic material at the crack tip. Griffith used energetic approach
to the problem. Creation of two new surfaces (crack) increases the surface energy of the body.
Now the question whether a body will remain stable after crack growth, depends on the fact
2.2 Griffith’s Work

whether the body has sufficient energy to afford formation of new surfaces. In order to find constant C of equation 2.2, Griffith make use of energy balance of a body. He took a reference state of a glass fiber with no crack or flaw and loaded it with a uniform tension. He then calculated the potential energy stored in the body. Then he fixed the remote boundary so that the applied load does not do extra work and then he introduced a flaw of length a into the specimen. The formation of the crack and the two new surfaces relaxes the stresses and hence the stored elastic strain energy, $U_e$, reduces near the crack faces. At the same time, creation of two new surfaces increases the surface energy, $\Gamma$, of the body. The change of total free energy from reference state due to crack is thus “Surface energy minus elastic strain energy “, that is $\psi = \Gamma - U_e$. where $\psi$ represents the total or free energy.

let us consider an infinite uniformly loaded plate with an elliptical crack of length 2a as shown in figure 2.1(a). we can now define the total energy of the system as consisting of three parts (1) the amount of work done by the applied loads, W (2) the elastic energy, $U_E$ and (3) the energy required to form the crack surface, $\Gamma$. The total energy is

$$U_{tot} = -W + U_E + \Gamma$$  \hspace{1cm} (2.3)

According to linear elastic theory, a body under constant applied loads obeys $W = 2U_E$. The total energy of the system is then

$$U_{tot} = -U_E + \Gamma$$  \hspace{1cm} (2.4)
Griffith used the stress solution by Inglis (1913) to show that the increase in strain energy is given as

\[ U_E = \frac{\pi a^2 \sigma^2 B}{E} \]  

(2.5)

where B is the thickness of the plate. The surface energy is given as

\[ \Gamma = 4aB\gamma \]  

(2.6)

where \( \gamma \) is the surface energy per unit area and a material constant. Thus the total energy of the system can be given as

\[ U_{tot} = -\frac{\pi a^2 \sigma^2 B}{E} + 4aB\gamma \]  

(2.7)

Figure 2.1(b) below shows the plot of the above equation. Maximization of the above equation yields

\[ a_c = \frac{2\gamma E}{\pi \sigma^2} \]  

(2.8)

where \( a_c \) is the critical crack length. Now defining the crack area \( A = 2aB \), we can see from the figure that the point \( \partial \psi / \partial A = 0 \) defines the equilibrium point and the crack length associated with it is known as the critical crack length. For crack lengths below the critical length, the crack would remain stable.

Observing equation 2.8, it is clear that the critical crack length below which the crack would remain stable decreases quickly with stress level. Alternatively, the critical stress level that a cracked body can sustain is given as

\[ \sigma_c = \sqrt{\frac{2\gamma E}{\pi \sigma}} \]  

(2.9)

Observing equations 2.2 and 2.9, the Constant C of Griffith’s equation is then simply

\[ C = \sqrt{\frac{2\gamma E}{\pi}} \]  

(2.10)

It is now clear that

- the critical stress level for a given crack length varies with material,

- the critical stress level decreases with crack length, i.e the larger the crack, the easier it may become unstable

hence the material strength is not only dependent on material properties but also depends upon the flaws present in the body.
2.3 Irwin’s Work

2.2.1 Energy Release Rate

According to law of conservation of energy the work done per unit time by the applied loads($\dot{W}$) must be equal to the rates of change of the internal elastic energy($\dot{U}_E$), plastic energy($\dot{\Gamma}_p$), kinetic energy($\dot{K}$) of the body and the energy per unit time($\dot{\Gamma}$) spent in increasing the crack area. Assuming the propagation is slow and plastic deformations are negligible, the conservation of energy can then be written in mathematical form as

\[
\frac{\partial W}{\partial t} = \frac{\partial U_E}{\partial t} + \frac{\partial \Gamma}{\partial t}
\]

(2.11)

\[
\dot{W} = \dot{U}_E + \dot{\Gamma}
\]

(2.12)

Lets define $\Pi = U_E - W$ be the potential energy of the system, then above equation becomes

\[
-\dot{\Pi} = \dot{\Gamma}
\]

(2.13)

As all the changes with respect to time are caused by change in flaw size we have.

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial A} \frac{\partial A}{\partial t}
\]

(2.14)

\[
\frac{\partial}{\partial A} = \frac{\partial}{\partial t}
\]

(2.15)

where $A$= is crack Area. The equation 2.13 can now be written as

\[
-\frac{\partial \Pi}{\partial A} = \frac{\partial \Gamma}{\partial A} = G
\]

(2.16)

where $G$ is known as energy release rate. It characterizes the amount of energy available for crack propagation. The crack propagation is said to occur when the energy release rate, $G$ reaches a critical value,$G_{cr}$. This is the basic failure criteria in an energy release rate criteria for mixed mode fracture of materials [Nuismer 1975]

2.3 Irwin’s Work

Till 1950, the Griffith’s work [Griffith 1921] was largely ignored due to the fact that the Griffith’s theory does not give good solutions for all materials and especially for metals, where the realistic energy required for the fracture was orders of magnitude than the surface energy.

The studies conducted by Orawan and Irwin during 1948 [Erdogan 2000] showed that even the fracture in brittle materials, there is extensive plastic deformation at the crack
2.3 Irwin’s Work

surface and hence a source of energy dissipation. The effect of plastic zone in brittle materials
will be small as compare to the strain energy dissipated by the formation of the crack, but in case
of ductile materials, it plays a vital role. As the load on the body is increased, the plastic zone
develops behind the crack tip, the size of the plastic zone increases with the increase in load
and at critical load the material starts unloading. Cycles of loading and unloading releases the
energy in the form of heat. All these thoughts led to an important modification in the Griffith’s
work where a plastic work term is added into the energy balance equation to take into account
the plastic work at the crack front.

The energy lost/released can now be considered as consisting of two parts

1. The elastic energy which is released as the crack grows, i.e. surface energy, $\gamma$

2. Plastic energy dissipation, $\gamma_p$

Hence we can write now

$$\Gamma = \gamma + \gamma_p$$

(2.17)

Similarly the Constant C of Griffith’s model can now be expressed as:

$$C = \sqrt{\frac{E\Gamma}{\pi}}$$

(2.18)

$$\Rightarrow C = \sqrt{\frac{E(2\gamma + \gamma_p)}{\pi}}$$

(2.19)

2.3.1 Modes of failure

Before going further, it is worthy to introduce here three basic failure modes of the material,
namely Mode I, Mode II and Mode III. Mode I is an opening mode. It corresponds to an
opening of the crack faces normal to each other under the action of tensile load. Mode II is
in-plane shear/sliding failure mode. The shear stresses acts parallel to the plane of the crack
and perpendicular to the crack front. Mode III failure mode is classified as out of plane tearing
mode. The shear stresses are applied parallel to the plane of the crack and crack front. The three
modes of failures are shown schematically in the figure (2.2).

2.3.2 Stress Intensity Factor

Another important contribution of Irwin and his colleges in the field of fracture mechanics is,
they developed a method for evaluating the amount of energy available for the crack propagation
2.3 Irwin’s Work

(a) Mode I: Opening  (b) Mode II: in-plane shear  (c) Mode III: Out of plane shear

Figure 2.2: Modes of failures

in terms of asymptotic stress and displacement field. The method requires the loading and geometry conditions to evaluate the energy release rate. The stress field for linear elastic solid in terms of asymptotic stress in the neighborhood of crack tip in its generic form is given as

\[
\sigma_{ij} \approx \frac{K_m}{\sqrt{2\pi r}} f_{ij}(\theta)
\]  \hspace{1cm} (2.20)

where

- \(\sigma_{ij}\) is the Cauchy stress tensor.
- \(r\) is radial distance of point of query from the crack tip.
- \(\theta\) is the angle w.r.t plane of the crack.
- \(f_{ij}(\theta)\) are functions independent of loading and crack geometry.
- The coefficient of the singular term \(K\) is called as stress Intensity factor.

The generalized expression for the asymptotic displacement field is

\[
u_i \approx \frac{K_m}{2\mu} \sqrt{\frac{r}{2\pi}} g(\theta)
\]  \hspace{1cm} (2.21)

The asymptotic stress field for the three modes of failure is given as

\[
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left[2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right]
\]  \hspace{1cm} (2.22)

\[
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}
\]  \hspace{1cm} (2.23)

\[
\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right]
\]  \hspace{1cm} (2.24)

The Displacements field is given as

\[
u_x = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[\kappa - 1 + 2\sin^2 \frac{\theta}{2}\right] + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[\kappa - 1 + 2\cos^2 \frac{\theta}{2}\right]
\]  \hspace{1cm} (2.25)

\[
u_y = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[\kappa - 1 + 2\cos^2 \frac{\theta}{2}\right] - \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[\kappa - 1 - 2\sin^2 \frac{\theta}{2}\right]
\]  \hspace{1cm} (2.26)
where $\kappa = \text{kolsov constant}$

$$\kappa = \begin{cases} 
3 - 4\nu & \text{for plane strain} \\
\frac{3-\nu}{1+\nu} & \text{for plane stress}
\end{cases}$$

### 2.4 Elasto Plastic Fracture Mechanics

The theories and laws of the linear elastic fracture mechanics (LEFM) can only be applicable to materials which behaves in a linear elastic manner. But all the materials do not follow the same rule and specially the ductile materials, like steel. In ductile materials due to increase in load, a plastic zone develops behind the crack tip which might be of the same order of magnitude as the crack size. Thus, in that case as the load increases the crack size increases, at the same time the plastic zone increases, which increase the plastic energy dissipation. hence the fracture resistance of the material also increases with increasing crack size as is obvious from the energy balance equation $\Gamma = \gamma + \gamma_p$. Therefore it was necessary to take into account plasticity effects in evaluating the fracture strength of the material.

### 2.4.1 J-Integral

Later in the 1960s, Rice [1968] developed a way to compute the energy release rate, the so-called J-integral. The J-integral also known as conservation integral represents a way to compute the strain energy release rate for the material where the crack tip deformation is such that it does not obey linear elastic laws. The approach is to identify a line integral which has the same value for all integration paths surrounding the crack tip. Rice showed that J-integral is path independent, hence evaluating the J-integral in a far field around a crack tip can be related to the near-tip deformations. In this way crack tip complications can be avoided by evaluating the energy release rate in the domain where the results are reliable. J-integral was developed for non-linear elastic solids but is also valid for elasto-plastic materials as nonlinear elasticity is equivalent to the deformation theory of plasticity (provided there is no unloading). The J-integral thus provided an alternative approach to calculate the G or K (stress intensity factors).

The Rice’s integral in its original form can be written as:

$$J = \int_{\Gamma} \left( Wdy - T \frac{\partial u}{\partial x} ds \right)$$

(2.27)
where $\Gamma$ is a curve surrounding the notch/crack tip. The integral being evaluated in a counterclockwise sense starting from the lower flat notch surface and continuing along the path $\Gamma$ to the upper flat surface. $T$ is the traction vector defined according to outward normal along $\Gamma$, $T_i = \sigma_{ij}n_j$. $u$ is the displacement vector, and $ds$ is an element of an arc length along $\Gamma$. $W$ is the strain energy density given by

$$W(\epsilon) = \int_0^\epsilon \sigma_{ij} d\epsilon_{ij}$$

(2.28)

See also [Banks-sills and Sherman 1992] in the above spirit.

### 2.4.2 Interaction Integral

As has been explained earlier that J-integral is way of calculating the strain energy release rate and its path independent property helps to relate the integral evaluated in the far field with the crack tip field. The J-integral is related to the stress intensity factors ($K_I, K_{II}$) as:

$$J = \frac{K_I^2}{E^*}, \quad J = \frac{K_{II}^2}{E^*}$$

(2.29)

and for mixed mode failure we have

$$J = \frac{K_I^2}{E^*} + \frac{K_{II}^2}{E^*}$$

(2.30)

where

$$E^* = \begin{cases} E & \text{Plane stress} \\ \frac{E}{1-\nu^2} & \text{Plane strain} \end{cases}$$
2.4 Elasto Plastic Fracture Mechanics

For multi mode fracture it is thus clear that stress intensity factors for the two modes cannot be obtained independent of each other. The goal is then achieved by defining two equilibrium states of the body, state 1 and state 2. state 1 being the actual state of the body and state 2 being an auxiliary state. Field variables associated with the two states are denoted with superscripts 1 and 2. Superposition of the two equilibrium states leads to another equilibrium state denoted by $J^{(1+2)}$.

\[
J^{(1+2)} = \int_{\Gamma} \left[ \frac{1}{2}(\sigma_{ij}^1 + \sigma_{ij}^2)(\epsilon_{ij}^1 + \epsilon_{ij}^2)\delta_{ij} - (\sigma_{ij}^1 + \sigma_{ij}^2)\frac{\partial(u_i^1 + u_i^2)}{\partial x_j} \right] n_j d\Gamma
\]  

(2.31)

simplifying the above equation we can write as:

\[
J^{(1+2)} = J^{(1)} + J^{(2)} + M^{(1,2)}
\]

where $M^{(1,2)}$ is called the interaction integral, expressed as

\[
M^{(1,2)} = \int_{\Gamma} \left[ W^{(1,2)} \delta_{ij} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_j} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_j} \right]
\]

(2.32)

where $W^{(1,2)}$ is the interaction/mutual strain energy of the body given by

\[
W^{(1,2)} = \sigma_{ij}^{(1)} \epsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \epsilon_{ij}^{(1)}
\]

(2.33)

recalling the relationship between J and K we can write the expression for mixed mode failure as:

\[
J^{(1+2)} = J^{(1)} + J^{(2)} + \frac{2}{E^*} \left( K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right)
\]

(2.34)

\[
\Rightarrow M^{(1,2)} = \frac{2}{E^*} \left( K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right)
\]

(2.35)

The M-integral shown above deals with interaction terms only and will be used for evaluating the stress intensity factors (SIFs) independently. Important thing to note here is that, M-integral is related to the crack-tip fields (i.e $K_I$ and $K_{II}$) but yet may be evaluated in the region away from the crack tip, where such calculations (stress and deformations) can be performed with greater accuracy and convenience as compare to the crack tip region.

In order to solve for mixed mode fracture problem we make a judicious choice of auxiliary state. Considering state 2 as pure mode I we have

\[
K_I^{(2)} = 1 \quad \text{and} \quad K_{II}^{(2)} = 0
\]
2.4 Elasto Plastic Fracture Mechanics

The equation 2.35 simplifies to

\[ K_I^{(1)} = \frac{E^*}{2} M^{(1,2i)} \]  

(2.36)

where \( 2i \) represents first auxiliary state. The M-integral is then evaluated by determining the state 1 parameters from the usual finite element analysis along the predefined integration path \( \Gamma \) around the crack tip in the far field. The state 2 parameters are evaluated using the asymptotic stress and displacement fields expressions of LEFM by inserting the appropriate values of \( K_I^{(2)} = 1 \) and \( K_{II}^{(2)} = 0 \).

In the next step considering state 2 as pure mode II, we have

\[ K_I^{(2)} = 0 \quad \text{and} \quad K_{II}^{(2)} = 1 \]

then the stress intensity factor for the state 1 can be given as

\[ K_{II}^{(1)} = \frac{E^*}{2} M^{(1,2ii)} \]  

(2.37)

where \( 2ii \) represents second auxiliary state. The M-integral is then evaluated by determining the state 1 parameters from the usual finite element analysis, and the state 2 parameters are evaluated using the asymptotic stress and displacement fields expressions of LEFM by inserting the appropriate values of \( K_I^{(2)} = 0 \) and \( K_{II}^{(2)} = 1 \).

### 2.4.3 Domain Form of Interaction Integral

The contour integral mentioned above is not in a form best suited to finite element calculations. For numerical purposes it is more advantageous to recast the conservation integral which is actually a line/contour integral into an area/domain integral. This is done by introducing a weighting function \( q \) such that, it has a value equal to unity on the contour \( \Gamma \) and zero at the outer contour \( \Gamma_o \) (refer to figure 2.4). Within the area enclosed by a closed path \( \Gamma, \Gamma_o, C^+ \) and \( C^- \), the weighting function \( q \) is an arbitrary smooth function varying in between zero and unity.

The interaction integral for a closed path \( C = \Gamma \cup \Gamma_o \cup C^+ \cup C^- \) can be written as

\[ M^{(1,2)} = \int_C \left[ W^{(1,2)} \delta_{ij} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_j} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_j} \right] q m_j d\Gamma \]  

(2.38)

where \( m_j \) are components of unit normal vector to the closed curve \( C \) acting outward to the area \( A \). It should be noted here that \( m_j = -n_j \) on the contour \( \Gamma \) and \( m_j = n_j \) on \( \Gamma_o, C^+, C^- \). The
2.4 Elasto Plastic Fracture Mechanics

**Figure 2.4:** Conventions for domain J: domain A is enclosed by $\Gamma, C^+, C^-$ and $\Gamma_o$; unit normal $m_j = n_j$ on $\Gamma_o$ and $m = -n_j$ on $\Gamma$

Crack faces are considered to be traction free. Now using the divergence theorem and passing the limit to the crack tip we get

$$M^{(1,2)} = \int_A \left[ -W^{(1,2)} \delta_{ij} + \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_j} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_j} \right] \frac{\partial q}{\partial x_j} dA \quad (2.39)$$

**Figure 2.5:** Weight function $q$ on elements

For numerical evaluation of the integral the domain A is set from the collection of elements about the crack tip. This is done by selecting all elements which have nodes within a
ball of radius \( r_d \) centered at the crack tip. As the J-integral is path independent, hence integral can be evaluated in the far field, so radius \( r_d \) for the domain \( A \) could be selected large enough to avoid complications of the crack tip. Usually radius \( r_d \) is selected to be 2 to 3 time the square root of the area of an element.

It is interesting to note that, within the domain the value of \( \frac{\partial q}{\partial x_j} \) is equal to zero and hence automatically the integral is evaluated only at the boundary elements where \( \frac{\partial q}{\partial x_j} \neq 0 \). Thus evaluating a domain form of interaction integral is an alternative way of evaluating a contour integral best suited to finite element framework. More details on computation of domain form of interaction integral can be found in [Shih and Asaro 1988].
Chapter 3

Extended Finite Element Method - Realization in 1D

3.1 Introduction

Extended finite element (XFEM) method offers an elegant way to model discontinuities and singularities independently of the mesh. This is made possible due to the notion of partition of unity. Before exploring XFEM, we shall first put few comments on standard finite element method (FEM) and partition of unity methods.

3.2 Finite Element Method, FEM

In order to set the basic ideas of the finite element method, we shall make use of a 1-D model example for illustration of FEM. Consider a 1-D body with domain $\Omega$ (figure 3.1(a)). The finite element approximation begins with discretizing the domain $\Omega$ into sub-domains $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ (figure 3.1(b)).

Then we put nodes at the vertices of each element, where the coordinates of the nodes are $x_i = \{x_1, x_2, x_3, x_4, x_5\}$. We then associate with each node an interpolation function (shape function) $\phi_i$, which can better approximate the field. These shape functions have a compact support $\omega_i = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, which consist of union of elements connected to the node i (figure 3.1(c)).
Let $u^h|_{\omega_i}$ be the local approximation of the field, such that $u^h|_{\omega_i} \in U^h$, and is able to approximate the local field $u|_{\omega_i} \in U$ well. Then the FEM approximation reads as:

$$u^h(X) = \sum_{i=1}^{n} \phi_i(X) u_i$$  (3.1)

In order to reduce the error $\|u - u^h\|_{\Omega}$ i.e the difference between the field variable and the approximation, we try to bound the error locally such that $\|u - u^h\|_{\omega_i}$ is minimum. This is achieved by selecting the interpolation functions $\phi_i$ on its support $\omega_i$ such that it satisfies the condition, equation (3.2)

$$\sum_{i=1}^{n} \phi_i(X) = 1$$  (3.2)

Below are few of the characteristics of the standard finite element approximation function of the form, equation (3.2)
3.3 Partition of Unity Finite Element Method, PUFEM

- Let us say that the elements are satisfying the condition that rigid body motion does not cause any strains in the elements, then a constant value of $u_i$ specified at all nodes must result in a constant value of displacement $u^h = u_o$ at all points.

$$u^h(X) = \left( \sum_{i=1}^{i=4} \phi_i(X) \right) u_o = u_o \hspace{1cm} (3.3)$$

Hence equation 3.2 is also known as reproducing condition due to the fact that shape functions possess a property of Partition of Unity.

- The approximation function interpolates in the values $u_i$ such that for any point $x_a$

$$u^h(x_a) = \sum_{i=1}^{i=4} \phi_i(x_a) u_i \hspace{1cm} (3.4)$$

$$u^h(x_a) = u_a \hspace{1cm} (3.5)$$

- Equation 3.2 also shows that the global approximating space inherits the smoothness of basis function(shape function) as well as local approximation properties of the space $u^h|_{\Omega_i}$.

3.3 Partition of Unity Finite Element Method, PUFEM

The key idea in the partition of unity methods, is the use of partition of unity functions which are functions whose values sums up to unity at each point in the domain $\Omega_PoU$. Let us consider a body $B$ which belongs to a 1D space $\mathbb{R}$ with domain $\Omega$. Let $\tau_N$ be the open cover of the domain composed of $N$ supports (clouds) $\omega_i$ centered at $x_i$. This is shown in figure 3.2.

$$\tau_N = \{ \omega_i \}_{i=1}^{N} \hspace{1cm} (3.6)$$

$$\bar{\omega} \subset \bigcup_{i=1}^{N} \omega_i \hspace{1cm} (3.7)$$

It becomes obvious from the above equations that $\Omega \subseteq \bar{\omega}$. Further, the important thing to note here is that no mesh has been generated rather nodes have been placed arbitrarily in the domain $\Omega$ to approximate the field at these points.

Let $g_i$ be the local approximation of $u$ and belongs to local approximating space $v_i(\omega_i)$ defined on support $\omega_i$, which is so selected that $g_i \in v_i(\omega_i)$ and can approximate the field $u|_{\omega_i}$ well, see figure 3.2. More detail on the nature of local approximation space $v_i(\omega_i)$ is given in chapter 5.
The local approximation has to combine somehow to give the global approximation space $U^h$ of U. This global space has to be built in such a way that the difference between $U^h$ and U is minimum. This is usually done by bounding the local error $\| u - u^h \|_{\omega_i}$ using functions on the supports $\omega_i$ such that

$$\sum_{i=1}^{N} \phi_i = 1 \quad \forall \ x \in \Omega$$

(3.8)

The function $\phi_i$ has a non-zero value only in $\omega_i$. The functions $\phi_i$ are called as partition of unity subordinate to the open covering $\tau_N$.

In the passage below we shall discuss the methodology of partition of unity within
the framework of finite element method, using the same 1D model example, as we have used for the FEM analysis in the previous section.

Consider a body with domain $\Omega$ as shown in figure3.1(a). The analysis begins with discretizing the domain into sub domains $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ with nodes having coordinates $x_i = \{x_1, x_2, x_3, x_4, x_5\}$. We then associate with each node an interpolation function (shape function) $\phi_i$. These shape functions have a compact support $\omega_i = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, consists of union of elements connected to the node $i$ (figure3.1(c)). It is noteworthy that till now the implementation in PUFEM is essentially the same as in standard FEM. Let $g_i$ be the local approximations which best describe the field, as defined earlier. Consider now the element $\Omega_3$ with nodes $x_3$ and $x_4$, associated with them the shape function $\phi_3$ and $\phi_4$ having support $\omega_3$ and $\omega_4$, which is the union of elements $\{\Omega_2, \Omega_3\}$ and $\{\Omega_3, \Omega_4\}$ respectively (figure3.3). In order to approximate the field $U$, let the following basis functions are used

$$\{\phi\} = \{\phi_3, \phi_4\} \times \{g_3, g_4\}$$

$$= \{\phi_3g_3, \phi_4g_3, \phi_3g_4, \phi_4g_4\} \quad (3.9)$$

The approximation to the field $U$, on element $\Omega_3$ can now be written as

$$u^h(X) = \sum_{i=3}^{4} \phi_i \left( \sum_{l=3}^{4} g_l(X) a_i \right) \quad (3.10)$$

The element basis function spans to six functions, three for each node. First two of them are the standard basis functions, the last four functions are the extended or enriched basis functions which are obtained by the multiplication of standard basis function $\phi_i$ with the local approximation functions $g_i$. We already know that the finite element shape functions possess the property...
of partition of unity, hence we can also use them as partition of unity functions. It is now intuitive to see that the linear combination of shape functions defined in equation 3.9 can reproduce the local approximation functions as

\[ \phi_3 g_3 + \phi_4 g_3 = g_3 \]
\[ \phi_3 g_4 + \phi_4 g_4 = g_4 \]

It is worthwhile to mention here that the span of the basis function can be enhanced and multiple number of local approximation functions can be added into the approximating space, equation 3.9.

3.4 eXtended Finite Element Method, X-FEM

XFEM is a partition of unity based method, where special functions describing the field behavior are incorporated locally into the finite element approximation space and the resulting space is fully capable of capturing all the features of interest. In the passage below we shall demonstrate, using a 1D model example, the XFEM methodology to locally enrich the field, where a local partition of unity is satisfied to capture the desired features of interest in the solution.

The XFEM procedure begins with standard FEM procedural steps. Referring to figure 3.4, discretize the 1-D domain \( \Omega \) into sub domains \( \Omega_i = \{ \Omega_1, \Omega_2, \Omega_3, \Omega_4 \} \) with nodes at the vertices having coordinates denoted by \( x_i = \{ x_1, x_2, x_3, x_4, x_5 \} \). Next we associate with each node an interpolation function \( \phi_i \). The interpolation functions have a compact support \( \omega_i = \{ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \} \). The standard FEM approximation now reads

\[ u_h(X) = \sum_{i=1}^{i=5} \phi_i(X) u_i \quad (3.11) \]

In order to approximate the field correctly, needs a search for a function that can capture local features of interest. Let \( g(X) \) be the local approximating function that can approximate the field \( U \) well in the region of discontinuity. The function \( g(X) \) could take a form of a Heaviside \( H(X) \) function as it also represents a jump. Next we define a region ’D’ with domain \( \Omega^{PoU} \) in the interval \( [x_3, x_4] \), where we want the enrichment of the field, i.e element \( \Omega_3 \) is required to be enriched with function \( g(X) = H(X) \).
In order to incorporate the local enrichment into the Finite element approximation, we exploit the notion of partition of unity. We enrich the nodes of the element $\Omega_3$ with enrichment function $g(X)$. As a consequence the enrichment will be active on the support (clouds) $\omega_3$ and $\omega_4$. Then the next step is to find the partition of unity function i.e the functions which sum up to 1 in the region D. Using FEM shape functions, we already know that they possess PoU property, so we shall use the same interpolation functions, i.e $\phi_3$ and $\phi_4$ as our partition of unity functions. However it should be noted here that any other different function could also be used as a partition of unity function until and unless it satisfies equation 3.8. Now using the notion of partition of unity the enrichment function $g(X)$ can be included as

$$\sum_{j=3}^{4} \phi_j g(X) a_j$$

(3.12)

where $a_j$ are the enriched degrees of freedoms and are unknowns. The extended approximation of FEM now reads as

$$u^h(X) = \sum_{i \in I} \phi_i u_i + \sum_{i \in J} \phi_i g(X) a_i$$

(3.13)

where $I$ contains set of all nodes, $I = \{x_1, x_2, x_3, x_4, x_5\}$ and $J$ contains set of enriched nodes, $J = \{x_3, x_4\}$. Further it can be noticed that $J \subset I$.

Some Important characteristics of the above equation can be seen

- when $i \in I$

$$u^h(x_i) = \sum \phi_i u_i = u_i$$

It is a standard FEM case, showing that FEM approximation interpolates in the nodal values.

- when $i \in J$

$$u^h(x_i) = \sum \phi_i u_i + \sum \phi_i g(x_i) a_i$$

$$u^h(x_i) = u_i + g(x_i) a_i$$

The FEM approximations are now, no more nodal interpolants, i.e $u(x_k) \neq u_k$
It is worth mentioning that, the extended finite element method is similar to the partition of unity finite element method of Melenk and Babuska [1996] except the fact that, XFEM is a local partition of unity method. By local here we mean that, only a region near the discontinuities such as cracks, holes, material interfaces is enriched with enrichment functions, using the notion of partition of unity.
3.4 eXtended Finite Element Method, X-FEM

(a) Discontinuous field $U$ to be approximated on $\Omega$

(b) XFEM mesh discretization

(c) XFEM enriched local domain with enriched nodes $x_4$ and $x_5$

(d) Standard plus PoU linear interpolation functions over the domain

Figure 3.4: XFEM implementation steps
Level Set Representation of Discontinuities

4.1 Introduction

Level set is a method introduced by Osher and Sethian [1988] for tracking moving interfaces. The key point in the level set method is to represent the interfaces at any time \( t \), with a zero level set function i.e \( \phi(x,t) = 0 \). where \( \phi(x,t) \) is the level set function.

Let \( \Gamma \) be an open or closed interface that divides the domain into two distinct domains \( \Omega_A \) and \( \Omega_B \) (figure 4.1) , and is moving outward with a velocity \( F \) normal to the interface then the evolution equation for the interface using the material time derivative (total derivative) can be written as:

\[
\frac{\partial \phi}{\partial t} + F |\nabla \phi| = 0
\]

(4.1)

The initial condition is given as

\[
\phi(X, t = 0) = given
\]

(4.2)

\[\text{Figure 4.1: a:Domain } \Omega \text{ with an open discontinuity, b:Domain } \Omega \text{ with a closed discontinuity}\]

The initial condition is usually taken as the signed distance function such that the
level set function has positive values on one side of the interface and negative values at the other side of the interface and the interface is identified by the zero level set function.

In order to construct a level set function using a signed distance function, we find the closest point on the discontinuity $\Gamma$ such that $|X - X_\Gamma|$ is a minimum. where $X$ is any query point and $X_\Gamma$ is a point on the discontinuity $\Gamma$ as shown in figure 4.2. This actually defines a vector $\vec{d} = (X - X_\Gamma)$ which is orthogonal to the discontinuity $\Gamma$ at point $X_\Gamma$. Next in order to construct the level set function we define a normal to the discontinuity at point $X_\Gamma$. The level set function is then given as:

$$\vec{d} = (X - X_\Gamma)$$  \hspace{1cm} (4.3)

$$\phi(X, t = 0) = \vec{d} \cdot \hat{n}$$  \hspace{1cm} (4.4)

![Figure 4.2: Signed distance function](image)

### 4.2 Modeling cracks using Level set method

Level set method offers an elegant way of modeling discontinuities. Level set method has been successfully applied for modeling cracks. In this section details regarding the crack modeling using level set functions and its coupling with XFEM will be discussed. Further, later in the section some of the key advantages of using level set functions in the framework of XFEM will be highlighted.

The key point in modeling of crack and any discontinuity using level set method is to represent the discontinuity as a zero level set function. For the modeling of crack we define the level set function as a signed distance function. As the crack is a discontinuity which
does not divide the domain into two distinct parts completely, rather a portion of the domain is divided, hence to fully characterize a crack we define two level set functions (i) a normal level set function $\phi$ and (ii) a tangential level set function $\psi$. Both the two level set functions are defined as a signed distance functions. For the evaluation of signed distance function, let $\Gamma_c$

![Figure 4.3: Construction of Level set functions](image)

be the crack surface. Then for any point $x$ we find the closest point $x_{\Gamma}$ on the crack such that $|x - x_{\Gamma}|$ is a minimum. Where $x$ is any point of query and $x_{\Gamma}$ is a closest point on the crack surface $\Gamma_c$. We then define a unit normal vector $\hat{n}$ to the crack segment at $x_{\Gamma}$ as shown in figure 4.2. The signed distance function is then computed as follows:

$$\phi = (X - X_{\Gamma}).\hat{n} \quad (4.5)$$

Initially as the level set function is to be computed over the whole domain. We followed the approach proposed in Stolarska et al. [2001]. Consider an interior crack as shown in figure 4.3. In order to construct a level set function over the whole domain, we extend the crack tip segment to meet the boundary of the domain. In case a crack is an interior crack both the tips should be extended. The normal level set function is then computed using the original crack segment $\Gamma_c$ and the virtual segments(extensions). The figure 4.4 shows the plot of normal level set function for an interior crack.
4.2 Modeling cracks using Level set method

The tangential level set function $\psi$ is computed by finding the minimum signed distance to the normal drawn at the crack tip. If the crack is an interior crack then we need to define two tangential level set functions $\psi_1$ and $\psi_2$ corresponding to each crack tip. Even in this case we are able to define a unique tangential level set function, in order to make further computations easy, as:

$$\psi = \max [\psi_1, \psi_2] \quad (4.6)$$

Construction of normal and tangential level set functions for an interior crack is explained in figure 4.3. Figure 4.5 a,c shows the tangential level set function $\psi_1$ corresponding to the crack tip 1, while figure 4.5 b,d shows the tangential level set function $\psi_2$ corresponding to crack tip 2. Figure 4.6 shows the unique tangential level set function for an interior crack.

It is now evident that the crack can now be fully characterized by the two level set functions $\phi$ and $\psi$ such that $\phi = 0$ and $\psi < 0$ at the crack surface $\Gamma_c$. The crack tip is identified by an intersection of normal and tangential zero level set functions i.e intersection of $\phi = 0$ and $\psi = 0$. Formally we can write it as:

$$\begin{cases} 
For ~ X \in \Gamma_c & \phi = 0 \ AND \ \psi \leq 0 \\
For ~ X \in \Gamma_{\text{tip}} & \phi = 0 \ AND \ \psi = 0 
\end{cases}$$
4.2 Modeling cracks using Level set method

![Image of 2D and 3D contours of \( \psi \) for crack tips 1 and 2](image)

**Figure 4.5:** Tangential level set functions \( \psi_1 \) and \( \psi_2 \) corresponding to crack tip 1 and 2

Within the rest of the domain \( \phi \) will have a positive value above the crack and a negative value below the crack. The function \( \psi \) will have a positive value to the right of the normal at the crack tip. The \( \phi \) and \( \psi \) functions for an interior crack is shown in figure 4.7.

Level set method proves to be very useful in defining the crack as seen above. Within the framework of finite elements the level set functions \( \phi \) and \( \psi \) just defined above can be interpolated within an element using the standard shape functions as:

\[
\phi(X) = N_i(X)\phi_i
\]  

(4.7)

where \( \phi_i \) are the nodal values of the level set function.

Furthermore, within the frame work of XFEM, the level set functions can also be used for determining the Heaviside enrichment function. where we can define the enrichment function
4.2 Modeling cracks using Level set method

Figure 4.6: Unique Tangential level set function $\psi$ for an interior crack

(a) 2D contour of $\phi$  (b) 2D contour of $\psi$

Figure 4.7: Normal and tangential level set functions characterizing the crack

$H$ as:

$$H = \text{sign}(\phi)$$  \hspace{1cm} (4.8)

where

$$\text{sign}(\phi) = \begin{cases} +1 & \phi > 0 \\ -1 & \phi < 0 \end{cases}$$

The crack tip field is enriched with near-tip enrichment functions in the framework of XFEM. Where the position of a point is expressed in polar coordinates $(r, \theta)$, of the crack tip coordinate system. In order to align the discontinuity in the enrichment function a sequence of mapping was required to rotate each crack segment onto the crack [Belytschko and Black 1999]. However the use of level set functions alleviate the need for such mapping. With the
the use of level set functions it is intuitive to state the position of a point in the crack tip polar coordinate system as:

\[ r = \sqrt{\phi^2 + \psi^2} \]  
\[ \theta = \arctan \left( \frac{\phi}{\psi} \right) \]

Figure 4.8 below shows the plot of \( r \) and \( \theta \) around a crack tip of an edge crack. It can be seen from the figure that the \( \theta \) is align with the discontinuity and varies from \(-\pi\) to \(+\pi\).

Figure 4.8: Level sets with the method of Stolarska et al. [2001]
4.2 Modeling cracks using Level set method

4.2.1 Issues regarding crack modeling using level set functions

Level Set methods offers an elegant way of modeling cracks. Modeling cracks using Level set functions within the framework of XFEM was introduced by Stolarska et al. [2001]. Later on, it was implemented very successfully by many authors. These functions can also be used to identify the Elements completely cut by the crack into two and the elements containing the crack tip as mentioned in Stolarska et al. [2001]. The elements that are completely cut by the crack can be found by

$$\phi_{min} \times \phi_{max} \leq 0 \text{ AND } \psi_{max} < 0$$

The elements containing the crack tip can be found by

$$\phi_{min} \times \phi_{max} \leq 0 \text{ AND } \psi_{min} \times \psi_{max} \leq 0$$

The demonstration of the above mentioned criteria is shown in the figure 4.9. Where the nodes of the element containing the crack tip is shown by squares while nodes of cut elements are shown with circles around nodes.

![Selection of enriched elements using level sets](image)

(a) Enriched nodes  
(b) level set functions

**Figure 4.9:** Selection of enriched elements using level sets

4.2.1.1 Issue (A)

The criteria mentioned above did not perform well for all conditions of crack growth and is found to be insufficient in identifying the cut elements and tip elements. The criteria was seen to be affected from certain crack geometries. To support the argument we present the following
4.2 Modeling cracks using Level set method

examples. The figure 4.10a shows a zoom of the propagating crack on a 2D finite element mesh. The cut elements and the tip elements were found using the criteria mentioned above. The nodes

![Figure 4.10: Selection of enriched elements using level sets](image)

needed to be enriched by the Heaviside function are shown with circles while those needed to be enriched by the asymptotic near tip function are shown with squares around nodes. Hence the element whose all four nodes are encircled with squares will be regarded as tip element, while the elements whose all four nodes are encircled with circles will be designated as cut elements. From the figure4.10 it can be seen that using the criteria mentioned above, two elements are selected as tip elements. Further it can also be observed that amongst the two tip elements, there is one element which contains the actual crack tip, while the other element is neither cut by the crack nor contains the crack tip. Failure in detection of correct cracked elements or correct domain for enrichment one one hand will not only result in an increase in computational cost but on the other hand will also make other routines of the XFEM programing file to work improperly, such as element partitioning routines etc. It is interesting to see that how it happens. This could be better understood from figure 4.10b. The Normal level set functions are shown by circles while the Tangential level set functions are shown by squares. Green and red colors of the level set functions shows whether the function bears a positive value or a negative value. It is clear from the figure that inclination of crack tip segment also satisfied the condition $\psi_{\min} \times \psi_{\max} < 0$ in an element which is neither a tip nor a cut element.

Now let us see the figure 4.11a and b. Once again for a different crack configuration
4.2 Modeling cracks using Level set method

(a) Enriched nodes

(b) Level set functions

Figure 4.11: Selection of enriched elements using level sets

the criteria failed by not only detecting correctly the tip element but also causes the cut element’s
detection criteria to fail. As we can see that now the element cut by the crack is detected as a
tip element.

The above examples clarified that the criteria mentioned above for the determination of cut
elements and tip elements is not a satisfactory criteria to be used.

4.2.1.2 Issue (B)

This has been demonstrated earlier that level set function is very useful when it comes to model-
ing cracks. Using the level set function position of a point in polar crack tip coordinates system
(r and \( \theta \)) could be easily computed as:

\[
\begin{align*}
    r & = \sqrt{\phi^2 + \psi^2} \\
    \theta & = \tan^{-1}\left(\frac{\phi}{\psi}\right)
\end{align*}
\]

(4.11) \quad (4.12)

However using the simple signed distance function to construct the normal and tangential level
set function introduces a discontinuity in the r and \( \theta \) field itself, as shown in the figure 4.12.
Figure4.12a shows the contour plot of r around the crack tip and figure4.12b shows the contour
plot of \( \theta \) around the crack tip. It can be seen as the cracked curved there is a discontinuity in the
r and \( \theta \) field as identified also by Duflot [2007], which is actually due to the discontinuity in the
\( \phi \), the normal level set function. It can be inferred from the analysis that such type of disconti-
uinities will result for highly curved or kink cracks. The discontinuity in the r field is shown by
red dashed line in figure 4.12a. Similarly the discontinuity in the $\theta$ could be seen within a circle drawn in the figure 4.12b. If one moves along the line, shown by an arrow, away from the crack tip, several contour lines are passed by line, which clearly shows several different values of $\theta$ along the same radii which is actually wrong.

Such discontinuities becomes important while computing the domain form of interaction integral/J-Integral, where if such discontinuities in $r$ or in $\theta$ or in both, if present within the domain selected for the computation of J-integral, will result in wrong computations of $r$ and $\theta$ values and consequently results in quantitatively wrong estimates of the parameters i.e Interaction integral, stress intensity factors, and hence the accuracy of the whole solution will be influenced. This in turn also influences the crack propagation direction, if the criteria is based on computing stress intensity factors.

Similarly, in some cases we do not apply the near tip enrichment only to the nodes whose support contains the crack tip rather enrichment is extended to all the nodes that fall within certain radius around the crack tip. In this case such discontinuities, if present within the domain of enrichment, causes wrong estimates of $r$ and $\theta$ values for the computation of enrichment functions. Consequently the enriched shape function (N) and gradient operator (B) matrices will be quantitatively inaccurate, thus affecting the stiffness matrix and ultimately the overall solution. However it is worth mentioning here, that modified forms of level set functions are also proposed like smoothed Level set method by Duflot [2007], which adresses the issues of discontinuities in $r$ and $\theta$.

4.3 Modeling closed discontinuities using level set functions

Using the same basic concept of Level set function i.e to represent a discontinuity with a zero level set function, one can also model closed discontinuities. Within the framework of XFEM, where the discontinuities such as cracks, voids, material interfaces etc. are not required to be aligned with finite element mesh, this methodology proves to be an elegant and efficient choice to characterize the discontinuity. Below in the passage, a method to construct the level set functions for closed discontinuities of different shapes is explained.
4.3 Modeling closed discontinuities using level set functions

(a) function $r$ in crack tip coordinates  
(b) function $\theta$ in crack tip coordinate

Figure 4.12: crack tip polar coordinates $r$ and $\theta$

4.3.1 Circular discontinuity

For a circular hole/void we find the minimum signed distance to construct the level set function as follows.

$$\phi(X, 0) = \|X - X_c\| - R_c$$  \hspace{1cm} (4.13)

where $X$ is any query point, $X_c$ denotes the center coordinates of the circle and $R_c$ is the radius of the circle. The level set function can then simply be the signed distance function to the circular discontinuity. In this case the level set function will have a positive value outside the circle and a negative distance value for any point inside the circle. The points that lies on the circle will be represented as a zero value of the function.

If a body contains number of circular discontinuities than a single level set function for all the discontinuities can be defined as:

$$\phi(X, 0) = \min_{i=\{1,2,3,...,n_c\}} \{\|X - X^i_c\| - R^i_c\}$$  \hspace{1cm} (4.14)

where $n_c$ is the total number of circles. $X^i_c$ and $R^i_c$ refers to center and radius of $i$th circle.

4.3.2 Elliptical discontinuity

Consider an ellipse defined by its locus $x_E$ in local coordinate system. Let $a$ and $b$ be the semi-major and semi-minor axes of an ellipse. The mapping between local and global coordinate
system is given by $X = Rx$. where $R$ is the rotation matrix. $x = \{x, y\}$ and $X = \{X, Y\}$ are the local and global coordinate system. The radial distance of any point from an ellipse is given in local coordinate system as $(x - x_E)$. The radial vector in the global coordinate system can then be stated as:

$$X = R(x - x_E)$$

The level set function can then be defined as:

$$\phi(X, 0) = \left\| \left( \frac{X^2}{a^2} + \frac{Y^2}{b^2} \right) \right\| - 1 \quad (4.15)$$
4.3 Modeling closed discontinuities using level set functions

For multiple elliptical discontinuities one can write:

\[ \phi(X, 0) = \min_{i=\{1,2,3,\ldots,n_E\}} \phi(X_i, 0) \quad (4.16) \]
\[ \phi(X_i, 0) = \left\| \left( \frac{X_i^2}{a_i^2} + \frac{Y_i^2}{b_i^2} \right) \right\| - 1 \quad (4.17) \]

where \( i \) denotes the \( i \)th ellipse. It should be noted here that the kind of level set function \( \phi \) formulated above is not a signed distance function. But off course, it is a level set function. Another important thing to note here is that, we cannot use the linear basis functions to interpolate in the values of \( \phi \) within the element using the equation 4.7 mentioned above. This is due to the fact that the stated level set function for an elliptical discontinuity is of higher order.

In our approach, as we were interested only in the sign of the level set function and not in the numeric values of the function, hence we relied on the same linear basis functions to interpolate in the values. Furthermore it is of interest to know that a signed distance function for the above mentioned problem could be found by using the fast marching method defined by Sethian [1996].

**Figure 4.15:** Level set function for multiple elliptical discontinuities

4.3.3 Arbitrary polygonal discontinuity

In order to construct the level set function for any generic closed polygonal discontinuity \( \Gamma_p \), a polygon is considered to consist of number of small segments say \( N \). The approach is then to
find the minimum signed distance function to these segments, similar to the way, we constructed the normal level set function for an open crack. The difference here is that the discontinuity $\Gamma_p$ is a closed discontinuity. The level set function is then defined as:

$$\phi(X, t = 0) = \vec{d} \cdot \hat{n}$$  \hspace{1cm} (4.18)$$

where

$$\vec{d} = (X - X_{\Gamma_p})$$  \hspace{1cm} (4.19)$$

$$\|X - X_{\Gamma_p}\| = \min_{i=1,2,...,N} \|X - X_i\|$$  \hspace{1cm} (4.20)$$

Where $X_i$ is the nearest point on the segment i. $X_{\Gamma_p}$ is the nearest point on the discontinuity $\Gamma_p$. $X_{\Gamma_p}$ can also be seen as the orthogonal projection of a point $X$ on the discontinuity $\Gamma_p$. $\hat{n}$ is the unit normal to the discontinuity $\Gamma_p$ at point $X_{\Gamma_p}$. In case no unique normal is defined than the sign of the level set function will be positive if $(X - X_{\Gamma_p})$ belongs to cone of normals and negative otherwise.

Figure 4.16: Illustration of evaluating minimum signed distance to a polygon
Figure 4.17: Level set function for a hexagon
Chapter 5

Extended Finite Element Method - Realization in 2D

5.1 Mechanics of Cracked body

5.1.1 Kinematics

Consider a continuum body $\mathbb{B}$ which belongs to a 2D space $\mathbb{R}^2$ with domain $\Omega$

$$\mathbb{B} \equiv \Omega \subset \mathbb{R}^2$$  \hspace{1cm} (5.1)

Then any material point $X$ in the body can be defined as

$$X \in \Omega \subset \mathbb{R}^2$$  \hspace{1cm} (5.2)

Let the time instant $t_o$ represents the initial or reference state of the body $\mathbb{B}$ with domain represented as $\Omega_o$, and time $t$ represents the current state/configuration of the body $\mathbb{B}$ with domain represented as $\Omega$. Mathematically we can then write

$$\mathbb{B} \equiv \Omega_o \text{ and } X \equiv \mathbf{X} \text{ with } X \in \mathbb{B}, \mathbf{X} \in \Omega_o$$  \hspace{1cm} (5.3)

$$\mathbb{B} \equiv \Omega \text{ and } X \equiv \mathbf{x} \text{ with } X \in \mathbb{B}, \mathbf{x} \in \Omega$$  \hspace{1cm} (5.4)

Let the body $\mathbb{B}$ is crossed by a discontinuity $\Gamma$ with normal $\mathbf{N}$ to the boundary of the discontinuity, such that it divided the body domain into two distinct domains represented as $\Omega_o^-$ and $\Omega_o^+$. where $\Gamma$ is essentially an internal boundary in case of a 2D and a surface in case of a 3D
5.1 Mechanics of Cracked body

\[
\Omega = \Omega^- \cup \Omega^+ \tag{5.5}
\]

This results in the creation of two new boundaries/surfaces denoted as \( \Gamma^- \) and \( \Gamma^+ \). This is shown in the figure 5.1, where \( N^+ \) and \( N^- \) are the normals to the crack faces on \( \Gamma^+ \) and \( \Gamma^- \) in the reference configuration of the body with domain \( \Omega_o^+ \) and \( \Omega_o^- \) respectively. Similarly \( n^+ \) and \( n^- \) are the normals to the crack faces on \( \gamma^+ \) and \( \gamma^- \) in the current configuration of the body with domain \( \Omega^+ \) and \( \Omega^- \) respectively.

We can now define a deformation mapping function \( \varphi \) independently on both sides of the discontinuity \( \Gamma \) as \( \varphi^- \) and \( \varphi^+ \) which maps the particles \( X \in B, X \in \Omega_o \) from the reference configuration to current configuration \( X \in B, x \in \Omega \).

\[
\varphi(X) = \begin{cases} 
\varphi^-(X) & \forall X \in \Omega_o^- \\
\varphi^+(X) & \forall X \in \Omega_o^+ 
\end{cases}
\]

Intuitively we can write

\[
x^- = \varphi^-(X) \tag{5.6}
\]

\[
x^+ = \varphi^+(X)
\]

Accordingly the deformation gradient \( F \) can be defined as

\[
F^- = \nabla_X \varphi^-(X) \quad \forall X \in \Omega_o^- \tag{5.7}
\]

\[
F^+ = \nabla_X \varphi^+(X) \quad \forall X \in \Omega_o^+
\]

It is now intuitive to express the jump in the deformation map at the discontinuity as

\[
[F] = F^+ - F^-
\]
It is worthwhile to mention here that the change of configuration can also be expressed in terms of displacement field $u$.

$$u(X) = x(X) - X \quad (5.8a)$$

$$u^-(X) = \varphi^- - X \quad \forall X \in \Omega^- \quad (5.8b)$$

$$u^+(X) = \varphi^+ - X \quad \forall X \in \Omega^+ \quad (5.8c)$$

and the displacement jump at the discontinuity will be

$$[u] = u^+ - u^- \quad \forall X \in \Gamma \quad (5.9a)$$

$$[u] = [\varphi] \quad \forall X \in \Gamma \quad (5.9b)$$

For completeness, the Green/Lagrangian strain tensor for the body $B$ is expressed as

$$E = \frac{(\nabla_X u + (\nabla_X u)^T)}{2} + \frac{(\nabla_X u)(\nabla_X u)^T}{2} \quad (5.10)$$

where $E$ is the Green/Lagrangian strain tensor, $\nabla_X u$ is the gradient of displacement, $\epsilon$ is the symmetric part of displacement gradient and $E_2$ represents the non-linear part of the strain tensor $E$.

### 5.2 XFEM Enriched Basis

Extended finite element method helps us to model discontinuities arbitrarily aligned with the mesh. This becomes possible by making use of partition of unity. Due to the notion of partition of unity, any function typically non-polynomial can be incorporated into the FEM approximation. The basic idea lies in defining functions which sums up to one on the domain $\Omega^{POU}$.

$$\sum_{i \in I} \phi_i = 1 \quad (5.11)$$

Using the partition of unity functions it can be observed that

$$\sum_{i \in I} \phi_i g(x) = g(x) \quad (5.12)$$

meaning any function when multiplied with the partition of unity function can be reproduced. Additionally it inherits the smoothness of partition of unity function.
5.2 XFEM Enriched Basis

5.2.1 Explanation

Let's consider a body $\mathbb{B}$ with domain $\Omega^{POU}$. Let $\{\omega_i\}$ be the system of overlapping patches/clouds covering the whole domain and centered at nodes $i \in I$, figure 5.2. Further let's define a set of functions $\phi_i$ associated with the nodes $i$ on the patch $\omega_i$, such that they possess the following property

$$\sum_{i \in I} \phi_i = 1 \quad (5.13)$$

Now let $v_i$ be the space of functions by which the field $u^h|\omega_i$ on the patch/cloud $\omega_i$ can be described well, then the global space $U^h|\Omega$ can be approximated well with $V^h$ such that

$$V = \sum \phi_i v_i \quad (5.14)$$

Key points to note from the above discussion are, firstly the global space $V$ now inherits the smoothness of partition of unity functions. Secondly the global space inherits the local properties, thirdly the equation above possess a reproducing property. This has a greater advantage, meaning any function typically non-polynomial can be reproduced exactly, which will possess the smoothness of partition of unity functions.

Within the framework of finite element method (FEM), it is interesting to see that the finite element standard shape functions possess the property of partition of unity and hence can be used as partition of unity functions. For example, the iso-parametric shape functions for
a quadrilateral finite element are given as:

\[ N_1 = \frac{1}{4}(1 - \xi)(1 - \eta) \]  
(5.15)

\[ N_2 = \frac{1}{4}(1 + \xi)(1 - \eta) \]  
(5.16)

\[ N_3 = \frac{1}{4}(1 + \xi)(1 + \eta) \]  
(5.17)

\[ N_4 = \frac{1}{4}(1 - \xi)(1 + \eta) \]  
(5.18)

where \( \xi \) and \( \eta \) are the coordinates of a point in the parent element coordinate system. Next we group all the elements sharing the same node \( i \). The union of all those elements now can be regarded as the cloud or a patch \( \omega_i \) centered at \( i \). Grouping the shape functions of these elements forms the partition of unity function \( \phi_i \), which has a unity value at node \( i \) and zero at the boundary. The function \( \phi_i \) is called the partition of unity subordinate to the cover \( \omega_i \). In future we shall call \( \omega_i \) as the support of the nodal shape function or simply the nodal support, figure 5.3.

Next, we know the reproducing property of the partition of unity function, as mentioned above (equation 5.11 and 5.12) and observed by Duarte and Oden [1996] and Melenk and Babuska [1996]. Multiplying any function typically non-polynomial with the basis function will give rise to enriched basis function, which now inherits the local approximation properties and smoothness of the partition of unity shape function \( \phi_i \).

\[ \sum \phi_i g(x) = g(x) \]

This is illustrated in figure 5.4. For more details refer to [Strouboulis et al. 2000].
It is now time to define the nature of \( v_i \) (space of enrichment functions). As has been explained earlier, \( v_i \) is the space of functions by which \( u^h|_{\omega_i} \) can be approximated well locally on \( \omega_i \). Due to the two basic properties of partition of unity (i.e., they sum to 1, and they are able to reproduce any function) explained earlier, any function of any nature and in any number can be added into the local approximation space. The key idea being to paste together the local approximation spaces using the notion of partition of unity. Mathematically, we can define the enriched functional space as \( \{g_1(x), g_2(x), ..., g_i(x)\} \). where \( g(x) \) is the local approximation function. This local approximation function can be a polynomial or a non-polynomial function, discontinuous functions (Heaviside, Dirac delta, absolute value), singular functions, trigonometric functions, logarithmic functions or any other available priori knowledge on the characteristic properties of the expected solution. For example, in case of crack growth problems, we use four enrichment functions for regions containing the crack tips, obtained from the analytical solution in linear elastic fracture mechanics (LEFM), of displacement field around a
5.2 XFEM Enriched Basis

For the crack tip, the enriched approximation space of functions will be

\[ v_i = \{g_1(X), g_2(X), g_3(X), g_4(X)\} \]  
(5.19)

For regions that are completely cut by the crack, the Heaviside enrichment function is used, hence the enriched space of function will be

\[ v_i = \{H_1(X)\} \]  
(5.20)

Partition of unity basis function can be obtained by simply multiplying the standard FEM interpolation function with the local enrichment function as

\[ \phi_{ENR} = \{\phi_j \ast g_l(X)\}_{j=1}^{n^E} \]  
(5.21)

where \( j = 1 \) to \( n^E \), \( n^E \) = number of enriched nodes, \( l = 1 \) to \( n^{Enr} \), \( n^{Enr} \) = number of enrichment functions for each enriched node. This is illustrated in figure 5.4. Further it is to mention here that any function which forms the partition of unity can used and enriched basis is then obtained by multiplying the enrichment function with the partition of unity function. However, as the classical finite element shape functions also satisfies the partition of unity condition, so they could also be used as partition of unity functions.

The extended or enriched finite element basis can now be written as

\[ \{\phi_i\}_{i=1}^{N} = \{\phi_{STD}\} \bigcup \{\phi_{ENR}\} \]  
(5.22)

\[ \{\phi_i\}_{i=1}^{N} = \{\phi_i\}_{i=1}^{N} \bigcup \{\phi_j g_1(X), \phi_j g_2(X), \phi_j g_3(X), \ldots, \phi_j g_l(X)\}_{j=1}^{n^E} \]  
(5.23)

where \( N \) is number of nodes.

Within the framework of Finite element method if the nodal shape functions are defined by \( N \), then the XFEM Shape function matrix can be express as

\[ [N] = \begin{bmatrix} N_{STD} & N_{ENR} \end{bmatrix} \]  
(5.24)

The \( B \) matrix (discretized gradient operator matrix) will be

\[ [B] = \begin{bmatrix} B_{STD} & B_{ENR} \end{bmatrix} \]  
(5.25)

The XFEM displacement approximation can be expressed as

\[ u_{XFEM} = \sum_{i=1}^{N} N_i u_i + \sum_{j=1}^{n^E} N_j \left( \sum_{l=1}^{n^{Enr}} g_l(X) a_j \right) \]  
(5.26)

\[ u_{XFEM} = u_{STD} + u_{ENR} \]  
(5.27)
5.3 Modeling strong discontinuities in XFEM

In Matrix form we can then write as

\[ u_{XFEM} = \begin{bmatrix} N_{STD} & N_{ENR} \end{bmatrix} \begin{bmatrix} u_{STD} \\ u_{ENR} \end{bmatrix} \]

5.3 Modeling strong discontinuities in XFEM

As we have seen in the previous section, that any generic function representing the behavior of the approximating field can be easily incorporated into the approximation space, in this sense XFEM offers an elegant modeling methodology for modeling arbitrary discontinuities. Strong discontinuities shows a jump in the field, hence in such cases enriching the approximation space with a Heaviside or a step function is a reasonable choice. The resulting enriched basis function formed by multiplication of the enrichment function and the partition of unity shape function contains a jump at the interface and thus gives a better approximation to the field variable.

Referring to figure 5.5, let us consider a body with domain \( \Omega \). The domain is discretized into three elements 1,2 and 3 with domains \( \Omega_1, \Omega_2, \Omega_3 \). Let there be a crack in an element 2, such that it incorporates a strong discontinuity at \( x=x_o \) in the field variable. let \( N_2 \) and \( N_3 \) are the classical linear finite element shape functions associated with nodes 2 and 3 respectively, which also satisfy the property of partition of unity and hence can also be used as partition of unity functions. In order to model this discontinuity, let us use a step function \( H(X) \) as an enrichment function, defined as

\[
H(X) = \begin{cases} 
+1 & x - x_o > 0 \\
-1 & x - x_o < 0 
\end{cases}
\]

where \( x \) is any query point in the domain \( \Omega \). Then the XFEM approximation to the field variable \( u \), reads as:

\[
u_{XFEM} = \sum_{i=1}^{N} N_i u_i + \sum_{j=1}^{n_E} N_j H(X) a_j \quad (5.28)
\]

It can be seen from the figure 5.5, that the enriched basis function thus formed by the multiplication of enrichment function and the shape functions, possess a strong discontinuity at \( x=x_o \) required to approximate the behavior of \( u \).
5.4 Modeling weak discontinuities in XFEM

Intuitively the jump at the interface can be given as

\[
[u] = u^+ - u^-
\]

(5.29)

\[
= \sum_{i=1}^{N} N_i^+ u_i + \sum_{j=1}^{n_e} N_j^+ H(X^+) a_j - \sum_{i=1}^{N} N_i^- u_i - \sum_{j=1}^{n_e} N_j^- H(X^-) a_j
\]

\[
= \sum_{j=1}^{n_e} N_j^+ H(X^+) a_j - \sum_{j=1}^{n_e} N_j^- H(X^-) a_j
\]

\[
= \sum_{j=1}^{n_e} \bar{N}_j a_j
\]

where \( N_i^+ \) and \( N_i^- \) are the shape functions evaluated just to the left and just to the right of the interface. \( \bar{N}_j \) is the difference of the enriched shape functions on two sides of the discontinuity.

In our case using the step function \( H(X) \) of the form mentioned above, we can see that \( \bar{N}_j = 2N(X_o)j \).

\[
[u] = \sum_{j=1}^{n_e} 2N_j(X_o) a_j
\]

(5.30)

5.4 Modeling weak discontinuities in XFEM

Referring to figure 5.6, consider a domain \( \Omega \), which is discretized into three elements with nodes 1, 2, 3, and 4. Let N2 and N3 be the linear shape functions associated with nodes 3 and 4. A weak discontinuity in the field variable \( u \) shows a kink at the interface \( x = x_o \) and has discontinuous derivative. For modeling fields having discontinuous derivatives usually a ramp function in the form of absolute distance function is used [Krongauz and Beytchko 1998].

\[
\psi(X) = |\phi(X)|
\]

(5.31)

where \( \phi(X) \) is the signed distance function. In our 1D problem we can define \( \phi(X) \) as

\[
\phi(X) = x - x_o
\]

(5.32)

where \( x \) is any point of query and \( x_o \) is the location of the interface. The enriched basis function thus formed by the multiplication of enrichment function and the partition of unity function is
5.5 Extended finite element method for modeling cracks and crack growth problems

5.5.1 Introduction

The key point in X-FEM implementation is to carefully select the region/domain where it is required the enrichment of the field for close approximation to the accurate solution and the proper selection of enrichment function that can best describe the field. In its earlier version...
by Belytschko and Black [1999], the crack was modeled with a near-tip enrichment functions throughout its length and required a sequence of mapping that rotates each segment of the crack onto the crack model, to align the discontinuity in the enriching functions with the crack. Later on a more efficient enrichment scheme was proposed in [Moes et al. 1999], where the approximation near the crack tip was enriched with near-tip enrichment functions and Heaviside function was used to model the discontinuity along the rest of the crack. In the section below, we shall discuss the strong and weak forms of the Galerkin method along with discrete equations for XFEM. Later on the enrichment scheme, including the proper selection of the enrichment zone and evaluation of enrichment function will be reviewed in detail.

### 5.5.2 XFEM Problem Formulation

In this section we shall first review the basic governing equations for the elastostatic analysis. Consider a body $\mathbb{B}$ with domain denoted by $\Omega$ and outer boundary $\Gamma$. The body is subjected to a uniform body/volume forces $\mathbf{b}$, the traction forces are applied at the boundary $\Gamma_t$. The displacement boundary conditions are applied at the boundary surface $\Gamma_u$. We can readily write
5.5 Extended finite element method for modeling cracks and crack growth problems

that $\Gamma = \Gamma_u \cup \Gamma_t$. In addition to these external boundaries, a body contains a crack surface boundary inside the domain $\Omega$, denoted by $\Gamma_d$. The crack boundary is considered to be traction free and consists of two coincident boundaries $\Gamma_d^+$ and $\Gamma_d^-$. The details of the domain $\Omega$ and the boundary $\Gamma$ is given in the figure 5.7.

**Figure 5.7:** Body with internal crack subjected to loads

The strong form of the equilibrium equation is:

$$\nabla \cdot \sigma + b = 0$$  \hspace{1cm} (5.35)

The boundary conditions for the body $\mathbb{B}$ are:

$$\sigma \cdot n = \vec{t} \text{ on } \Gamma_t$$  \hspace{1cm} (5.36)
$$\sigma \cdot n = 0 \text{ on } \Gamma_d^+$$  \hspace{1cm} (5.37)
$$\sigma \cdot n = 0 \text{ on } \Gamma_d^-$$  \hspace{1cm} (5.38)
$$u = \vec{u} \text{ on } \Gamma_u$$  \hspace{1cm} (5.39)

where $\sigma$ is the Cauchy stress tensor and $n$ is the unit outward normal.

Considering the case of small strains and displacements we can express strain displacement relationship as:

$$\epsilon = \nabla_s u$$  \hspace{1cm} (5.40)

where the $\epsilon$ is the linear strain tensor and $\nabla_s u$ is the symmetric part of the displacement gradient. Considering the material to be linear elastic the constitutive equation is given by the Hook’s law.
5.5 Extended finite element method for modeling cracks and crack growth problems

as:

\[ \mathbf{\sigma} = \mathbf{\sigma}(\mathbf{\epsilon}) = \mathbf{C}\mathbf{\epsilon} \]  \hspace{1cm} (5.41)

where \( \mathbf{C} \) is the elastic material stiffness tensor. The space of the admissible displacement field is given as

\[ u \in U = \{ u | u \in H^1(\Omega), u = \bar{u} \text{ on } \Gamma_u, u \text{ is discontinuous on } \Gamma_d \} \]  \hspace{1cm} (5.42)

The test function is given as

\[ w \in W = \{ w | w \in H^1(\Omega), w = 0 \text{ on } \Gamma_u, w \text{ is discontinuous on } \Gamma_d \} \]  \hspace{1cm} (5.43)

The weak form of the above mentioned strong form of equilibrium equation is then given as:

\[ \int_{\Omega} [\nabla w : \mathbf{\sigma}(u)] \ d\Omega - \int_{\Gamma_t} w \mathbf{t} \cdot d\Gamma - \int_{\Omega} w \mathbf{b} \ d\Omega = 0 \]  \hspace{1cm} (5.44)

5.5.3 Discrete form of equilibrium Equation

Using the bubnov-Galerkin method the trial functions \( u^h \) as well as the test function \( w^h \) are represented as the linear combination of the same interpolation functions. The X-FEM test and trial function can be given as:

\[ u^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) u_i + \sum_{j \in J} N_j(\mathbf{x}) H(\mathbf{x}) a_j + \sum_{k \in K} N_k(\mathbf{x}) \left( \sum_{l=1}^{4} F_l(\mathbf{x}) b^l_k \right) \]  \hspace{1cm} (5.45)

\[ w^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) w_i + \sum_{j \in J} N_j(\mathbf{x}) H(\mathbf{x}) c_j + \sum_{k \in K} N_k(\mathbf{x}) \left( \sum_{l=1}^{4} F_l(\mathbf{x}) e^l_k \right) \]  \hspace{1cm} (5.46)

where \( N_i \) are the classical finite element shape functions, \( H(\mathbf{x}) \) is the Heaviside enrichment function for the discontinuous field along the length of the crack and \( F_l \) are the four near-tip enrichment functions for the crack front.

Substituting the test and trial function into the weak form mentioned above and writing the equilibrium for an \( i \)th node, we get (for brevity ignoring the integrand quantities and writing in matrix form)

\[ - \int_{\Omega} (\nabla w)^T \mathbf{\sigma} + \int_{\Gamma_t} w^T \mathbf{t} + \int_{\Omega} w^T \mathbf{b} = 0 \]
5.5 Extended finite element method for modeling cracks and crack growth problems

\[- \int_{\Omega} (B_{std}^u w_i + B_{enr}^a c_i + B_{enr}^b e_i^l)^T \sigma \]
\[+ \int_{\Gamma_t} (N_{std}^u w_i + N_{enr}^a c_i + N_{enr}^b e_i^l)^T \tilde{t} \]
\[+ \int_{\Omega} (N_{std}^u w_i + N_{enr}^a c_i + N_{enr}^b e_i^l)^T b = 0 \] (5.47)

Knowing the arbitrariness of \( w_i, c_i \) and \( e_i^l \) we can thus write

\[- \int_{\Omega} (B_{std}^u)^T \sigma + \int_{\Gamma_t} (N_{enr}^a)^T \tilde{t} + \int_{\Omega} (N_{enr}^b)^T b = 0 \]
\[+ c_i \left[ - \int_{\Omega} (B_{std}^a)^T \sigma + \int_{\Gamma_t} (N_{enr}^a)^T \tilde{t} + \int_{\Omega} (N_{enr}^b)^T b \right] \]
\[+ e_i^l \left[ - \int_{\Omega} (B_{std}^b)^T \sigma + \int_{\Gamma_t} (N_{enr}^b)^T \tilde{t} + \int_{\Omega} (N_{enr}^b)^T b \right] = 0 \] (5.48)

Any Constitutive law can be incorporated into the above discrete equations. For our case we shall use linear elastic case obeying Hook’s law, thus we get the following system of equations

\[K_{ij} d_j = f_i \] (5.49)

where

\[K_{ij}^{\alpha\beta} = \int_{\Omega} (B_{\alpha}^i)^T C (B_{\beta}^j) \quad d\Omega \] (5.50)

\((\alpha, \beta = u,a,b)\)

In the matrix form we can write as

\[K_{ij} = \begin{bmatrix}
K_{uu}^{ij} & K_{ua}^{ij} & K_{ub}^{ij} \\
K_{au}^{ij} & K_{aa}^{ij} & K_{ab}^{ij} \\
K_{bu}^{ij} & K_{ba}^{ij} & K_{bb}^{ij}
\end{bmatrix} \]

where the gradient operator B, considering that the same classical shape functions are also used as partition of unity functions, is given as

\[B_i^u = B_{std}^u = \begin{bmatrix}
(N_{std}^u, x) & 0 \\
0 & (N_{std}^u, y) \\
(N_{std}^u, y) & (N_{std}^u, x)
\end{bmatrix} \]
5.5 Extended finite element method for modeling cracks and crack growth problems

\[ B_i^a = B_{enr_i}^a = \begin{bmatrix} (H N_{std_i}),_x & 0 \\ 0 & (H N_{std_i}),_y \end{bmatrix} \]

\[ B_i^{bl} = B_{enr_i}^{bl} = \begin{bmatrix} (F_i N_{std_i}),_x & 0 \\ 0 & (F_i N_{std_i}),_y \\ (F_i N_{std_i}),_y & (F_i N_{std_i}),_x \end{bmatrix} \]

(l=1 to 4)

The external nodal force vector can then be expressed as

\[ f_i = \{ f_i^u, f_i^a, f_i^{bl1}, f_i^{bl2}, f_i^{bl3}, f_i^{bl4} \} \]  

(5.51)

\[ f_i^u = f_{std_i}^u = \int_{\Gamma_i^b} (N_{std_i})^T \bar{t} d\Gamma + \int_{\Omega_i^b} (N_{std_i})^T b d\Omega \]  

(5.52)

\[ f_i^a = f_{enr_i}^a = \int_{\Gamma_i^b} (H N_{std_i})^T \bar{t} d\Gamma + \int_{\Omega_i^b} (H N_{std_i})^T b d\Omega \]  

(5.53)

\[ f_i^{bl} = f_{enr_i}^{bl} = \int_{\Gamma_i^b} (F_i N_{std_i})^T \bar{t} d\Gamma + \int_{\Omega_i^b} (F_i N_{std_i})^T b d\Omega \]  

(5.54)

\( N_{std_i} \) denotes the classical finite element shape functions, while \( N_{enr}^a = H N_{std_i} \) and \( N_{enr}^{bl} = F_i N_{std_i} \).

5.5.4 Enrichment Scheme for 2D crack Modeling

For modeling of cracks two types of enrichment functions are used

1. Heaviside/step function function.

2. Asymptotic near-tip enrichment functions from LEFM.

5.5.4.1 Step function

The elements which are completely cut by the crack, such that they split into two, are enriched with the Heaviside/step function. The splitting of the domain by the crack causes a displacement jump, the step/Heaviside function gives the desired behavior to approximate the true field. Mathematically the step function is defined as

\[ H(x,y) = \begin{bmatrix} 1 & \text{for} (x - x^*) \cdot n > 0 \\ -1 & \text{for} (x - x^*) \cdot n < 0 \end{bmatrix} \]
5.5 Extended finite element method for modeling cracks and crack growth problems

where $x$ is the point under query, $x^*$ is nearest point to the crack segment $\Gamma_d$ and $n$ is the unit normal vector at $x^*$. The whole aim is to determine, whether the point is above or below the crack segment. The step function can also be determined by finding the minimum signed distance function to the crack geometry. The function will have a positive value if the dot product of the two vectors is a positive value and negative otherwise. If no unique normal is defined then the function will have a positive value if $(x - x^*)$ belong to the cone of normals. The two conditions are explained in the figure(5.9) below.

(a) Evaluation of orientation of a point for a smooth crack
(b) Evaluation of orientation of a point for a kink crack

Figure 5.9: Evaluation of Heaviside function
5.5 Extended finite element method for modeling cracks and crack growth problems

5.5.4.2 Asymptotic Near-tip Enrichment function

Step function or a Heaviside enrichment is good when the element is totally cut by the crack, such that it divides the element into two. In case the element contains the crack tip, then part of the element is cut and part of it not. Hence in such cases the step function cannot be used to enrich the domain. In Linear elastic fracture mechanics (LEFM), the exact solution of the stress and displacement field is available. Where the displacement field around the crack tip is given by

\[ u_x = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[ \kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right] + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[ \kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right] \]  
\[ (5.55) \]

\[ u_y = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[ \kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right] - \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[ \kappa - 1 - 2 \sin^2 \frac{\theta}{2} \right] \]  
\[ (5.56) \]

where \( \kappa = \frac{3 - 4\nu}{3 - \nu} \) plane strain  
\[ \kappa = \frac{3 - \nu}{1 + \nu} \] plane stress

where \( K_I \) and \( K_{II} \) are the stress intensity factors for mode I and II fracture. \( r \) and \( \theta \) are the polar coordinates of a query point in crack tip coordinate system. It was shown in Flemming et al. [1997] that the displacement field is contained within the span of following four functions.

\[ \{F_i(r, \theta)\}_{i=1}^4 = \left\{ \sqrt{r} \cos \left( \frac{\theta}{2} \right), \sqrt{r} \sin \left( \frac{\theta}{2} \right), \sqrt{r} \sin \left( \frac{\theta}{2} \right) \sin \theta, \sqrt{r} \cos \left( \frac{\theta}{2} \right) \sin \theta \right\} \]  
\[ (5.57) \]

In other words these functions forms the basis of the asymptotic field around the crack tip. These four functions are used for enriching the field near the crack tip, thus giving rise to four additional degrees of freedom in each direction at a node, in addition to standard degrees of freedom. It should be noted here that among the four the second function \( \sqrt{r} \sin(\theta/2) \) is discontinuous along the crack surfaces, thus giving the effect of required discontinuity in the approximation along the crack. The rest of the three functions are used for improving the solution near the crack tip. The four enrichment functions are shown in the figure 5.10. The inclusion of \( \sqrt{r} \) term in the enrichment function gives the required singularity in the stress field. The important thing to note here is that, with the use of the above mentioned near-tip enrichment functions an element partially cut by the crack could be modeled, as shown in figure 5.11. Further to mention here, that enrichment functions are discontinuous along the ray, \( \theta = \pm \pi \). If the crack segment within an element tries to rotate, then the same mapping
5.6 Crack initiation and growth

as mentioned in [Belytschko and Black 1999; Dolbow et al. 2000a] is required to align the discontinuity in the enriching functions with the crack, by rotating each segment of the crack onto the crack model. However using the level set functions to describe the crack geometry, no mapping is then required as defining $r$ and $\theta$ for the enriching functions by

$$ r = \sqrt{\phi^2 + \psi^2}, \quad \theta = \text{atan} \frac{\psi}{\phi} $$

This ensures that the discontinuity does coincide with the geometry of the crack. This is explained in section 4.2 and figure 4.8.

5.6 Crack initiation and growth

The accuracy and reliability of the analysis of a cracked body primarily depends upon the accurate determination and continuity of the crack path. It is therefore very much important to select the crack growth criteria very carefully. Some of the commonly used crack growth criteria are:

1. Minimum strain energy density criteria, [Sih 1974]
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![Enrichment Function](image)

**Figure 5.11:** Enrichment function \( \sqrt{r \sin \left( \frac{\theta}{2} \right)} \), for a crack tip element

2. Maximum energy release rate criteria, [Nuismer 1975]

3. Maximum hoop stress or maximum principal stress criteria, [Erdogan and Sih 1963]


It should be noted here that Meschke and Dumstorff [2007] proposed a global energy based criterion for cohesive and cohesion cracks, where the crack propagation angle \( \theta_{cr} \) of the new crack segment relative to existing crack is determined by minimizing the total energy of the body. For more details see also [Dumstorff and Meschke 2007]. In addition to this, virtual crack extension method by Hwang and Ingraffea [2007] is proposed for modeling multiply cracked systems.

In the section some of the crack growth criteria widely used in LEFM coupled with XFEM are discussed briefly. The crack growth criteria will be discussed primarily for a crack in 2D.

### 5.6.1 Minimum strain energy density criteria

The minimum strain energy criteria [Sih 1974] relies on determining the strain energy density function \( S \), which is a measure of the strength of elastic energy field in the vicinity of crack tip. The function \( S \) is a quadratic function of mode I and mode II stress intensity factors, which vary with angle \( \theta \), where \( \theta \) is the angle of the crack segment. The criteria assumes that (1) the crack initiation will occur when the minimum of \( S \) reaches to some critical value \( S_{cr} \), which is a material characteristic and can be determined experimentally. (2) The crack will extend in a direction in which strain energy density factor possess a minimum value. It states that the...
5.6 Crack initiation and growth

direction of crack initiation coincides with the minimum strain energy density along a constant radius around a crack tip. The minimum strain energy density factor $S$ is given by

$$S = a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2$$  \hspace{1cm} (5.58)

where $K_I$, $K_{II}$ and $K_{III}$ are the mode I, II and III stress intensity factors. The constants are given as

$$a_{11} = \frac{\kappa + 1}{16\mu\lambda \kappa^2 \cos \theta} \left[ 2(1 - 2\nu) + \frac{\kappa - 1}{\kappa} \right]$$

$$a_{12} = \frac{(\kappa^2 - 1)^{1/2}}{8\mu\lambda \kappa^2 \cos \theta} \left[ \frac{1}{\kappa} - (1 - 2\nu) \right]$$

$$a_{22} = \frac{1}{16\mu\lambda \kappa^2 \cos \theta} \left[ 4(1 - \nu)(\kappa - 1) + \frac{1}{\kappa}(\kappa + 1)(3 - \kappa) \right]$$

$$a_{12} = \frac{1}{4\mu\lambda \kappa \cos \theta}$$

Then the direction of propagation is determined such that

$$\left( \frac{\partial S}{\partial \theta} \right)_{\theta = \theta_{cr}} = 0 \hspace{1cm} \left( \frac{\partial^2 S}{\partial \theta^2} \right)_{\theta = \theta_{cr}} > 0$$  \hspace{1cm} (5.59)

It is worth mentioning that the criteria works well for linear elastic fracture mechanics.

5.6.2 Maximum energy release rate criteria

The maximum energy release rate criteria [Nuismer 1975] is based on determining the energy release rate at the crack tip. According to this criteria it is assumed that (1) the crack propagation will initiate, when the maximum energy release rate reaches some critical value $G_{cr}$, where $G_{cr}$ is a material characteristic. (2) The crack will grow in a radial direction from the crack tip along which the energy release rate is maximum. The direction of propagation is then given by

$$\theta_{cr} = 2\arctan \left( \frac{K_I}{K_{II}} \right)$$  \hspace{1cm} (5.60)

where $K_I$ and $K_{II}$ are the mode I and mode II stress intensity factors at the old crack tip. This representation of the criteria requires the evaluation of the two stress intensity factors. Nishioka [1983] presented the energy release rate as a function of the path independent J-integral, where the energy release rate is given as

$$G = J_1 \cos \theta + J_2 \sin \theta$$  \hspace{1cm} (5.61)
maximization of the above yields the direction of propagation of crack.

\[ \theta_{cr} = \arctan \left( \frac{J_1}{J_2} \right) \]  \hspace{1cm} (5.62)

The J-integral is evaluated as

\[ J_k = \int_{\Gamma} \left[ \frac{1}{2} \sigma_{ij} \epsilon_{ij} \delta_{jk} - \sigma_{ij} \frac{\partial u_i}{\partial x_k} \right] n_j d\Gamma \]  \hspace{1cm} (5.63)

The conservation integral can be expressed in a more convenient form by representing it as a domain form of integral which best suits the finite element analysis framework. Where the integral is evaluated on an area \( A_\Gamma \) surrounded by a closed contour \( C = \Gamma \cup C^+ \cup \Gamma_o \cup C^- \). This is explained in detail in section 2.4.3 and is illustrated graphically in figure 5.13. The integral is then evaluated as

\[ J_k = -\int_A \left[ W \delta_{jk} - \sigma_{ij} \frac{\partial u_i}{\partial x_k} \right] \frac{\partial q}{\partial x_j} n_j dA \]  \hspace{1cm} (5.64)

where \( q \) is a smooth weighting function.

It is worth mentioning here, that the criteria works best for traction free cracks.

### 5.6.3 Maximum hoop(circumferential) stress criterion or maximum principal stress criterion

The most commonly used crack growth criteria in LEFM is the maximum hoop stress criteria [Erdogan and Sih 1963]. The criteria is based on the evaluation of mixed mode stress intensity factors \( K_I \) and \( K_{II} \). According to this criteria it is assumed that (1) the crack initiation will occur when the maximum hoop stress reaches to a critical value, (2) the crack will grow in a direction \( \theta_{cr} \) in which circumferential stress \( \sigma_{\Theta\Theta} \) is maximum. The direction is determined by
5.6 Crack initiation and growth

Figure 5.13: Conventions for domain J: domain A is enclosed by $\Gamma$, $C^+$, $C^-$ and $\Gamma_o$; unit normal $m = n$ on $\Gamma_o$ and $m = -n$ on $\Gamma$

Evaluating the stress intensity factors $K_I$ and $K_{II}$ using the domain form of interaction integral around the crack tip assuming the crack surfaces are traction free. The circumferential stress in the direction of crack propagation is a principal stress, hence the crack propagation direction is determined by setting the shear stress equal to zero.

\[
\sigma_{r\theta} = \frac{1}{2\pi r} \cos \left( \frac{\theta}{2} \right) \left( \frac{1}{2} K_I \sin \theta + \frac{1}{2} K_{II} (3 \cos \theta - 1) \right) = 0
\]  

(5.65)

This leads to the equation for the crack propagation direction $\theta_{cr}$ in local crack tip coordinate system as

\[
\theta_{cr} = 2 \arctan \left( \frac{K_{II}}{K_I} \pm \sqrt{\frac{K_I^2}{K_{II}^2} + 8} \right)
\]  

(5.66)

It is worth mentioning that according to this criteria maximum propagation angle $\theta_{cr}$ is limited to 70.5° for pure mode II cracks. The criteria basically works well for traction free crack surfaces. However it has been used for cohesive crack growth problems with the assumption that the size of the investigated structure has a minor influence on crack path and influence only the global deflection curve. A more efficient expression for $\theta_{cr}$ is implemented in Liang et al. [2003].

\[
\theta_{cr} = 2 \arctan \frac{-2 K_{II}/K_I}{1 + \sqrt{1 + 8(K_{II}/K_I)^2}}
\]  

(5.67)

5.6.4 Average stress criteria

Wells and Sullays [2001] used the average stress criteria for modeling cohesive cracks within the framework of XFEM. According to this criteria, the crack is initiated when a maximum tensile
stress at any point exceeds the maximum tensile strength of the material. The normal to the crack extension \( n_{cr} \) is taken to be the principal axis of the averaged stress tensor corresponding to the maximum principal stress. The motivation for this criteria was using non-local stress quantity instead of using local stress, to improve the reliability of the computed stress in the vicinity of the crack as the crack tip is not located at a point where stresses are known accurately. Non-local/ average stress tensor \( \sigma_m \) is calculated as a weighted average of stresses using a Gaussian weight function ,\( w \).

\[
\sigma_m = \int w \sigma dV \quad (5.68)
\]

\[
w = \frac{1}{(2\pi)^{3/2}l^2} e^{\frac{-r^2}{2l^2}} \quad (5.69)
\]

where \( l \) determines how quickly weight function decays away from crack tip. However it was observed [Dumstorff and Meschke 2007] that the criteria does not represent fully the correct crack path as compare to other crack tracking criterions.

### 5.6.5 Global tracking algorithm

Global tracking algorithm proposed by [Oliver et al. 2004] showed good results [Dumstorff and Meschke 2007; Areias and Belytschko 2005] in predicting crack paths and can be easily and elegantly be incorporated into the finite element program. However this comes at the cost of solving additional global system of equations with one degree of freedom per node.

In contrast to other tracking algorithms, global tracking algorithm does not need to be evaluated for each individual crack segment rather it traces all discontinuity paths at once. The basic idea is to construct a function \( \vartheta \) whose iso-lines run perpendicular to the direction of principal stresses in all integration points of the investigated structure. An iso-line is then define as

\[
S_i = \{ x \in \Omega | \vartheta(x) = \vartheta_i \} \quad (5.70)
\]
where $S_i$ is the isoline for which $\vartheta(x) = \vartheta_i^s$.

## 5.7 Numerical Integration

For the elements that are cut by the crack, a modified integration scheme is practiced in XFEM. This is due to the fact that usual Gauss quadrature rule fails to integrate the field in the elements cut by the crack accurately and correctly. Hence it is necessary to partition the element, in order to integrate the field properly on both sides of the discontinuity. In order to integrate the field properly on both sides of the crack, the elements cut by the crack are partitioned into sub triangles where usual Gauss quadrature could be used. The partitioning scheme in XFEM is illustrated in figure 5.15. After the element is partitioned, the integration of the weak form for element cut by crack is performed by replacing the loop over an element by loop over sub polygons/sub-triangles. Hence the integrand $f(x)$ is integrated as

$$ F(X) = \int_{\Omega^+} f(x) dx + \int_{\Omega^-} f(x) dx $$

(5.71)

Numerically the function $f(x)$ over the whole domain is integrated as follows

$$ F(X) = \sum_{\Omega_e} \sum_{\Omega_s} f(gp) gw $$

(5.72)

where $\Omega_e$ is the domain of an element and $\Omega_s$ is the domain of a sub-triangle.

In order to understand this, let us recall the basic methodology of finite elements. The first step in the solution of equilibrium equation in finite element methodology is to convert the strong form of equilibrium equation into an equivalent weak form. This is done by multiplying a weighting function with the strong form and then integrating over the whole domain. The next step is then to use a divergence theorem which shifts the gradient operator from the Cauchy stress tensor to the test function/weighting function, hence relaxing the continuity requirement from the stress field. Alternatively the application of divergence theorem requires that the field $u$ is sufficiently regular and does not contain any discontinuity. This condition demands, that if a discontinuity is present then the domain must be discretized into non-overlapping sub domains, such that the element boundaries match with the boundaries of the discontinuity, so that the field become continuous and regular. Thus making the weak form equivalent to the strong form. In the usual finite element method crack modeling is done by re-meshing the domain to match the element boundaries with the crack geometry, in order to satisfy the above mentioned
5.7 Numerical Integration

(a) Sub-triangulation of cut element

(b) Sub-triangulation of tip element

(c)

Figure 5.15: Sub-triangulation of elements cut by a crack

conditions. Not only that, re-meshing also requires that the new elements thus formed should be well conditioned and no badly shaped elements will be formed.

The partitioning of elements cut by the crack in the extended finite element method is different from the re-meshing in finite element method in the following ways

1. The partitioning of an element is done only for the integration purpose and no extra degrees of freedoms are added to the system unlike the usual finite element method, where such re-meshing becomes not only cumbersome but also computationally costly in case of crack growth problems.

2. As the partitioning of elements in XFEM is done only for the purpose of integration, no conditions on the shape of sub-polygons or sub-triangles is imposed. On the other hand re-meshing in finite element method requires the construction of well shaped elements.

It worth mentioning here that Ventura [2006] proposed another method for inte-
5.8 Blending Elements

Incorporation of an enrichment function and ultimately modeling of arbitrary discontinuities in the framework of extended finite method (XFEM) is made possible due to the notion of partition of unity, as discussed in detail in previous sections. XFEM is a local partition of unity based method where the special function/enrichment functions are added locally into the domain. By local we mean that enrichment is added only to a portion of the whole domain where they it is needed to capture discontinuities or singularities etc using the notion of partition of unity. This results in a region of the domain which blends the enriched domain with the rest of the domain. This blending region causes problems in getting optimal results. This is discussed in detail in the coming paragraphs.

Consider a body with domain $\Omega$ which is discretized into finite elements for numerical analysis and is crossed by an arbitrary discontinuity $\Gamma$ as shown in the figure 5.16. Within the framework of XFEM, in order to model such arbitrary discontinuity the nodes whose nodal support is cut by the discontinuity will be enriched. The enrichment $g(x)$ is incorporated into the finite element approximation space using the concept of partition of unity. This requires defining functions $\phi_j$ (partition of unity functions) over the domain $\Omega_{ENR}$, intended to be enriched, such that it satisfies the partition of unity

$$\sum_j \phi_j = 1 \quad (5.73)$$

Usually the same standard finite element shape functions are also used as partition of unity functions, as they also satisfy the partition of unity condition. Due to the property of partition
5.8 Blending Elements

![Blending Elements Diagram]

**Figure 5.16:** Typical discretization illustrating $\Omega_{ENR}$, Blending domain $\Omega_{BLEND}$ and standard domain $\Omega_{STD}$

of unity any function can be exactly reproduced in $\Omega_{ENR}$

$$\sum_j \phi_j g(x) a_j = g(x), \quad a_j = 1$$  \hspace{1cm} (5.74)

It is interesting to see that enriching the domain with some kind of enrichment function has actually divided the domain into three distinct domains. (1) Domain $\Omega_{STD}$ consists of elements where no enrichment is applied and the approximation to the field variable is constructed in a usual way, as we do in FEM. Elements belonging to the domain $\Omega_{STD}$ will be called as standard elements or FEM elements in the subsequent paragraphs. (2) Domain $\Omega_{ENR}$ consists of elements enriched with an enrichment function $g(x)$ to capture special behaviors of the solution. Elements belonging to domain $\Omega_{ENR}$ will be called as enriched elements. (3) Domain $\Omega_{BLEND}$ consists of elements whose some of the nodes are enriched and some of them are not. These elements are actually the transitional elements which blends the enriched region $\Omega_{ENR}$ with the rest of the domain $\Omega_{STD}$. Elements belonging to the domain $\Omega_{BLEND}$ will be called as blending elements. These three different domains are illustrated in the figure5.16.

It is very important to see that in blending elements the partition of unity is not satisfied i.e

$$\sum_j \phi_j \neq 1, \quad \text{in } \Omega_{BLEND}$$  \hspace{1cm} (5.75)
5.8 Blending Elements

As a result of this the enrichment function is not reproduced exactly in blending elements. If this is the only consequence of not satisfying the partition of unity in blending elements then it might not be a problem, as we only wished to enrich the domain $\Omega_{ENR}$ and wanted to reproduce the enrichment function in $\Omega_{ENR}$. The non-satisfaction of partition of unity in blending elements has a secondary effect on the approximating field. The approximation in the blending element can be written as

$$u(x) = N_i u_i + \phi_j g(x) a_j \quad (5.76)$$

As the partition of unity is not satisfied for the function $\phi_j$ in the blending elements, there will remain unwanted terms in the approximation, resulting from $\phi_j g(x)$ until and unless $a_j$ is zero. As a consequence it is not possible to reproduce a polynomial of the order of the FE shape
5.8 Blending Elements

function in blending elements. This results in wrong approximation of the field variable inside the blending element.

Another very severe effect of the unwanted terms in blending elements due to the non-satisfaction of partition of unity is that, it may seem in the first appearance that the error induced by the unwanted terms in the approximation field is only affecting the blending elements, however the affects are more pronounced as it affects the overall solution and ultimately the convergence of the XFEM.

The effects of blending element is explained by a numerical example of a bi-material bar under uniaxial tension. The bar consists of length \( L = 25 \) units. The elastic moduli of the bar from length \( L = 0 \) to \( 12.5 \) units is \( E = 5 \) and from \( L = 12 \) to \( 25 \) units \( E = 1.5 \). A uniaxial tension load is applied at the end of the bar, while the other end of the bar is clamped. Due to the change in material properties a weak discontinuity is introduced into the displacement filed \( u \) at \( L = 12.5 \) units. Solution by usual FEM requires the mesh to align with the material interface boundaries, however using XFEM discontinuities could be aligned arbitrarily with the mesh. For the purpose of numerical analysis, the bar is discretized into five elements. In the discussion below we shall refer the location of node and displacements at the node corresponding to its spatial coordinates, for example \( x_{10} \) will refer to a node located at \( L = 10 \) units and similarly \( u_{10} \) will refer to the displacement at \( x_{10} \). The nodes are shown with bold squares in figure (5.17a,b,c). In our case element 3 is an enriched element (figure5.17(a)) and elements attached on both sides of this enriched element are the blending elements (i.e element 2 and 4). Figure5.17(a) shows the analysis results, where the dotted line shows the standard \( (u_{std}) \) and enriched \( (u_{enr}) \) displacements over the elements and solid line shows the sum of the standard and enriched displacement (i.e \( u_{fem} = u_{std} + u_{enr} \)). It can be seen from the figures (5.17(a),5.17(b)), that the displacement field in the enriched element is approximated correctly due to the fact that the enrichment function is reproduced exactly, however in the neighboring elements, which are infact the blending elements, enrichment function is not reproduced exactly due to which the field is not approximated correctly. Figure5.17(b) shows the comparison of exact solution and numerical XFEM solution. Few things are obvious from the graph, first the weak discontinuity is modeled exactly and thus depicting the correct structural behavior. Secondly due to the unwanted terms, the approximation of field over the blending elements shows deviation from the
exact solution. Thirdly and the most important thing to note is that, due to the blending element effect the error progresses over the whole domain, even though the elements which are neither enriched elements nor blending elements are also affected. This results in an overall error in the solution and thus degrades the convergence. The error computed in the solution is 3.2 percent. Where the error in the solution is computed as

$$ERROR = \frac{\int |u_{x,fem} - u_{exact}|}{\int |u_{exact}|}$$

Figure 5.17(c) shows the displacement field computed using XFEM in comparison with the exact solution. The analysis was carried out with a finer mesh consisting of 10 elements this time. It can be seen from the graph that solution converges to the exact solution as compare to the previous case, where a coarse mesh was used. The error in the solution was found to be 0.4 percent, which is quite below than the previous result. From this exercise, it is now clear that blending elements not only harm the solution locally but also degrades the overall convergence of the XFEM.

There are several ways to correct the approximation for blending elements. Simple step function enrichment does not cause problem in the blending element because of the fact that the function is a constant and as long as $\phi_j$ is the same or of lower order than $N_i$ blending element effect is compensated. However using shifted enrichment automatically removes the enrichment from the domain which is not required to be enriched. For other functions Chessa et al. [2003] proposed an enhanced strain method and more recently Fries [2008] proposed corrected XFEM method for the problems in blending elements. For a few application in the above spirit see also [Tarancon et al. 2009; Gracie et al. 2008].

5.9 Cohesive Crack Growth

5.9.1 XFEM Problem formulation

Consider a body $\mathbb{B}$ with domain $\Omega$ and surface $\Gamma$. Let $\tilde{t}$ be the surface forces/tractons acting on the surface of the body $\Gamma_t$. The Dirichlet boundary conditions are applied at $\Gamma_u$. Let a crack denoted by $\Gamma_c$ partially divides the domain into $\Omega^+$ and $\Omega^-$. The crack faces are not traction free. Let $\Gamma_{coh}$ be the portion of the crack $\Gamma_c$ where the cohesive tractions are acting. This
5.9 Cohesive Crack Growth

is shown in figure 5.18. The equilibrium and other kinematics of the body, ignoring the body forces for the time, in mathematical form can be expressed as

\[
\text{div}\sigma = 0 \quad \text{in} \quad \Omega \\
\mathbf{u} = \bar{\mathbf{u}} \quad \text{in} \quad \Gamma_t \\
\sigma \cdot \mathbf{n} = \bar{t} \quad \text{in} \quad \Gamma_t \\
\sigma \cdot \mathbf{n}^+ = t^+ \quad \text{in} \quad \Gamma_{coh}^+ \\
\sigma \cdot \mathbf{n}^- = t^- \quad \text{in} \quad \Gamma_{coh}^- 
\]

where \( t^+ \) and \( t^- \) are the cohesive tractions acting on the crack surface \( \Gamma_{coh}^+ \) and \( \Gamma_{coh}^- \) respectively. \( n^+ \) and \( n^- \) are the outward normals to the crack faces at \( \Gamma_{coh}^+ \) and \( \Gamma_{coh}^- \) respectively, refer to figure 5.18. Further we can say that

\[
\sigma \cdot \mathbf{n}^+ = -\sigma \cdot \mathbf{n}^- = t^+ = -t^- = t_c \quad \text{on} \quad \Gamma_{coh}
\]

Considering the small strain problem the strain displacement relationship is given by

\[
\epsilon = \nabla^* \mathbf{u} \quad \text{(5.78)}
\]

we define here another field i.e the separation vector \( \mathbb{U} \) which is defined as the difference of the displacements on the two sides of the crack as shown in the figure 5.18. In mathematical form it is given as

\[
\mathbb{U} = \mathbf{u}^- - \mathbf{u}^+ \quad \text{(5.79)}
\]

5.9.2 Traction separation law

In case of cohesive crack growth problem, the crack surfaces are no more traction free. Hence a traction separation law needs to be defined. The variation of the tractions on the surface \( \Gamma_{coh} \) depends upon the problem under study. However for the moment, we define a generic traction separation law as

\[
t_c = \mathbf{KU} \quad \text{(5.80)}
\]
5.9 Cohesive Crack Growth

![Figure 5.18: Body with a cohesive crack](image)

where $\mathcal{K}$ is the interface material matrix or in other words slope of the cohesive law. Further we can also write that

$$t_c = t_c \cdot n$$

$$\bar{U} = U \cdot n$$

### 5.9.3 weak form

The space of the admissible displacement field is given as

$$u \in U = \{ u | u \in H^1(\Omega), u = \bar{u} \text{ on } \Gamma_u, u \text{ is discontinuous on } \Gamma_c \}$$  \hspace{1cm} (5.81)

The test function is given as

$$w \in W = \{ w | w \in H^1(\Omega), w = 0 \text{ on } \Gamma_u, w \text{ is discontinuous on } \Gamma_c \}$$  \hspace{1cm} (5.82)

As the cohesive forces/tractions are present on some portion of the crack, the total potential of the body has to be modified to take account for the cohesive tractions transfered through the crack faces. The weak form of the above mentioned strong form of equilibrium equation (equation 5.77) is then given as:

$$\int_{\Omega} [\nabla w : \sigma(u)] \, d\Omega - \int_{\Gamma_t} w \cdot \vec{f} \, d\Gamma - \int_{\Gamma^+_{coh}} w t^+_c \, d\Gamma - \int_{\Gamma^-_{coh}} w t^-_c \, d\Gamma = 0$$  \hspace{1cm} (5.83)

where the additional terms $\int_{\Gamma^+_{coh}} w t^+_c \, d\Gamma + \int_{\Gamma^-_{coh}} w t^-_c \, d\Gamma$ represents the contribution from the cohesive tractions. The above equation can be written in a more compact form noting the above
mentioned fact that \( t^+ = t^- = t_c \) as
\[
\int_{\Omega} \left[ \nabla w : \sigma(u) \right] \; d\Omega + \int_{\Gamma_{\text{coh}}} t_c \mathbf{U} \; d\Gamma = \int_{\Gamma_t} \bar{t} \mathbf{w} \; d\Gamma \tag{5.84}
\]

### 5.9.4 Discrete form of equilibrium Equation

In the sections below, we shall formulate the discrete form for the cohesive crack propagation problem in 2D, where the domain is enriched with two kinds of enrichment functions, i.e the Step function and the near tip enrichment functions.

Using the bubnov-Galerkin method the trial function \( u^h \) as well as the test function \( w^h \) are represented as the linear combination of the same interpolation functions. The X-FEM test and trial functions can then be given as.

\[
u^h(x) = \sum_{i \in I} N_i(x)u_i + \sum_{j \in J} N_j(x)H(x)a_j + \sum_{k \in K} N_k(x) \left( \sum_{l=1}^{4} F_l(x)b^l_k \right) \tag{5.85}
\]

\[
w^h(x) = \sum_{i \in I} N_i(x)w_i + \sum_{j \in J} N_j(x)H(x)c_j + \sum_{k \in K} N_k(x) \left( \sum_{l=1}^{4} F_l(x)e^l_k \right) \tag{5.86}
\]

where \( N_i \) are the finite element shape functions, \( H(x) \) is the Heaviside enrichment function for the discontinuous field along the length of the crack and \( F_l \) are the four near-tip enrichment functions for the crack front.

For brevity ignoring the integrand quantities and writing the weak form in matrix form we get
\[
\int_{\Omega} \nabla w^T \sigma(u) + \int_{\Gamma_{\text{coh}}} \mathbf{U}(w)^T t_c - \int_{\Gamma_t} w^T \bar{t} = 0 \tag{5.87}
\]

substituting the test and trial function into the weak form mentioned above and writing the equilibrium for the ith node, we get
\[
\int_{\Omega} (\mathbf{B}^u_{\text{std}} w_i + \mathbf{B}^a_{\text{enr}} c_i + \mathbf{B}^b_{\text{enr}} e^4_i)^T \sigma + \int_{\Gamma_{\text{coh}}} (\tilde{\mathbf{N}}^a_{\text{enr}} c_i + \tilde{\mathbf{N}}^b_{\text{enr}} e^4_i)^T t_c - \int_{\Gamma_t} (\mathbf{N}^u_{\text{std}} w_i + \mathbf{N}^a_{\text{enr}} c_i + \mathbf{N}^b_{\text{enr}} e^4_i)^T \bar{t} = 0 \tag{5.88}
\]

where we incorporated the test function in \( \mathbf{U}(w) \) as

\[
\mathbf{U}(w) = w^- - w^+ \tag{5.89}
\]

\[
\mathbf{U}(w) = \mathbf{N}^u_{\text{std}} w_i + (H \mathbf{N}^a_{\text{enr}}) c_i + (F_l \mathbf{N}^b_{\text{enr}}) e^4_i - \mathbf{N}^u_{\text{std}} w_i - (H \mathbf{N}^a_{\text{std}})^c_i - (F_l \mathbf{N}^b_{\text{enr}}) e^4_i \tag{5.90}
\]
As the discontinuity is present in the enriched basis function so the standard part will automatically vanish and we are left with difference in the enriched part

\[
\mathbb{U}(w) = \left[(HN_{enr}^a) - (HN_{enr}^a)^+\right] c_i + \left[(F_i N_{enr}^b) - (F_i N_{enr}^b)^+\right] e_i^t
\]  

(5.91)

\[
\mathbb{U}(w) = (\tilde{N}_{enr}^a)c_i + (\tilde{N}_{enr}^b)e_i^t
\]

(5.92)

where \( \tilde{N}_{enr}^a \) and \( \tilde{N}_{enr}^b \) shows the difference of the shape functions computed on two sides of the crack.

Rearranging the terms we can get

\[
w_i \left[ \int_{\Omega} (B_{std}^a)^T \sigma - \int_{\Gamma_t} (N_{std}^u)^T \dot{t} \right] + c_i \left[ \int_{\Omega} (B_{enr}^a)^T \sigma - \int_{\Gamma_t} (N_{enr}^a)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^a)^T t_c \right] + e_i^t \left[ \int_{\Omega} (B_{enr}^b)^T \sigma - \int_{\Gamma_t} (N_{enr}^b)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^b)^T t_c \right] = 0
\]  

(5.93)

Knowing the arbitrariness of \( w_i \), \( c_i \) and \( e_i^t \) we can thus write

\[
\left[ \int_{\Omega} (B_{std}^a)^T \sigma - \int_{\Gamma_t} (N_{std}^u)^T \dot{t} \right] = 0
\]  

(5.94)

\[
\left[ \int_{\Omega} (B_{enr}^a)^T \sigma - \int_{\Gamma_t} (N_{enr}^a)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^a)^T t_c \right] = 0
\]  

(5.95)

\[
\left[ \int_{\Omega} (B_{enr}^b)^T \sigma - \int_{\Gamma_t} (N_{enr}^b)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^b)^T t_c \right] = 0
\]  

(5.96)

We can see that, the above set of equations is not much different from standard XFEM, inspite the fact that cohesive terms are present. If the crack surfaces are traction free \( t_c = 0 \), then the above equations simplifies to standard XFEM equations.

We can now write the above equations in matrix form as follows

\[
\begin{bmatrix}
\int_{\Omega} (B_{std}^a)^T \sigma \\
\int_{\Omega} (B_{enr}^a)^T \sigma + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^a)^T t_c \\
\int_{\Omega} (B_{enr}^b)^T \sigma + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^b)^T t_c
\end{bmatrix} =
\begin{bmatrix}
\int_{\Gamma_t} (N_{std}^u)^T \dot{t} \\
\int_{\Gamma_t} (N_{enr}^a)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^a)^T t_c \\
\int_{\Gamma_t} (N_{enr}^b)^T \dot{t} + \int_{\Gamma_{coh}} (\tilde{N}_{enr}^b)^T t_c
\end{bmatrix}
\]

or more compactly we can write

\[
[f_{int}] = [f_{ext}]
\]

It is to mention here that the above mentioned equation can now be used for the cohesive crack growth problem with any generic material stress strain relationship \( \sigma = \sigma(\epsilon) \) and a generic traction separation law \( t_c = t_c(U) \).
5.10 Modeling Voids in XFEM

Presence of flaws in a structure such as holes, voids, inclusions, cracks etc. requires careful analysis of the structure or a structural component to judge the integrity, strength and its durability. Careful analysis of critical structural components such as joints and connections becomes even more important as the structural integrity and behavior is deeply affected by their response. Modeling holes/voids and inclusions requires the finite element mesh to match the geometry of the void or an inclusion interface. This not only becomes cumbersome and time consuming, when modeling arbitrary number of defects and inclusions but also sometimes computationally costly. In this context XFEM offers an elegant way of modeling arbitrary discontinuities, where the mesh does not require to be aligned with the boundaries of voids or material interfaces. This is done by incorporating proper enrichment function into the finite element approximation space where these functions are pasted together using the notion of partition of unity. The resulting extended or an enriched finite element approximating space is then capable of well approximating the field variable with less computational cost. In the section below modeling voids and inclusions within the framework of XFEM will be discussed in detail.

5.10.1 XFEM problem formulation

Consider a body $B$ with domain $\Omega \subset \mathbb{R}^2$ and a surface represented by $\Gamma$. let $\bar{t}$ be the traction applied on the surface $\Gamma_t$. The Dirichlet boundary conditions are applied on the surface $\Gamma_u$ such that $\Gamma_t \subset \Gamma, \Gamma_u \subset \Gamma$. Let the body contains a void represented as $\Gamma_h$ and an inclusion $\Gamma_I$. It is further stated that the holes are considered to be traction free while the continuity of tractions hold on material interfaces $\Gamma_I$. The equilibrium and kinematics of the body ignoring the body forces for the time, is given as

$$
\text{div} \sigma = 0 \quad \text{in} \ \Omega
$$

$$
u = \bar{u} \quad \text{in} \ \Gamma_t
$$

$$
\sigma \cdot n = \bar{t} \quad \text{in} \ \Gamma_t
$$

$$
\sigma \cdot n_h = 0 \quad \text{in} \ \Gamma_h
$$

$$
[\sigma \cdot n_I] = 0 \quad \text{in} \ \Gamma_I
$$

where $n$, $n_h$ and $n_I$ are the unit outward normals to the $\Gamma_t$, $\Gamma_h$ and $\Gamma_I$ respectively. $\bar{u}$ and $\bar{t}$ are the prescribed displacements and tractions. The body is shown in the figure 5.19.
5.10 Modeling Voids in XFEM

5.10.2 XFEM weak formulation

Let $U$ be the displacement field which we want to approximate. Next we define the functional spaces in which we search for the solution. Let the space of the admissible displacement field is defined as

$$ u \in U = \{ u | u \in H^1(\Omega), u = \bar{u} \text{ on } \Gamma_u \} $$

(5.97)

The test function is given as

$$ w \in W = \{ w | w \in H^1(\Omega), w = 0 \text{ on } \Gamma_u \} $$

(5.98)

The weak form is then obtained by multiplying the differential equation i.e the strong form of equilibrium equation with a weighting function $w$ and then integrating over the domain $\Omega$. The problem can now be stated as, find $u \in U$ such that $\forall w \in W$ with the boundary conditions $u = \bar{u}$ on $\Gamma_u$ and $\sigma \cdot n = \bar{t}$ on $\Gamma_t$.

$$ \int_{\Omega} [\nabla w : \sigma(u)] \, d\Omega - \int_{\Gamma_t} w \bar{t} \, d\Gamma = 0 $$

(5.99)

5.10.3 XFEM Discrete formulation

Using the Bubnov-Galerkin method the trial function $u^h$ as well as the test function $w^h$ are represented as the linear combination of the same interpolation functions. Then the XFEM
displacement approximation for the test and trial functions can be written as

\[
\begin{align*}
    u^h(x) &= \sum_{i \in I} N_i(x) u_i + \sum_{j \in J} N_j(x) g(x) a_j \\
    w^h(x) &= \sum_{i \in I} N_i(x) w_i + \sum_{j \in J} N_j(x) g(x) c_j
\end{align*}
\] (5.100)

where \( N_i \) are the standard finite element shape functions, \( g(x) \) is the enrichment function incorporated into the FEM space to model arbitrary discontinuities. We shall discuss the nature of the enrichment function for voids and material interfaces in the coming sections.

Incorporating the test and trial functions into the above mentioned weak form and knowing the arbitrariness of the weighting function we can come up with the result

\[
\begin{bmatrix}
    \int_\Omega (B_{std}^u)^T \sigma \\
    \int_\Omega (B_{enr}^a)^T \sigma
\end{bmatrix} = \begin{bmatrix}
    \int_{\Gamma_t} (N_{std}^u)^T \bar{t} \\
    \int_{\Gamma_t} (N_{enr}^a)^T \bar{t}
\end{bmatrix}
\]

or more compactly we can write

\[
[f_{\text{int}}] = [f_{\text{ext}}]
\]

For linear elastic material where the material obeys the Hook’s law (\( \sigma = C \epsilon \), with \( C \) as elastic material constitutive matrix) we have

\[
\begin{bmatrix}
    \int_\Omega (B_{std}^u)^T C (B_{std}^u) \\
    \int_\Omega (B_{enr}^a)^T C (B_{enr}^a)
\end{bmatrix} \begin{bmatrix}
    u_j \\
    a_j
\end{bmatrix} = \begin{bmatrix}
    \int_{\Gamma_t} (N_{std}^u)^T \bar{t} \\
    \int_{\Gamma_t} (N_{enr}^a)^T \bar{t}
\end{bmatrix}
\]

where

\[
B_{std}^u = \begin{bmatrix}
    N_{i,x} & 0 \\
    0 & N_{i,y}
\end{bmatrix}
\]

and

\[
B_{enr} = \begin{bmatrix}
    g(x) N_{i,x} + g(x)_{,x} N_i & 0 \\
    0 & g(x) N_{i,y} + g(x)_{,y} N_i \\
    g(x) N_{i,y} + g(x)_{,y} N_i & g(x) N_{i,x} + g(x)_{,x} N_i
\end{bmatrix}
\]

5.10.4 Enrichment function for voids

The enrichment function \( g(x) \) used for the nodes whose nodal support is intersected by the void is a Heaviside function (\( V(x) \)) as proposed by Sukumar et al. [2001]. A level set function \( \phi(x) \) is constructed over the domain to help in defining the geometry of the discontinuity, evaluation
of enrichment function and also for the selection of elements intersected by the interface. Construction of level set function is described in detail in chapter 4. More generally we can say that, a node that lies outside the void will have \( V(X) = 1 \) and a node that lies inside the void will have \( V(X) = 0 \). Mathematically using the level set function the enrichment function is evaluated as

\[
V(X) = \begin{cases} 
+1 & \phi(x) > 0 \\
0 & \phi(x) < 0 
\end{cases}
\]

The nodes that lies inside the void and whose nodal support is not intersected by the void are removed from the calculations. This is done usually by removing the degrees of freedoms associated with those nodes from the system of equations and solve the system only for the remaining degrees of freedoms.

### 5.10.5 Enrichment function for inclusions

Inclusions such as inhomogeneities in material properties within a body, produces discontinuities in the gradient of the displacement field. In order to incorporate discontinuities in the derivatives of the function an enrichment function \( g(x) \) in the form of a ramp function was proposed in the frame work of enriched element free galerkin method(Enriched EFG) by Flemming et al. [1997] and later, it was used by Sukumar et al. [2001] within XFEM. However it is worth mentioning that on the other hand due to the problems in blending elements Sukumar et al. [2001] have also proposed a modified enrichment function for inclusions. The ramp enrichment function is given as

\[
g(x) = \varphi(x) = |\phi(x)|
\]

The modified enrichment function of Sukumar et al. [2001] is

\[
\varphi(x) = \left| \sum_i N_i \phi_i \right|
\]

where in this case \( \varphi_i = \phi_i \) and \( \phi_i \) is the value of the level set function at node \( i \).
Chapter 6

XFEM Implementation

6.1 Introduction

Extended finite element method (XFEM) is an extension of standard finite element method (FEM), where the field is enriched with functions that capture the local behavior well, using the notion of partition of unity. As it is an extension of standard FEM, hence large body of finite element technology and software can be exploited. In the section below the XFEM implementation is discussed in detail.

6.2 Selection of enriched nodes

As has been mentioned earlier, one of the key steps in the implementation of XFEM is the selection of proper region, where the field is required to be enriched. This automatically requires Selection of correct nodes, where additional degrees of freedoms are added to the system. The strategy for the selection of enriched nodes is straight forward. In the passage below the support of the nodes will refer to the support of the nodal shape function which consists of union of all elements connected to that node. whereas the support closure will refer to the boundary of the nodal support (refer to figure6.1).

For the purpose of selection of enriched nodes, we define three sets of nodes \( I, J \) and \( K \). Where set \( I \) contains all the nodes of the body, set \( K \) the contains the nodes whose support closure contains crack tip 1 and/or 2 and set \( J \) contains the nodes whose support is intersected by crack and excluding the nodes contained in set \( K \). Mathematically we can express
6.2 Selection of enriched nodes

Figure 6.1: Nodal support and closure

as

\[ J = \{ j \in I : \omega_j \cap C \neq \emptyset, j \notin K1, j \notin K2 \} \]  \hspace{1cm} (6.1)

\[ K1 = \{ k \in I : x1 \in \bar{\omega}_k \} \]  \hspace{1cm} (6.2)

\[ K2 = \{ k \in I : x2 \in \bar{\omega}_k \} \]  \hspace{1cm} (6.3)

where \( x1 \) and \( x2 \) denotes the position of crack tip and \( C \) denotes the crack geometry.

we begin with creating a set \( I \), which contains all the nodes of the domain \( \Omega \). Then we construct Set \( K1 \) and \( K2 \) such that they contains the nodes whose support closure contains crack tip 1 or 2. Roughly speaking, set \( K1 \) and \( K2 \) contains the nodes of the element which contains the crack tip. In case of interior crack whose both tips are within the domain, the set \( K1 \) will contains the nodes of the element which contains the first crack tip and set \( K2 \) contains the nodes of the element which contains the second crack tip. In case of an edge crack one of the two sets \( K1 \) or \( K2 \) will remain empty. In our implementation, set \( K1 \) and \( K2 \) are constructed by explicitly finding the element containing crack tip 1 or 2, by making loop over all finite elements. It is worth mentioning here that, the number of crack-tip enriched nodes can be increased by including all those nodes which fall within certain radius of influence, in which region the asymptotic near-tip field is assumed to dominate the solution [Laborde et al. 2005a]. Then we construct set \( J \), which contains the nodes whose support is intersected by the crack. Now construct set \( J \) whose support is intersected by the crack. This can be done by using the level set functions, however we found that the criteria based on level set function is not fully able to find the cut elements and tip elements as explained in detail in chapter 4. We used explicit criteria to select the elements and nodal support cut by the crack. This is done by
6.2 Selection of enriched nodes

making a loop over the element’s edges to see, whether the edges are intersected by the crack or not. If more than one element’s edge is cut by the crack the element is classified as cut element and nodes of that element is put into the set \( J \). Next remove from set \( J \) the nodes that are either present in set \( K_1 \) or in set \( K_2 \). Then remove the nodes from set \( J \) whose closure is intersected by the crack and not the support. This is usually done by finding the area above and below the crack and if the ratio of the two is less than certain tolerance then the node is removed. This is usually done to avoid ill conditioned stiffness matrix which may results in spurious fields. Figure 6.2 shows the enriched nodes on a finite element mesh containing an interior crack. The

Figure 6.2: Enriched Nodes: circular nodes belongs to set \( J \), square nodes belongs to set \( K \)

nodes that are encircled with a circle, are the nodes contained in set \( J \) and will be enriched with a step function. The nodes which are encircled with squares, are the ones contained in set \( K_1 \) and \( K_2 \). Set \( K_1 \) contains enriched nodes with respect to crack tip 1 and set \( K_2 \) contains nodes with respect to crack tip 2 and will be enriched with crack-tip asymptotic functions.

6.2.1 Selection of enriched elements

In order to select the elements cut by the crack and the elements containing the crack tip we used an explicit criteria. To select the elements cut by the crack a loop on element edges is
made for each crack segment. If an intersection of an element edge with the crack segment is found, the edge is classified to be cut edge. After looping on all the edges of an element, if there exists more than one edge to be cut by the crack, the element is classified to be a cut element. Tip elements were found by simply finding the element which contains the crack tip, by making loop over all elements for a single crack tip.

6.3 Evaluation of enrichment functions

6.3.1 Step function

As has already explained, the jump function will take a value of +1 if the query point is above the crack and -1 when the query point is below the crack and will take a value of zero if it is on the crack or within certain tolerance limit. The jump function can easily be computed in many ways. Use of geometric predicates is very common in computational mechanics. Among these Orientation and incircle test are used quite frequently.

6.3.1.1 Orientation test

Orientation test determines whether the point under consideration is above or below the given line segment. The test is performed by evaluating a sign of the determinant. In reality it calculates the area of a triangle. We define a triangle with nodes consists of nodes of

![Figure 6.3: Orientation Test](image-url)
6.3 Evaluation of enrichment functions

Line segment and query point as being the third node of the triangle (refer to figure 6.3). Evaluation of determinant will give the twice the area of a triangle. It is obvious that, if the nodes of the triangle are in counter clock wise direction sign of the determinant will be positive (figure 6.3a) and if clockwise then the sign will be negative (figure 6.3b), and if the point falls on the line, the determinant will have a zero value. Mathematically it can be expressed as:

\[
\Delta = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix}
\]

Or

\[
\Delta = \begin{vmatrix} a_x - c_x & a_y - c_y \\ b_x - c_x & b_y - c_x \end{vmatrix}
\]

Then the position of a point can be determined as (table 6.1). The procedure is repeated for each crack segment, the sign of the function(\(\Delta\)) with minimum magnitude will be the sign of the jump function.

Table 6.1: Algorithm: Orientation test

<table>
<thead>
<tr>
<th>Condition</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta &gt; \text{tol})</td>
<td>Point is left to the line, (H(X) = +1)</td>
</tr>
<tr>
<td>(\Delta &lt; \text{tol})</td>
<td>Point is right to the line, (H(X) = -1)</td>
</tr>
<tr>
<td>(\Delta = \text{tol})</td>
<td>Point is on the line, (H(X) = 0)</td>
</tr>
</tbody>
</table>

6.3.1.2 Level Set representation of crack

Here we discuss the procedure to find the signed distance function and can also be called as a level set function. The evaluation of signed distance function is as follows:

Define a parameter \(r\) as

\[
r = \frac{\vec{AC} \cdot \vec{AB}}{\|\vec{AB}\|^2}
\]
6.3 Evaluation of enrichment functions

Figure 6.4: Signed distance evaluation

To evaluate the parameter $r$, let's define $L = \text{length of line } AB$ (refer to figure 6.4)

$$L = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2} \quad (6.5)$$

Hence we can write $r$ as

$$r = \frac{(C_x - A_x)(C_y - A_y) + (B_x - A_x)(B_y - A_y)}{L^2} \quad (6.6)$$

Then $r$ has the following meaning as given in table 6.2. Next we define another parameter $s$,

Table 6.2: Interpretation of parameter $r$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 0$</td>
<td>$P = A$</td>
</tr>
<tr>
<td>$r = 1$</td>
<td>$P = B$</td>
</tr>
<tr>
<td>$r &lt; 0$</td>
<td>$P$ is on the backward extension of $\vec{AB}$</td>
</tr>
<tr>
<td>$r &gt; 1$</td>
<td>$P$ is on the forward extension of $\vec{AB}$</td>
</tr>
<tr>
<td>$0 &lt; r &lt; 1$</td>
<td>$P$ is on the line $\vec{AB}$</td>
</tr>
</tbody>
</table>

such that

$$s = \frac{\vec{AB} \times \vec{AC}}{||\vec{AB}||^2} \quad (6.7)$$

$$s = \frac{(B_x - A_x)(C_y - A_y) - (C_x - A_x)(B_y - A_y)}{L^2} \quad (6.8)$$

where $s$ has the following meaning as given in table 6.3. The signed distance function can now be evaluated as given in table 6.4. The step function will then easily be computed by finding the sign of the normal level set function $\phi$ as defined in chapter 4.
6.3 Evaluation of enrichment functions

Table 6.3: Interpretation of parameter $s$

| $s > 0$ | $C$ is left of $\vec{AB}$ |
| $s < 0$ | $C$ is right of $\vec{AB}$ |
| $s = 0$ | $C$ is on the $\vec{AB}$ |

Table 6.4: Algorithm Determining signed distance function

1. Find $r$

2. **IF** segment is a tip segment **Then**
   
   $P = A - r(B - A)$

   **Else if** $r \geq 1$ **Then** $P = B$

   **Else if** $r \leq 0$ **Then** $P = A$

   **Else** $P = A - r(B - A)$

   **End IF**

3. Find $\vec{CP}$, the distance

4. Repeat the process for all crack segments and find the minimum $|CP|$ distance

5. **IF** $P$ belongs to tip nodes or within tip segment $(0 < r > 1)$
   
   Find $s$

   Signed distance will be $\text{sign}(s) \cdot |CP|$

   **Else**

   Find $s$ for the two segments sharing that node

   $\text{sign}(s) = +1$ if and only if $s > 0$ for cone of normals at the common node

   Signed distance will be $\text{sign}(s) \cdot |CP|$

   **End IF**
6.3 Evaluation of enrichment functions

6.3.2 Near-Tip enrichment function

Nodes whose nodal closure contains the crack tip are enriched with the four enrichment functions. For review they are mentioned below

\[
\{ F_i(r, \theta) \}_{i=1}^{4} = \left\{ \sqrt{r} \cos \left( \frac{\theta}{2} \right), \sqrt{r} \sin \left( \frac{\theta}{2} \right), \sqrt{r} \sin \theta, \sqrt{r} \cos \left( \frac{\theta}{2} \right) \sin \theta \right\}
\] (6.9)

The enrichment functions are calculated in crack tip polar coordinate system \( r \) and \( \theta \) (figure 6.5). This requires the transformation of gauss points into the crack-tip coordinate system, which is as follows:

![Crack Tip coordinate system](image)

**Figure 6.5:** Crack Tip coordinate system

### 6.3.2.1 Conversion of Gauss points into Global coordinate system

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
x_e \\
y_e
\end{bmatrix} N_I(\xi), \xi = \begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
\]

where \( N_I(\xi) \) is a matrix of element’s shape functions evaluated at gauss points(\( \xi \)). \( x_e, y_e \) denotes element’s coordinates. \( X, Y \) denotes the global coordinates.

### 6.3.2.2 Conversion of global coordinates into crack tip Cartesian coordinates

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{bmatrix} \begin{bmatrix}
X - X_{tip} \\
Y - Y_{tip}
\end{bmatrix}
\]

where \( \alpha \) is the angle of a crack tip segment in global coordinate system with the horizontal (figure 6.5).
6.4 Formation of XFEM N and B matrix

6.3.2.3 Conversion of Cartesian coordinates into polar coordinates

\[ r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left( \frac{y}{x} \right) \]  \hspace{1cm} (6.10)

It is worth mentioning here that, \( \theta \) varies in \( \pm \pi \)

6.4 Formation of XFEM N and B matrix

Construction of XFEM shape functions \( \mathbf{N} \) and discretized gradient operator matrix \( \mathbf{B} \) is straightforward. Where \( \mathbf{N} \) and \( \mathbf{B} \) matrix is given as the concatenation of standard FEM part and an enriched part as explained in section 5.2.

\[
[\mathbf{N}] = \begin{bmatrix} \mathbf{N}_{STD} & \mathbf{N}_{ENR} \end{bmatrix} \]  \hspace{1cm} (6.11)

The \( \mathbf{B} \) matrix will be

\[
[\mathbf{B}] = \begin{bmatrix} \mathbf{B}_{STD} & \mathbf{B}_{ENR} \end{bmatrix} \]  \hspace{1cm} (6.12)

6.4.1 Shape functions

![Figure 6.6: Physical and parent 4 nodded element](image)

For a four nodded isoparametric quadrilateral element, the standard FEM bilinear shape functions associated with each node are given as [Belytchko 1996; Zienkiewicz and
6.4 Formation of XFEM N and B matrix

Taylor 1996; Hjelmstad 2005]

\[ N_1 = \frac{1}{4}(1 - \xi)(1 - \eta) \]  
\[ N_2 = \frac{1}{4}(1 + \xi)(1 - \eta) \]  
\[ N_3 = \frac{1}{4}(1 + \xi)(1 + \eta) \]  
\[ N_4 = \frac{1}{4}(1 - \xi)(1 + \eta) \]  

The displacement approximation can then be written as

\[
\begin{bmatrix}
N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u_x \\ v_y \\ \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy}
\end{bmatrix}
\end{bmatrix} = N_{std} \mathbf{q}
\]

where \( \mathbf{q} \) represents a listing of nodal displacements for a particular element. The standard FEM shape function matrix is given as

\[
N_{STD} = \begin{bmatrix}
N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\]

for a generic enrichment function \( g(X) \), the enriched shape function matrix will be

\[
N_{ENR} = \begin{bmatrix}
N_{1g(X)} & 0 & N_{2g(X)} & 0 & N_{3g(X)} & 0 & N_{4g(X)} & 0 \\
0 & N_{1g(X)} & 0 & N_{2g(X)} & 0 & N_{3g(X)} & 0 & N_{4g(X)}
\end{bmatrix}
\]

where \( g(X) \) could take the following form as given in table 6.5

6.4.2 B operator

The strain component in voigt notations is given as

\[
\epsilon = \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
2\epsilon_{xy}
\end{bmatrix} = Su
\]
### 6.4 Formation of XFEM N and B matrix

#### Table 6.5: Enrichment functions g(X)

<table>
<thead>
<tr>
<th>g(X)</th>
<th>Enrichment type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(X)</td>
<td>step function or Heaviside function</td>
<td>Incorporates strong discontinuity in the approximation field</td>
</tr>
<tr>
<td>(</td>
<td>\phi(x)</td>
<td>)</td>
</tr>
<tr>
<td>(F^4(r, \theta))</td>
<td>near-tip enrichment function</td>
<td>four Crack tip enrichment functions usually used, (equation 6.9)</td>
</tr>
</tbody>
</table>

where S is suitable linear operator. Using the approximation for u we have

\[
e = Bq
\]  

(6.17)

with

\[
B = SN
\]  

(6.18)

The operator S can be defined as

\[
e = \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial y} & 0 \\
\frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial y} & 0 \\
\frac{\partial u_y}{\partial y} & \frac{\partial u_x}{\partial x} & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial u_x}{\partial x} \\
\frac{\partial u_y}{\partial y} \\
\frac{\partial u_y}{\partial x}
\end{bmatrix} = \begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
\]

With the shape functions already determined, the matrix B can easily be obtained as

\[
B_{STD} = \begin{bmatrix}
N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0 & N_{4,x} & 0 \\
0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y} & 0 & N_{4,y} \\
N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & N_{4,y} & N_{4,x}
\end{bmatrix}
\]

The enriched discretized gradient operator is

\[
B_{ENR} = \begin{bmatrix}
(N_{1}g)_{,x} & 0 & (N_{2}g)_{,x} & 0 & (N_{3}g)_{,x} & 0 & (N_{4}g)_{,x} & 0 \\
0 & (N_{1}g)_{,x} & 0 & (N_{2}g)_{,y} & 0 & (N_{3}g)_{,x} & 0 & (N_{4}g)_{,y} \\
(N_{1}g)_{,x} & (N_{1}g)_{,x} & (N_{2}g)_{,y} & (N_{2}g)_{,x} & (N_{3}g)_{,y} & (N_{3}g)_{,x} & (N_{4}g)_{,y} & (N_{4}g)_{,x}
\end{bmatrix}
\]

For the case where g(x) is a Heaviside or step function H(x), the derivative of the enriched term will be

\[
(N_{1}H)_{,j} = N_{1,j}H
\]  

(6.19)
6.4 Formation of XFEM N and B matrix

where I=node number=1,2,3,4 and j=1,2 (indicial notation)

For the case where the enrichment function g(x) is a ramp function \( \varphi(x) = |\phi(x)| \) the derivative of the enriched term is given as

\[
(N_I \varphi)_j = N_{I,j} \varphi + \text{sign}(\phi)\phi_j N_I
\]

(6.20)

where \( \phi_j \) can be given as

\[
\phi(x)_j = \begin{bmatrix} N_{1,j} & N_{2,j} & N_{3,j} & N_{4,j} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}
\]

where \( \phi_1, \phi_2, \phi_3, \phi_4 \) are nodal values of the enrichment function.

For the case where the enrichment function g(X) is near tip enrichment functions \( \{F_l(r, \theta)\}_{l=1}^4 \) we have

\[
(N_I F_l)_j = N_{I,j} F_l + F_{l,j} N_I
\]

(6.21)

The derivatives of the enrichment functions is given in section 6.4.4

6.4.3 Derivatives of shape function

Finite element computation are done using the parent element coordinate. Hence it is necessary to express all computations in terms of parent element. The relation between the derivatives in the parent element and physical coordinates [Belytchko 1996] are

\[
\frac{\partial N_I}{\partial x} = \frac{\partial N_I}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_I}{\partial \eta} \frac{\partial \eta}{\partial x}
\]

(6.22)

\[
\frac{\partial N_I}{\partial y} = \frac{\partial N_I}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_I}{\partial \eta} \frac{\partial \eta}{\partial y}
\]

(6.23)

which can be written as

\[
N_{I,x}^T = \begin{bmatrix} N_I, x & N_I, y \end{bmatrix}
\]

\[
N_{I,x}^T = N_{I,\xi}^T J^{-1} = \begin{bmatrix} N_{I,\xi} & N_{I,\eta} \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}
\]

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6.4 Formation of XFEM N and B matrix

where $J$ is the Jacobian matrix.

The gradient of shape function in parent coordinates is given as

$$N_{1,\xi} = -\frac{1}{4} (1 - \eta) \quad N_{1,\eta} = -\frac{1}{4} (1 - \xi)$$
$$N_{2,\xi} = \frac{1}{4} (1 - \eta) \quad N_{2,\eta} = -\frac{1}{4} (1 + \xi)$$
$$N_{3,\xi} = \frac{1}{4} (1 + \eta) \quad N_{3,\eta} = \frac{1}{4} (1 + \xi)$$
$$N_{4,\xi} = -\frac{1}{4} (1 + \eta) \quad N_{4,\eta} = \frac{1}{4} (1 - \xi)$$

The jacobian of the current configuration w.r.t parent coordinates is then given as

$$J = \begin{bmatrix} x_{\xi} & x_{\eta} \\ y_{\xi} & y_{\eta} \end{bmatrix}$$

with

$$x_{\xi} = \sum_{l=1}^{4} x_l N_{l,\xi} \quad x_{\eta} = \sum_{l=1}^{4} x_l N_{l,\eta} \quad (6.24)$$
$$y_{\xi} = \sum_{l=1}^{4} y_l N_{l,\xi} \quad y_{\eta} = \sum_{l=1}^{4} y_l N_{l,\eta} \quad (6.25)$$

6.4.4 Derivatives of crack tip enrichment functions

The derivatives of the enrichment functions with regard to global coordinates can be evaluated using the chain rule

$$\frac{dF}{dX} = \frac{\partial F}{\partial r} \frac{\partial r}{\partial X} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial X} \quad (6.26a)$$
$$\frac{dF}{dY} = \frac{\partial F}{\partial r} \frac{\partial r}{\partial Y} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial Y} \quad (6.26b)$$

This involves the evaluation of derivatives of enrichment functions in local polar coordinate system. Which are given as

$$\frac{\partial F_1}{\partial r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2} \quad \frac{\partial F_1}{\partial \theta} = -\frac{\sqrt{r}}{2} \sin \frac{\theta}{2}$$
$$\frac{\partial F_2}{\partial r} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2} \quad \frac{\partial F_2}{\partial \theta} = \frac{\sqrt{r}}{2} \cos \frac{\theta}{2}$$
$$\frac{\partial F_3}{\partial r} = \frac{1}{2\sqrt{r}} \sin \frac{\theta}{2} \sin \theta \quad \frac{\partial F_3}{\partial \theta} = \sqrt{r} \left[ \frac{1}{2} \sin \theta \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \cos \theta \right]$$
$$\frac{\partial F_4}{\partial r} = \frac{1}{2\sqrt{r}} \cos \frac{\theta}{2} \sin \theta \quad \frac{\partial F_4}{\partial \theta} = \sqrt{r} \left[ -\frac{1}{2} \sin \theta \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \cos \theta \right]$$

where $r = f(x, y)$ and $\theta = f(x, y)$, hence the derivatives of $r$ and $\theta$ according to global coordinates $X$ and $Y$ can be evaluated as
6.5 Computation of SIFs

\[
\begin{align*}
\frac{\partial r}{X} &= \frac{\partial r}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial r}{\partial y} \frac{\partial y}{\partial X} \\
\frac{\partial \theta}{X} &= \frac{\partial \theta}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial \theta}{\partial y} \frac{\partial y}{\partial X} \\
\frac{\partial \theta}{Y} &= \frac{\partial \theta}{\partial x} \frac{\partial x}{\partial Y} + \frac{\partial \theta}{\partial y} \frac{\partial y}{\partial Y}
\end{align*}
\]

where the derivatives of r and \( \theta \) according to x, y can be given as

\[
\begin{align*}
\frac{\partial r}{\partial x} &= \cos \theta, & \frac{\partial \theta}{\partial x} &= -\frac{\sin \theta}{r} \\
\frac{\partial r}{\partial y} &= \sin \theta, & \frac{\partial \theta}{\partial y} &= \frac{\cos \theta}{r}
\end{align*}
\]

Using the transformation relationship between the global and crack tip coordinates we have

\[
\begin{align*}
\frac{\partial x}{\partial X} &= \cos \alpha, & \frac{\partial x}{\partial Y} &= \sin \alpha \\
\frac{\partial y}{\partial X} &= -\sin \alpha, & \frac{\partial y}{\partial Y} &= \cos \alpha
\end{align*}
\]

6.4.5 Element stiffness matrix

The element stiffness matrix for an enriched element is

\[
K^e = \int_{\Omega^e} B^e \mathbf{C} B^e d\Omega = \int_{-1}^{1} \int_{-1}^{1} B^e(\xi, \eta) \mathbf{C} B^e(\xi, \eta) \det J d\xi d\eta
\]  

(6.27)

Where \( \mathbf{C} \) is the constitutive tangent operator. The stiffness matrix can also be expressed as

\[
K^e = \begin{bmatrix}
\int_{\Omega^e} B^e_{std} \mathbf{C} B^e_{std} d\Omega & \int_{\Omega^e} B^e_{std} \mathbf{C} B^e_{enr} d\Omega \\
\int_{\Omega^e} B^e_{enr} \mathbf{C} B^e_{std} d\Omega & \int_{\Omega^e} B^e_{enr} \mathbf{C} B^e_{enr} d\Omega
\end{bmatrix}
\]

6.5 Computation of SIFs

The stress intensity factors (SIFs) are calculated using domain form of interaction integral. The detail procedure is explained in section 2.4, where we came up with the result

\[
K_I^{(1)} = \frac{E^*}{2} M^{(1, mode I)}
\]

(6.28)

\[
K_{II}^{(1)} = \frac{E^*}{2} M^{(1, mode II)}
\]

(6.29)

where the interaction integral is given as

\[
M = \int_A \left[ \sigma_{ij}^{(1)} \frac{\partial u_j^{(2)}}{\partial x} + \sigma_{ij}^{(2)} \frac{\partial u_j^{(1)}}{\partial x} - W^{(1,2)} \delta_{ij} \right] \frac{\partial q}{\partial x_j} dA
\]

(6.30)

W is the strain energy given as

\[
W = \int_{\Omega} \sigma_{ij} \epsilon_{ij} d\Omega
\]

(6.31)
6.5 Computation of SIFs

6.5.1 Finite element representation of interaction integral

Shape functions for an iso parametric quadrilateral element is given in section 6.4, where the integration is performed on gauss points \((\xi, \eta)\). The coordinates \((x_1, x_2)\) in the physical space and the displacements \((u_1, u_2)\) is then given as

\[
X_j = \sum_{I=1}^{4} N_I X_{jI}, \quad u_j = \sum_{I=1}^{4} N_I U_{jI}
\]  

(6.32)

where \(X_{jI}\) and \(U_{jI}\) are the nodal coordinates and and nodal displacements respectively. \(j\) is the indicial notation \((j=1,2)\). The interaction integral will then be given as

\[
M = \sum_{\text{domain \text{A \ text{elem \text{gp}}}}=1}^{n_{\text{gp}}} \sum_{\text{gp}} \left\{ \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x} - W^{(1,2)} \delta_{ij} \right\} \frac{\partial q}{\partial x_j} \det \left( \frac{\partial x_k}{\partial \xi} \right) (gw_p)
\]  

(6.33)

Within an element the weighting function is approximated using the same bilinear element’s shape functions as

\[
q = \sum_{I=1}^{4} N_I q_I
\]  

(6.34)

where \(N_I\) and \(q_I\) are the nodal values of the shape function and weighting function respectively. The gradient of weighting function is evaluated as follows

\[
\frac{\partial q}{\partial X_j} = \sum_{I=1}^{4} N_{I,j} q_I
\]  

(6.35)

\[
(6.36)
\]

\(N_{I,j}\) are the usual derivatives of the shape functions. It should be noted here that, the interaction integral requires quantities in local Cartesian crack tip coordinate system, hence the factor \(\partial q/\partial X_j\) will be transformed into local coordinate system as

\[
[q_{,x_j}] = [R] [q_{,X_j}]
\]  

(6.37)

where \(R\) is the rotation matrix. More explicitly, we can also write the above equation as

\[
\begin{bmatrix}
\frac{\partial q}{\partial x} \\
\frac{\partial q}{\partial y}
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\frac{\partial q}{\partial X} \\
\frac{\partial q}{\partial Y}
\end{bmatrix}
\]

Now considering state (2) as pure mode II, we shall now compute the state 1 and 2 parameters
6.5 Computation of SIFs

6.5.2 Parameters of state 1

The quantities $\sigma_{ij}^{(1)}, u_i^{(1)}, u_{ij}^{(1)}$ are known from the extended finite element analysis. The superscript (1) denotes state 1.

6.5.2.1 Displacement gradients

The obtained quantities from the extended finite element analysis are in global coordinate system. However the evaluation of interaction integral requires the quantities in the local crack tip coordinate system. The gradient of displacement will be transformed into crack tip coordinate system as

$$
[u_{i,j}]^{(1)} = [R] [u_{I,J}]^{(1)} [R]^T
$$

where $R$ is the rotation matrix, $\alpha$ is the angle of the crack tip segment with the global X-axis. The small subscripts (i,j and x,y) represents the local crack tip coordinate system, while the capital subscripts (I,J and X,Y) represents the global coordinate system.

6.5.2.2 Stresses

The transformation of stress field from global into local crack tip coordinate will be as follows

$$
\sigma_{xx} = \left( \frac{\sigma_{XX} + \sigma_{YY}}{2} \right) + \left( \frac{\sigma_{XX} - \sigma_{YY}}{2} \right) \cos2\alpha + \sigma_{XY}\sin2\alpha
$$

$$
\sigma_{yy} = \left( \frac{\sigma_{XX} + \sigma_{YY}}{2} \right) - \left( \frac{\sigma_{XX} - \sigma_{YY}}{2} \right) \cos2\alpha - \sigma_{XY}\sin2\alpha
$$

$$
\sigma_{xy} = -\left( \frac{\sigma_{XX} - \sigma_{YY}}{2} \right) \sin2\alpha + \sigma_{XY}\cos2\alpha
$$

6.5.2.3 Strains

Under the assumption of small displacement gradients the strain is approximated as the symmetric part of the displacement gradient

$$
\varepsilon_{ij}^{(1)} = \nabla_s u = \frac{1}{2} \left( u_{i,j}^{(1)} + u_{j,i}^{(1)} \right)
$$
where $u_{i,j}^{(1)}$ is the gradient of displacement in local crack tip coordinate system as obtained in the previous section. In matrix form we can write as

$$
\epsilon_{i,j}^{(1)} = \frac{1}{2} \begin{bmatrix}
u_{x,x}^{(1)} + u_{x,y}^{(1)} & u_{y,x}^{(1)} + u_{y,y}^{(1)} \\
u_{y,x}^{(1)} + u_{y,y}^{(1)} & u_{y,y}^{(1)} + u_{y,y}^{(1)}
\end{bmatrix}
$$

### 6.5.3 Parameters of state 2

State 2 parameters are obtained by assuming the state 2 to be pure mode II, hence

$$
K_{II}^{(2)} = 1, \quad K_{I}^{(2)} = 0
$$

The superscript (2) denotes state 2.

#### 6.5.3.1 Displacement and displacement gradient

Incorporating the above values of stress intensity factors into the crack displacement field described in chapter 1, we get

$$u_x^{(2)} = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} [\kappa + 2 + \cos \theta]$$

$$u_y^{(2)} = -\frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} [\kappa - 2 + \cos \theta]$$

It should be noted here that the displacement field is a function of crack tip polar coordinate system and we are required to find the spatial derivatives according to the local crack tip cartesian coordinate system. This will be evaluated as follows

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left( \frac{y}{x} \right)$$

$$r, x = \cos \theta, \quad \theta, x = -\frac{\sin \theta}{r}$$

$$r, y = \sin \theta, \quad \theta, y = \frac{\cos \theta}{r}$$

The gradients of displacements with respect to polar coordinate system, $r$ and $\theta$ will be

$$u_{x,r}^{(2)} = \frac{1}{4\mu \sqrt{2\pi r}} \sin \frac{\theta}{2} [\kappa + 2 + \cos \theta]$$

$$u_{x,\theta}^{(2)} = \frac{1}{2\mu \sqrt{2\pi}} \left[ \frac{1}{2} \cos \frac{\theta}{2} [\kappa + 2 + \cos \theta] - \sin \frac{\theta}{2} \sin \theta \right]$$

$$u_{y,r}^{(2)} = \frac{1}{4\mu \sqrt{2\pi r}} \cos \frac{\theta}{2} [\kappa - 2 + \cos \theta]$$

$$u_{y,\theta}^{(2)} = -\frac{1}{2\mu \sqrt{2\pi}} \left[ -\frac{1}{2} \sin \frac{\theta}{2} [\kappa - 2 + \cos \theta] - \cos \frac{\theta}{2} \sin \theta \right]$$
6.6 Modified domain for J-integral computation

The gradient of displacements according to local Cartesian crack tip coordinate system will be

\[ u_{x,x}^{(2)} = u_{x,r}^{(2)} r_x + u_{x,\theta}^{(2)} \theta_x \]  
(6.48a)

\[ u_{x,y}^{(2)} = u_{x,r}^{(2)} r_y + u_{x,\theta}^{(2)} \theta_y \]  
(6.48b)

\[ u_{y,x}^{(2)} = u_{y,r}^{(2)} r_x + u_{y,\theta}^{(2)} \theta_x \]  
(6.48c)

\[ u_{y,y}^{(2)} = u_{y,r}^{(2)} r_y + u_{y,\theta}^{(2)} \theta_y \]  
(6.48d)

6.5.3.2 Stresses

The stress field for the state 2 is obtained by incorporating the above values of stress intensity factors into the equations for stress field as explained in chapter 2.

\[ \sigma_{xx}^{(2)} = -\frac{1}{\sqrt{2\pi r}} \sin \theta \left[ 2 + \cos \theta \cos \frac{3\theta}{2} \right] \]  
(6.49a)

\[ \sigma_{yy}^{(2)} = \frac{1}{\sqrt{2\pi r}} \sin \theta \cos \frac{3\theta}{2} \]  
(6.49b)

\[ \sigma_{xy}^{(2)} = \frac{1}{\sqrt{2\pi r}} \cos \frac{3\theta}{2} \left[ 1 - \sin \frac{\theta}{2} \sin \frac{\theta}{2} \right] \]  
(6.49c)

6.5.3.3 Strains

Using the assumption of small displacement gradient the strains are given as

\[ \epsilon_{ij}^{(2)} = \nabla_s u = \frac{1}{2} \left( u_{i,j}^{(2)} + u_{j,i}^{(2)} \right) \]  
(6.50)

In matrix form we can write as

\[ \epsilon_{i,j}^{(2)} = \frac{1}{2} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} u_{x,x}^{(2)} + u_{x,y}^{(2)} \\ u_{x,y}^{(2)} + u_{y,x}^{(2)} \\ u_{y,y}^{(2)} + u_{y,y}^{(2)} \end{bmatrix} \]

In order to evaluate the mode I stress intensity factors, repeat the procedure where state 1 parameters will remain same however re-evaluate the state 2 parameters making a judicious choice

\[ K_{II}^{(2)} = 0, \quad K_I^{(2)} = 1 \]  
(6.51)

6.6 Modified domain for J-integral computation

It is important to mention here that, in our implementation, the interaction integral near the boundary of the domain is computed with a modified procedure. Usually in order to compute the
interaction integral a domain with radius $r_d$ around the crack tip is selected. A weight function $q$ is then defined, which gets a value of 1 inside the domain and 0 otherwise. The interaction integral equation (equation 6.33) contains a term $dq/dx_j$. This means integral will only be evaluated, when $dq/dx_j \neq 0$. Hence the integral is evaluated automatically around a contour surrounding the crack tip/point of singularity (figure 6.7(a), 6.7(b)). In case where the crack is near to the boundary of the body, the domain for the evaluation of interaction integral might not lie completely inside the body. In such case the integral will be evaluated along the contour which is not surrounding the crack tip or point of singularity (figure 6.7(c), 6.7(d)). The evaluated integral will not be a representative of the crack tip energy. This in turn will result in wrong computations of stress intensity factors, wrong estimates for the crack propagation criteria and crack propagation direction (if the method is based on J-integral computation), and hence the overall results of the analysis will be effected. In order to eliminate the above discrepancies, we used a modified path for the computation of interaction integral (figure 6.7(e), 6.7(f)). The approach is very simple in its implementation, which requires one extra step to perform. First step being the same as to assign a value of 1 to a weight function for a node lying inside the radius $r_d$ and 0 otherwise. Next assign a value of 0 to a weight function for all boundary nodes. This will automatically computes the integral over a contour surrounding the crack tip. This has been demonstrated in the figure (6.7). Left column of the figures shows the values of weight function, where the red asterisk denotes a node having $q = 0$ and green asterisk represents $q = 1$. Right column of the figure shows the contour with a rainbow of colors, along which the interaction integral will be evaluated.
6.6 Modified domain for J-integral computation

Figure 6.7: Modified Path for M-integral, figures (a),(c),(e) shows the weight function $q$ for different crack tip positions, Figures (b),(d), and (f) shows the Paths for evaluation of M-integral
Chapter 7

Numerical Examples

7.1 Cracked 1D truss member

Consider a 1D bar of length 3L. Let $E$ be the elastic moduli and $A$ be the cross-sectional area of the bar. The bar is subjected to a prescribed displacement at the end while the other end of the bar is fixed. The bar is cracked at its mid length, $L=1.5L$. The geometry of the bar with boundary conditions is shown in the figure 7.1.

![Figure 7.1: 1D Cracked truss member](image)

7.1.1 Standard FEM solution with non-aligned mesh

The problem could be solved using a standard finite element method. In that case the finite element mesh has to be aligned with the crack as shown in the figure 7.2a. But if there is non-aligned mesh as shown in figure 7.2b, the finite element method fails to locate and identify the discontinuity present within the domain and thus gives a wrong solution. Analyzing the cracked 1D truss example by FEM, using the non-aligned mesh, with four unknowns ($u_1$, $u_2$, $u_3$, $u_4$),
7.1 Cracked 1D truss member

![FEM discretization of 1D truss member](image1)

![XFEM discretization of 1D truss member](image2)

Figure 7.2: FEM and XFEM mesh discretization

Figure 7.3: Degrees of freedom associated with each node

u3 and u4) as shown in the figure7.3, the stiffness matrix of the structure is then given as

\[
K = \frac{EA}{L} \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

The linear system of equation in matrix form can then be written as

\[
\frac{EA}{L} \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix} = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4
\end{bmatrix}
\]
7.1 Cracked 1D truss member

Solving the system of equation by applying the natural and essential boundary boundary conditions we have

\[
\frac{EA}{L} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\bar{u} \\
\end{bmatrix}
\]

\[u_1 = 0, \quad u_2 = \frac{\bar{u}}{3}, \quad u_3 = \frac{2\bar{u}}{3}, \quad u_4 = \bar{u}\]

7.1.2 XFEM solution with non-aligned mesh

In order to solve the problem using XFEM, we use the same non-aligned mesh discretization with three elements as shown in the figure 7.2b. Since there is a strong discontinuity in the displacement field at the mid length of the bar, we used step function as an enrichment function. The standard degrees of freedom are denoted by \(u\) and enriched degrees of freedom are represented as \(a\) (figure 7.4). There are are four standard degrees of freedom, one at each node. The nodal support of node \(x_2\) and \(x_3\) are cut by the crack, hence we enriched these nodes with a step function. Therefore nodes \(x_2\) and \(x_3\) has two degree of freedoms each, one standard and one enriched. The structure now hold a total of six degrees of freedom, four standard and two enriched degree of freedoms. This is illustrated in figure 7.4. For any element the XFEM stiffness matrix can be written as

\[K_e = \begin{bmatrix}
K_{uu} & K_{ua} \\
K_{au} & K_{aa} \\
\end{bmatrix}\]

where

\[K_{uu} = \int_0^L (B_{std}^u)^T B_{std}^u \, dx\]

\[K_{ua} = \int_0^L (B_{std}^u)^T B_{enr}^a \, dx\]

\[K_{ua} = \int_0^L (B_{enr}^a)^T B_{std}^u \, dx\]

\[K_{aa} = \int_0^L (B_{enr}^a)^T B_{enr}^a \, dx\]
7.1 Cracked 1D truss member

(a) Discretized domain used for analysis

(b) Standard and enriched degrees of freedom associated with each node

(c) Step function over the support of node $x_2$

(d) Step function over the support of node $x_3$

Figure 7.4: 1D discretized truss member used for XFEM analysis

7.1.2.1 Element No.1, $\Omega_1$

The enrichment function $H(X) = +1$ for $x \in \Omega_1$

\[ N_{std}^u = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \quad N_{enr}^a = H \left[ \frac{x}{L} \right] = \left[ \frac{x}{L} \right] \]

\[ B_{std}^u = \begin{bmatrix} -1 \frac{1}{L} \end{bmatrix} \quad B_{enr}^a = H \left[ \frac{1}{L} \right] = \left[ \frac{1}{L} \right] \]

\[ K_{uu} = EA \int_0^L (B_{std}^u)^T B_{std}^u \, dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ K_{ua} = EA \int_0^L (B_{std}^u)^T B_{enr}^a \, dx = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]
7.1 Cracked 1D truss member

further $K_{uu} = K_{ua}^T$

$$K_{aa} = EA \int_0^L (B_{enr}^a)^T B_{enr}^a \, dx = \frac{EA}{L}$$
$$K_e = \frac{EA}{L} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

7.1.2.2 Element No.2, $\Omega_2$

The enrichment function $H(X)$ over the element no.2 for both the nodes can be defined as

$$H(X) = \begin{cases} +1 & \text{for } x \in \Omega_2^+ \\ -1 & \text{for } x \in \Omega_2^- \end{cases}$$

$$N_{std}^u = \left[ 1 - \frac{x}{L} \frac{x}{L} \right] \quad N_{enr}^a = H \left[ 1 - \frac{x}{L} \frac{x}{L} \right]$$

$$B_{std}^u = \left[ -1 \frac{1}{L} \right] \quad B_{enr}^a = H \left[ -1 \frac{1}{L} \right]$$

$$K_{uu} = EA \int_0^L (B_{std}^u)^T B_{std}^u \, dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

As the element no.2 contains a discontinuity, therefore in order to integrate properly on both sides of the discontinuity, Integration is performed separately on $\Omega_2^+$ and $\Omega_2^-$. It should be emphasized here that this partition is only for integration purpose and no extra degrees of freedoms are added to the system. However as $K_{uu}$ term of the stiffness matrix was a standard FEM part and did not contain any discontinuous/enriched term therefore the integration was performed over the whole domain $\Omega_2$ all at once, without dividing the domain into two domains i.e $\Omega_2^+$ and $\Omega_2^-$. In order to solve other terms of the stiffness matrix i.e $K_{uu}$, $K_{au}$, $K_{aa}$, the integration will be performed separately on the two domains $\Omega_2^+$ and $\Omega_2^-$. Then we can write

$$K_{uu} = K_{uu}^+ + K_{uu}^-$$
$$K_{uu} = K_{uu}^+ + K_{uu}^-$$
$$K_{aa} = K_{aa}^+ + K_{aa}^-$$
7.1 Cracked 1D truss member

Integrating on $\Omega_2^+$
Use $H(X) = +1$

\[
K_{ua}^+ = EA \int_0^{L/2} (B_{std}^u)^T B_{enr}^a \, dx = \frac{EA}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

\[
K_{aa}^+ = EA \int_0^{L/2} (B_{enr}^a)^T B_{enr}^a \, dx = \frac{EA}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

Integrating on $\Omega_2^-$
Use $H(X) = -1$

\[
K_{ua}^- = EA \int_{L/2}^L (B_{std}^u)^T B_{enr}^a \, dx = \frac{EA}{2L} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\]

\[
K_{aa}^- = EA \int_{L/2}^L (B_{enr}^a)^T B_{enr}^a \, dx = \frac{EA}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

Now combining the results of integration on both sides of discontinuity to find $K_{ua}$ and $K_{aa}$

\[
K_{ua} = \frac{EA}{2L} \begin{bmatrix} 1 & -1 & -1 + 1 \\ -1 + 1 & 1 & -1 \end{bmatrix} = \frac{EA}{2L} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
K_{ua} = \frac{EA}{2L} \begin{bmatrix} 1 + 1 & -1 & -1 \\ -1 & 1 + 1 & -1 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

and $K_{aa} = K_{ua}^T$. The element stiffness matrix can be written as

\[
K_{e2} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\]

7.1.2.3 Element No.3, $\Omega_3$

The enrichment function $H(X) = -1$ for $x \in \Omega_3$

\[
N_{std}^u = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \quad N_{enr}^a = H \begin{bmatrix} 1 - \frac{x}{L} \end{bmatrix} = \frac{x}{L} - 1
\]
7.1 Cracked 1D truss member

\[ B_{\text{std}}^u = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{T} \end{bmatrix} \quad B_{\text{enr}}^a = H \begin{bmatrix} -1 \\ \frac{1}{T} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ K_{uu} = EA \int_0^L (B_{\text{std}}^u)^T B_{\text{std}}^u \, dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ K_{ua} = EA \int_0^L (B_{\text{std}}^u)^T B_{\text{enr}}^a \, dx = \frac{EA}{L} \begin{bmatrix} -1 \end{bmatrix} \]

\[ K_{au} = K_{ua}^T \]

\[ K_{aa} = EA \int_0^L (B_{\text{enr}}^a)^T B_{\text{enr}}^a \, dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \]

The structure stiffness matrix can now be written as

\[ K = \frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 & -1 & 2 \end{bmatrix} \]

The system of equations to solve after the application of necessary and essential boundary condition are

\[ \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{u} \\ \bar{a} \\ -\bar{u} \end{bmatrix} \]

The nodal displacements are

\[ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\bar{u}}{2} \\ \frac{\bar{u}}{2} \\ \bar{u} \\ -\frac{\bar{u}}{2} \\ -\frac{\bar{u}}{2} \end{bmatrix} \]
7.1 Cracked 1D truss member

One can easily see from the results that, the XFEM approximation is not a nodal interpolant. The nodal displacements can now be found by using the XFEM enriched displacement approximation i.e.

\[ u(X) = N_i u_i + H N_j a_j \]

In our current problem representation \( N_i = N_{std}^u \) and \( H N_j = N_{enr}^a \)

\[
\begin{align*}
    u(x_1) &= u_1 = 0 \\
    u(x_2) &= u_2 + H(x_2)a_1 = \frac{\bar{u}}{2} - \frac{\bar{u}}{2} = 0 \\
    u(x_3) &= u_3 + H(x_3)a_2 = \frac{\bar{u}}{2} + (-1) \left( -\frac{\bar{u}}{2} \right) = \bar{u} \\
    u(x_4) &= u_4 = \bar{u}
\end{align*}
\]

The numerical results are shown in figure 7.5 and 7.6, where \( u_{std}(X) = N_i u_i \) and \( u_{enr}(X) = H N_j a_j \)

**Figure 7.5:** Numerical solution of displacement field using XFEM

\[ H N_j a_j \text{ and } u(X) = N_i u_i + H N_j a_j. \]
7.2 Cohesive crack in 1D truss member

Consider 1D truss member of length 3L. Let $E$ be the elastic moduli and $A$ be the cross-sectional area of the bar. The material is considered to be linear elastic, hence obeys the Hook’s law. The bar is subjected to load $P$ at one end, while the other end is fixed. The bar deforms under the action of increasing load $P$, and at a certain load $P = P_{cr}$ the member loses its initial stiffness properties and attains a reduced stiffness value say $k$ at point $L = 1.5L$. The material response in this case at point $L = 1.5L$ is therefore bilinear, as shown in the figure 7.8. The geometry of the problem is shown in figure 7.7. The numerical model for the analysis is shown in figure 7.4.

Analysis of the problem can be done using the standard finite element method (FEM). However it requires the mesh to conform the geometry of the discontinuity. For an accurate analysis the mesh is required to be aligned with the spring as shown in the figure 7.2a. If a non-aligned mesh is used then FEM analysis fails to incorporate the contribution of a spring and thus will give erroneous results.
7.2 Cohesive crack in 1D truss member

![Figure 7.8: 1D truss member with a cohesive crack at the middle](image)

7.2.1 XFEM solution with non-aligned mesh

The problem described above could be considered as cohesive crack analysis problem. XFEM methodology for solving cohesive crack problems in 2D domains has already been discussed in section 5.9, where we came up with the discretized form of equilibrium equation of the body. For revision it is given below as well

\[
\begin{bmatrix}
\int_{\Omega} (B^a_{\text{std}})^T \sigma \\
\int_{\Omega} (B^a_{\text{enr}})^T \sigma + \int_{\Gamma_{\text{coh}}} (\bar{N}^a_{\text{enr}})^T t_c \\
\int_{\Omega} (B^b_{\text{enr}})^T \sigma + \int_{\Gamma_{\text{coh}}} (\bar{N}^b_{\text{enr}})^T t_c 
\end{bmatrix} = \begin{bmatrix}
\int_{\Gamma_{\text{t}}} (N^a_{\text{std}})^T \bar{t} \\
\int_{\Gamma_{\text{t}}} (N^a_{\text{enr}})^T \bar{t} \\
\int_{\Gamma_{\text{t}}} (N^b_{\text{enr}})^T \bar{t}
\end{bmatrix}
\]

As we are dealing with a 1D element, incorporation of a crack inside a domain causes a strong discontinuity, hence a step function will be used as an only enrichment function for the analysis. Therefore the above equation is simplified to

\[
\begin{bmatrix}
\int_{\Omega} (B^a_{\text{std}})^T \sigma \\
\int_{\Omega} (B^a_{\text{enr}})^T \sigma + \int_{\Gamma_{\text{coh}}} (\bar{N}^a_{\text{enr}})^T t_c 
\end{bmatrix} = \begin{bmatrix}
\int_{\Gamma_{\text{t}}} (N^a_{\text{std}})^T \bar{t} \\
\int_{\Gamma_{\text{t}}} (N^a_{\text{enr}})^T \bar{t}
\end{bmatrix}
\]

As the material properties are considered to be linear elastic, therefore the material constitutive law can be expressed as

\[
\sigma = D\epsilon 
\]

where \(D\) is the elastic modulus tensor. In our 1D case \(D = EA\). Further assuming the problem to be a small strain, small displacement problem, the strain displacement relationship is given as

\[
\epsilon = \nabla^s u 
\]
7.2 Cohesive crack in 1D truss member

Additionally as we stated earlier, that the material behavior at the discontinuity is assumed to be a bilinear with a reduced stiffness k, hence we can define the traction separation law as

\[ t_c = kU \]  \hspace{1cm} (7.3)

Using the material constitutive and traction separation laws mentioned above the equilibrium equation specializes to

\[
\begin{bmatrix}
\int_\Omega (B_{\text{std}}^u)^T D \nabla u^h \\
\int_\Omega (B_{\text{enr}}^a)^T D \nabla u^h + \int_{\Gamma_{\text{coh}}} \bar{N}_{\text{enr}}^a)^T k \bar{U}(u^h)
\end{bmatrix}
= \begin{bmatrix}
\int_{\Gamma_t} \bar{N}_{\text{std}}^u)^T \bar{t} \\
\int_{\Gamma_t} \bar{N}_{\text{enr}}^a)^T \bar{t}
\end{bmatrix}
\]

Now incorporating the XFEM test function into the above equilibrium equations we get

\[
\begin{bmatrix}
\int_\Omega (B_{\text{std}}^u)^T D (B_{\text{std}}^u u_j + B_{\text{enr}}^a a_j) \\
\int_\Omega (B_{\text{enr}}^a)^T D (B_{\text{std}}^u u_j + B_{\text{enr}}^a a_j) + \int_{\Gamma_{\text{coh}}} \bar{N}_{\text{enr}}^a k (\bar{N}_{\text{enr}}^a a_j)
\end{bmatrix}
= \begin{bmatrix}
\int_{\Gamma_t} \bar{N}_{\text{std}}^u)^T \bar{t} \\
\int_{\Gamma_t} \bar{N}_{\text{enr}}^a)^T \bar{t}
\end{bmatrix}
\]

or we can write

\[
\begin{bmatrix}
\int_\Omega (B_{\text{std}}^u)^T D B_{\text{std}}^u \\
\int_\Omega (B_{\text{enr}}^a)^T D B_{\text{std}}^u \\
\int_\Omega (B_{\text{enr}}^a)^T D B_{\text{enr}}^a \\
\int_{\Gamma_{\text{coh}}} \bar{N}_{\text{enr}}^a k (\bar{N}_{\text{enr}}^a a_j)
\end{bmatrix}
\begin{bmatrix}
u_j \\
a_j
\end{bmatrix}
= \begin{bmatrix}
\int_{\Gamma_t} \bar{N}_{\text{std}}^u)^T \bar{t} \\
\int_{\Gamma_t} \bar{N}_{\text{enr}}^a)^T \bar{t}
\end{bmatrix}
\]

In a more compact or familiar form we can write then

\[
\begin{bmatrix}
K_{uu}^{ij} & K_{ua}^{ij} \\
K_{au}^{ij} & K_{aa}^{ij}
\end{bmatrix}
\begin{bmatrix}
u_j \\
a_j
\end{bmatrix}
= \begin{bmatrix}
f_u^i \\
f_a^i
\end{bmatrix}
\]

The term \( K_{aa} \) is of significance here. As one can note, if the cohesive integral term from \( K_{aa} \) is removed then the above mentioned stiffness matrix reduces to a standard XFEM matrix for traction free crack.

7.2.2 XFEM analysis for 1D truss member with cohesive crack

7.2.2.1 Element No.1, \( \Omega_1 \)

The enrichment function \( H(X) = +1 \) for \( x \in \Omega_1 \)

\[
N_{\text{std}}^u = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \quad N_{\text{enr}}^a = H \begin{bmatrix} \frac{x}{L} \end{bmatrix} = \begin{bmatrix} \frac{x}{L} \end{bmatrix}
\]

\[
B_{\text{std}}^u = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad B_{\text{enr}}^a = H \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}
\]

\[
K_{uu} = EA \int_0^L (B_{\text{std}}^u)^T B_{\text{std}}^u dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\
-1 & 1 \end{bmatrix}
\]

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7.2 Cohesive crack in 1D truss member

\[ K_{ua} = EA \int_0^L (B_{std}^a)^T B_{enr}^a dx = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

\[ K_{aa} = EA \int_0^L (B_{enr}^a)^T B_{enr}^a dx + \int_{\Gamma_{coh}} (\bar{N}_{enr}^a)^T k \bar{N}_{enr}^a d\Gamma = \frac{EA}{L} \]

As the element does not contain a discontinuity, hence the second term of \( K_{aa} \) becomes zero.

The element stiffness matrix can now be assembled and written as

\[ K_{e1} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \]

7.2.2.2 Element No.2, \( \Omega_2 \)

The enrichment function \( H(X) \) over the element no.2 for both the nodes can be defined as

\[ H(X) = \begin{cases} +1 & \text{for } x \in \Omega_2^+ \\ -1 & \text{for } x \in \Omega_2^- \end{cases} \]

\[ N_{std}^a = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \quad N_{enr}^a = H \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \]

\[ B_{std}^a = \begin{bmatrix} -1 \\ \frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \quad B_{enr}^a = H \begin{bmatrix} -1 \\ \frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \]

\[ K_{ua} = EA \int_0^L (B_{std}^a)^T B_{std}^a dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\( K_{ua} \) will be evaluated on both sides of the crack independently as done previously.

\[ K_{ua} = EA \int_0^L (B_{std}^a)^T B_{enr}^a dx = \frac{EA}{L} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ K_{aa} = \int_{\Omega} (B_{enr}^a)^T D B_{enr}^a d\Omega + \int_{\Gamma_{coh}} (\bar{N}_{enr}^a)^T k \bar{N}_{enr}^a d\Gamma \]

\[ K_{aa} = \int_{\Omega^+} (B_{enr}^a)^T D B_{enr}^a d\Omega + \int_{\Omega^-} (B_{enr}^a)^T D B_{enr}^a d\Omega + \int_{\Gamma_{coh}} (\bar{N}_{enr}^a)^T k \bar{N}_{enr}^a d\Gamma \]

\[ K_{aa} = \frac{EA}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EA}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \left\{ -2 \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \right\} k \left\{ -2 \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \right\} \]
\[ K_{aa} = \frac{EA}{2L} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} + 4k \begin{bmatrix} (1 - \frac{x}{L})^2 & (1 - \frac{x}{L})(\frac{x}{L}) \\ (\frac{x}{L})(1 - \frac{x}{L}) & (\frac{x}{L})^2 \end{bmatrix} \bigg\vert_{x=\frac{L}{2}} \]

\[ K_{aa} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

Now the element stiffness matrix can be assembled and written as

\[
K_{e2} = \begin{bmatrix}
\frac{EA}{L} & -\frac{EA}{L} & 0 & 0 \\
-\frac{EA}{L} & \frac{EA}{L} & 0 & 0 \\
0 & 0 & \frac{EA}{L} + k & -\frac{EA}{L} + k \\
0 & 0 & -\frac{EA}{L} + k & \frac{EA}{L} + k
\end{bmatrix}
\]

### 7.2.2.3 Element No.3

The enrichment function \( H(X) = -1 \) for \( x \in \Omega \)

\[ N_{std} = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \quad N_{enr} = H \begin{bmatrix} 1 - \frac{x}{L} \end{bmatrix} = \begin{bmatrix} \frac{x}{L} - 1 \end{bmatrix} \]

\[ B_{std} = \begin{bmatrix} \frac{1}{L} \\ \frac{-1}{L} \end{bmatrix} \quad B_{enr} = H \begin{bmatrix} \frac{-1}{L} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{L} \end{bmatrix} \]

\[ K_{uu} = EA \int_0^L B_{std}^T B_{std} dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ K_{ua} = EA \int_0^L B_{std}^T B_{enr} dx = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

\[ K_{aa} = EA \int_0^L B_{enr}^T B_{enr} dx + \int_{\Gamma_{coh}} \bar{N}^T \bar{k} \bar{N} d\Gamma = \frac{EA}{L} \]

As the element does not contain a discontinuity, hence the second term of \( K_{aa} \) becomes zero. The element stiffness matrix can now be assembled and written as

\[
K_{e3} = \frac{EA}{L} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\]
7.2 Cohesive crack in 1D truss member

The assembled structure stiffness matrix can be written as

\[
K_{Str} = \begin{bmatrix}
\frac{EA}{L} & -\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 \\
-\frac{EA}{L} & \frac{2EA}{L} & -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
0 & -\frac{EA}{L} & \frac{2EA}{L} & -\frac{EA}{L} & 0 & -\frac{EA}{L} \\
0 & 0 & -\frac{EA}{L} & \frac{EA}{L} & 0 & \frac{EA}{L} \\
-\frac{EA}{L} & \frac{EA}{L} & 0 & 0 & 2\frac{EA}{L} + k & -\frac{EA}{L} + k \\
0 & 0 & -\frac{EA}{L} & \frac{EA}{L} & -\frac{EA}{L} + k & 2\frac{EA}{L} + k
\end{bmatrix}
\]

The system of equations to be solved in matrix form is

\[
K_{Str} = \begin{bmatrix}
\frac{EA}{L} & -\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 \\
-\frac{EA}{L} & \frac{2EA}{L} & -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
0 & -\frac{EA}{L} & \frac{2EA}{L} & -\frac{EA}{L} & 0 & -\frac{EA}{L} \\
0 & 0 & -\frac{EA}{L} & \frac{EA}{L} & 0 & \frac{EA}{L} \\
-\frac{EA}{L} & \frac{EA}{L} & 0 & 0 & 2\frac{EA}{L} + k & -\frac{EA}{L} + k \\
0 & 0 & -\frac{EA}{L} & \frac{EA}{L} & -\frac{EA}{L} + k & 2\frac{EA}{L} + k
\end{bmatrix}\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2
\end{bmatrix} = \begin{bmatrix}
0 \\ 0 \\ 0 \\ P \\ a_1 \\ a_2
\end{bmatrix}
\]

Assuming \(\frac{EA}{L} = 1\), \(k = 3\) and \(P = 4\) and applying the natural and essential boundary conditions to solve the matrix for the required unknowns we have:

\[
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2
\end{bmatrix} = \begin{bmatrix}
0 \\ 4.6667 \\ 8.6667 \\ 13.3333 \\ -0.6667 \\ -0.6667
\end{bmatrix}
\]

One can easily see from the results the XFEM approximation is not a nodal interpolant. The nodal displacements can now be found by using the XFEM enriched displacement approximation i.e

\[
u(X) = N_i u_i + H_j a_j
\]
7.2 Cohesive crack in 1D truss member

In our current problem representation $N_i = N_{std}^u$ and $H N_j = N_{enr}^a$

$$u(x_1) = u_1 = 0$$
$$u(x_2) = u_2 + H(x_2)a_1 = 4.6667 - 0.6667 = 4$$
$$u(x_3) = u_3 + H(x_3)a_2 = 8.6667 + (-1)(-0.6667) = 9.333$$
$$u(x_4) = u_4 = 13.333$$

Figures 7.9 and 7.10 shows the analysis results, where $u_{std}(X) = N_i u_i$ and $u_{enr}(X) = H N_j a_j$

**Figure 7.9:** Numerical solution of cohesive cracked axial member using XFEM

**Figure 7.10:** Numerical solution of cohesive cracked axial member using FEM

and $u(X) = N_i u_i + H N_j a_j$. 

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7.3 Modeling 2D Crack problems

7.3.1 Center edge crack in finite dimensional plate under tension

In order to test the robustness and efficiency of the method standard test problems were performed. The first test example is a finite dimensional plate with a center edge crack. The dimensions of the plate are \( w = 1, L = 2 \). An initial crack of length \( a = w/2 \) is introduced at the center of the plate. The geometry of the body and crack is shown in the figure 7.11.

![Finite dimensional plate with edge crack](image1)

![Numerical model](image2)

**Figure 7.11:** Numerical model and geometry of edge crack problem

The problem is a typical mode I failure problem. Stress intensity factor \( K_I \) is evaluated using XFEM analysis and compared with the exact solution available in the literature [Yau et al. 1980]. The stress intensity factors for an infinite plate subjected to a uniform stress with a crack in the center is given by

\[
K_I = \sigma_o \sqrt{\pi a}
\]
7.3 Modeling 2D Crack problems

where \( a \) is the length of the crack.

In order to take into account the finiteness of the body a correction factor \( C \) given in Ewalds and Wanhill [1984], is used

\[
KI = C \sigma_0 \sqrt{\pi a} \quad \text{(7.4)}
\]

\[
C = 1.12 - 0.231 \left( \frac{a}{w} \right) + 10.55 \left( \frac{a}{w} \right)^2 - 21.72 \left( \frac{a}{w} \right)^3 + 30.39 \left( \frac{a}{w} \right)^4 \quad \text{(7.5)}
\]

The analysis was performed with a uniform mesh formed by quadrilateral elements. Step function and near-tip enrichment functions were used to enrich the field near the crack. The numerical integration within elements not cut by crack is performed using standard gauss quadrature with Legendre polynomials. The elements which are cut by the crack or contains a crack tip, a high order gauss quadrature is used for integration. A plain strain problem is performed with \( E = 1000 \) units and \( \nu = 0.3 \). Several tests were performed with different parameters, as will be explained below, to test the accuracy of the method.

![Figure 7.12: Enrichment scheme](image)

In order to see the convergence rate of the method, we used three different types of structured meshes. mesh1 consists of 12 X 24 evenly spaced nodes, mesh2 consists of 24 X 48 evenly spaced nodes and mesh3 consists of 30 X 60 evenly spaced nodes . Furthermore, in order to check the effect of domain size for the computation of interaction integral on numerical solution, five different domain sizes were selected. The domain for computation of interaction integral is selected as the domain that falls within a ball of radius \( r_d \), where \( r_d \) is a function of
the size of an element, $l_e$. The five selected domains for the test are:

\[
\begin{align*}
domain 1 &= 1.5 \sqrt{a_e} \\
domain 2 &= 2 \sqrt{a_e} \\
domain 3 &= 2.5 \sqrt{a_e} \\
domain 4 &= 3 \sqrt{a_e} \\
domain 5 &= 4 \sqrt{a_e}
\end{align*}
\]

The error in the stress intensity factors is calculated as:

\[
ERROR = \frac{K_{I_{num}} - K_{I_{exact}}}{K_{I_{exact}}}
\]

Table 7.1 below shows the results of the analysis.

<table>
<thead>
<tr>
<th></th>
<th>Domain 1</th>
<th>Domain 2</th>
<th>Domain 3</th>
<th>Domain 4</th>
<th>Domain 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>mesh 1</td>
<td>0.022523</td>
<td>0.050974</td>
<td>0.050692</td>
<td>0.052075</td>
<td>0.052046</td>
</tr>
<tr>
<td>mesh 2</td>
<td>0.005588</td>
<td>0.023031</td>
<td>0.024170</td>
<td>0.024217</td>
<td>0.024104</td>
</tr>
<tr>
<td>mesh 3</td>
<td>0.010584</td>
<td>0.018036</td>
<td>0.019277</td>
<td>0.019221</td>
<td>0.019108</td>
</tr>
</tbody>
</table>

Comments: -

Figure 7.13 shows the rate of convergence for the problem under investigation with different domain. It can be noticed, that the error in evaluating stress intensity factors numerically using XFEM is small even with a coarse mesh. Further, the convergence curve for domain 1 does not show a good behavior. This is due to the fact that the selected domain for the computation of interaction integral was quite close to the crack tip, where the stress and displacement fields are non-smooth. However for all other domain sizes the result is approximately the same due to the fact that interaction integral was evaluated in the far field where the field was smooth and hence the crack tip complications were avoided. Figure 7.14 also supports the results of figure 7.13. Figure 7.14 shows computed error in mode 1 stress intensity factor as a function of domain for interaction integral. It can be observed from the figure, that the solution shows a good convergence as the domain selected for the evaluation of interaction integral is far from
7.3 Modeling 2D Crack problems

the tip field, approximately \( r_d > 2 \). It is clear from the figure that for \( r_d < 2 \), for the current problem, the error in the solution is very high and the results are unstable, which is also depicted in figure 7.13, for domain 1 convergence curve. Once the domain \( r_d \) is greater than 2, the solution not only becomes stable but also accurate. It is therefore reasonable to take \( r_d = 2 - 3\sqrt{a_e} \) (domain for interaction integral evaluation) to get a better approximation of the energy at the crack tip.

![Graph showing error in KI versus number of nodes for different domains](image)

**Figure 7.13:** Rate of convergence for center edge cracked plate problem

It is once again emphasized here, that the crack was arbitrarily aligned with the mesh, furthermore, no mesh refinement was made near the crack, but even then the field is better approximated, as can be seen from the table (7.1) and from the stress plots 7.15.
Figure 7.14: Effect of different domains for computation of M-integral on accuracy of solution
Figure 7.15: Results of Edge cracked plate problem
7.3 Modeling 2D Crack problems

7.3.1.1 Improved enrichment scheme and convergence of solution

In order to obtain an improved convergence of the solution, Laborde et al. [2005b] suggested an alternative approach, where a fixed area around a crack tip, independent of the mesh size $l_e$ is enriched. This is done by enriching all nodes falling inside a radius of influence around the crack tip, where the field is considered to be dominated by the crack tip field. In the discussion below we shall call this improved enrichment scheme as $\text{Enr}_2$ and the previous enrichment scheme where only tip element was enriched will be referred to as $\text{Enr}_1$.

For the test example below, we selected radius of domain, $R$ in which all nodes will be enriched as $R=0.15$. The enrichment scheme $\text{Enr}_2$ is shown in the figure(7.16) below. The material properties as well as geometric properties are the same as defined in previous example. For the analysis purpose, the same numerical model was used as shown in figure(7.16). The

![Figure 7.16: Modified/fixed area enrichment scheme](image-url)
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Analysis was performed with three different meshes

\[
\begin{align*}
\text{mesh} 1 & = 288 \text{ nodes} \\
\text{mesh} 2 & = 1152 \text{ nodes} \\
\text{mesh} 3 & = 1800 \times 60 \text{ nodes}
\end{align*}
\]

and five different domain radii for the evaluation of conservation integral.

\[
\begin{align*}
\text{domain} 1 & = 1.5 \sqrt{a_e} \\
\text{domain} 2 & = 2 \sqrt{a_e} \\
\text{domain} 3 & = 2.5 \sqrt{a_e} \\
\text{domain} 4 & = 3 \sqrt{a_e} \\
\text{domain} 5 & = 4 \sqrt{a_e}
\end{align*}
\]

The error in the stress intensity factors is calculated as:

\[
\text{ERROR} = \frac{K_I_{num} - K_I_{exact}}{K_I_{exact}}
\]

The results of the analysis are given in table (7.2)

<table>
<thead>
<tr>
<th>Domain</th>
<th>Enr2 mesh 1</th>
<th>Enr2 mesh 2</th>
<th>Enr2 mesh 3</th>
<th>Enr2 mesh 4</th>
<th>Enr2 mesh 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain 1</td>
<td>0.044002</td>
<td>0.024471</td>
<td>0.026051</td>
<td>0.033728</td>
<td>0.052046</td>
</tr>
<tr>
<td>Domain 2</td>
<td>0.013971</td>
<td>0.013858</td>
<td>0.013689</td>
<td>0.005532</td>
<td>0.010048</td>
</tr>
<tr>
<td>Domain 3</td>
<td>0.007677</td>
<td>0.008016</td>
<td>0.007903</td>
<td>0.007818</td>
<td>0.003359</td>
</tr>
</tbody>
</table>

Comments :-

Convergence curves are shown in figure 7.17 on a log-log plot for different domains of interaction integral. Figure 7.18 shows the effect of domain size for computing interaction integral on the numerical solution. Once again it is seen that, using larger domains for interaction integral is a good choice, as the solution becomes stable and accurate when the domain for computation of interaction is large.
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Figure 7.17: Rate of convergence with different domain sizes of interaction integral for modified enriched cracked plate problem

Figure (7.19) shows the comparison between the convergence curves obtained with the two enrichment schemes. It is clear from the figures that expanding the domain of enrichment, where required, works better than previous enrichment scheme and significantly improves the results.

Figure (7.20) shows variation in error in stress intensity factors with respect to the ratio of \( r_d \) and \( R \), where \( r_d \) is the the radius of domain for the evaluation of interaction integral and \( R \) is the radius of domain where the solution was assumed to be dominated by near tip field and hence all the nodes within radius \( R \) were enriched with near tip enrichment functions. It can be seen from the figure that the error is minimum when the two domain are same i.e \( r_d = R \). The solution shows good convergence once \( r_d > R \). For \( r_d < R \) the results are inaccurate and quite unstable. It is therefore concluded from the above results that, when the a fixed area enrichment scheme is used, then the domain for computation of interaction integral \( r_d \) should be greater domain of near-tip enrichment \( R \), in order to obtain accurate, stable results.
7.3 Modeling 2D Crack problems

![Figure 7.18](image1.png)

**Figure 7.18:** Effect of different domains for interaction integral on the accuracy of the solution

![Figure 7.19](image2.png)

(a) Convergence plots for two enrichment schemes with domain 2  
(b) Convergence plots for two enrichment schemes with domain 5

**Figure 7.19:** Comparison of rate of convergence between $Enr_1$ and $Enr_2
Figure 7.20: Error in KI with changing rd/R
7.3 Modeling 2D Crack problems

7.3.2 Center edge crack in finite dimensional plate under shear

In order to see the robustness and accuracy of the method, and to check the accuracy of the XFEM programming code for the mixed mode failure, a center edge crack in a finite dimensional plate loaded in shear at the boundary is considered.

As an example, a rectangular plate of width, W = 7in, L/W = 16/7 and a/W = 1/2 is selected. where a is the crack length introduced at the edge of the domain at L/2. The material response is considered to be linear elastic. Young’s modulus for the material is E = 100 ksi and poise’s ratio \( v = 0.3 \). The analysis was performed with two mesh discretization:

\[
\text{mesh1} = 288 \text{ nodes} \\
\text{mesh2} = 1152 \text{ nodes}
\]

Five different domains of interaction integral were considered for the analysis:

\[
\begin{align*}
\text{domain1} & = 1.5 \sqrt{a_e} \\
\text{domain2} & = 2.5 \sqrt{a_e} \\
\text{domain3} & = 3 \sqrt{a_e} \\
\text{domain4} & = 4 \sqrt{a_e} \\
\text{domain5} & = 5 \sqrt{a_e}
\end{align*}
\]

The exact stress intensity factors for this case is given in reference [Yau et al. 1980] where for the problem under investigation we have

\[
KI = 34.0 \text{ psi} \sqrt{\text{in}} \quad KII = 4.55 \text{ psi} \sqrt{\text{in}}
\]

Again two enrichment schemes were considered \( \text{Enr}_1 \) and \( \text{Enr}_2 \). The enrichment scheme where only tip element is enriched with near-tip functions will be referred to as \( \text{Enr}_1 \), while in the other enrichment scheme \( \text{Enr}_2 \), we enriched all nodes around a crack tip within radius \( R \). The numerical model with \( \text{Enr}_2 \) and the geometry of the domain is shown in the figure7.21.
7.3 Modeling 2D Crack problems

(a) Geometry of the edge crack plate problem  

(b) Numerical model used for analysis

Figure 7.21: Numerical model and geometry of the center edge crack plate subjected to nominal shear stress $\tau_o$

Figure 7.22: Zoom at the enriched zone, where red square blocks shows the nodes enriched with naer-tip enrichment functions and black circles shows the nodes enriched with heaviside enrichment functions
7.3 Modeling 2D Crack problems

Error in the numerical values of stress intensity factors is calculated as:

\[ \text{ERROR} = \frac{K_{I_{\text{num}}} - K_{I_{\text{exact}}}}{K_{I_{\text{exact}}}} \]

Tables 7.3 and 7.5 shows error in the computed the mixed mode stress intensity factors KI, while tables 7.4 and 7.6 shows error in the KII for enrichment scheme \( \text{Enr}_1 \) and \( \text{Enr}_2 \) respectively.

**Table 7.3:** Error in KI with enrichment scheme \( \text{Enr}_1 \)

<table>
<thead>
<tr>
<th>Domain 1</th>
<th>Domain 2</th>
<th>Domain 3</th>
<th>Domain 4</th>
<th>Domain 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>mesh 1</td>
<td>0.029397</td>
<td>0.063297</td>
<td>0.062526</td>
<td>0.062894</td>
</tr>
<tr>
<td>mesh 2</td>
<td>0.006568</td>
<td>0.026456</td>
<td>0.02460</td>
<td>0.025703</td>
</tr>
</tbody>
</table>

**Table 7.4:** Error in KII with enrichment scheme \( \text{Enr}_1 \)

<table>
<thead>
<tr>
<th>Domain 1</th>
<th>Domain 2</th>
<th>Domain 3</th>
<th>Domain 4</th>
<th>Domain 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>mesh 1</td>
<td>0.006571</td>
<td>0.020066</td>
<td>0.021758</td>
<td>0.020813</td>
</tr>
<tr>
<td>mesh 2</td>
<td>0.018879</td>
<td>0.010154</td>
<td>0.012066</td>
<td>0.011143</td>
</tr>
</tbody>
</table>

**Table 7.5:** Error in KI with enrichment scheme \( \text{Enr}_2 \)

<table>
<thead>
<tr>
<th>Domain 1</th>
<th>Domain 2</th>
<th>Domain 3</th>
<th>Domain 4</th>
<th>Domain 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>mesh 1</td>
<td>0.056629</td>
<td>0.038021</td>
<td>0.042535</td>
<td>0.043212</td>
</tr>
<tr>
<td>mesh 2</td>
<td>0.010344</td>
<td>0.010197</td>
<td>0.010506</td>
<td>0.004041</td>
</tr>
</tbody>
</table>

**Comments :-**

Figures (7.23, 7.24) below shows the variation of error in stress intensity factors as a function of domain selected for the evaluation of interaction integral. The figures re-affirmed our previous conclusions derived in center edge crack plate in tension problem. From the figures one can see significant deviation of the numerical solution from the exact solution when the domain of integration is small and close to the crack tip region. However the numerical solution shows good
agreement with the exact solution as the domain of integration becomes larger. As we have already demonstrated earlier that the domain form of interaction integral is a way of evaluating a contour integral around a crack tip, where the integral has a real value only at the boundary of the domain and within rest of the domain the integral is zero. Hence it is no more than contour integral. So looking from this perspective we already know the fact that, the conservation/contour integral gives stable and accurate results when the contour is far away from the crack tip region, as the integral is evaluated in the region where the field variables are generally smooth. This signifies that the domain form of interaction integral is best evaluated when the domain of integration is large. Thus the obtained results are an obvious outcome of the phenomenon which we already knew. The results obtained also shows that the solution becomes independent and insensitive to the domain size as and when the selected domain becomes large.

Table 7.6: Error in KII with enrichment scheme $Enr_2$

<table>
<thead>
<tr>
<th>Domain</th>
<th>Domain 2</th>
<th>Domain 3</th>
<th>Domain 4</th>
<th>Domain 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>mesh 1</td>
<td>0.047077</td>
<td>0.012022</td>
<td>0.016462</td>
<td>0.016264</td>
</tr>
<tr>
<td>mesh 2</td>
<td>0.016044</td>
<td>0.008681</td>
<td>0.008857</td>
<td>0.000857</td>
</tr>
</tbody>
</table>

Figure 7.23: Effect of different domains $r_d$ for interaction integral on the accuracy of the solution with enrichment scheme $Enr_1$

The same behavior can also be seen when the near-tip enrichment is extended to a zone, within radius $R$, figures(7.23 , 7.24). However we see a sudden decrease in an error
for mesh2. This was also an expected behavior following the conclusions of example 1 of plate under tension. In order to see, what really happens, we refer to figures(7.25). Figure(7.25) shows error in computed stress intensity factors as function of the ratio of \( r_d \) and \( R \). The sharp decrease in error results when \( r_d R = 1 \). The same behavior was also observed in the previous example as well. The results becomes stable and accurate once \( r_d R \) becomes greater than 1, i.e \( r_d R > 1 \). mesh 1 did not show this behavior because no point was evaluated at \( r_d R = 1 \). However if any of its points were evaluated near \( r_d R = 1 \), it would certainly have shown the same behavior as shown by the mesh 2.

**Figure 7.24:** Effect of different domains \( r_d \) for interaction integral on the accuracy of the solution with enrichment scheme \( Enr_2 \)

**Figure 7.25:** Effect of ratio \( r_d/R \) on the accuracy of the solution
Hence summarizing we can say that in general the good and accurate results for stress intensity factors and interaction integral is obtained when a large domain for computation of integral is selected. As a lower limit in case where tip enrichment is extended to certain region, the domain for integration should be greater than domain of enrichment in order to get stable, accurate results.
Next we consider the example of an interior crack. In case where the crack is present in the interior of the domain, there are two tips of the crack that needs to be enriched with the tip-enrichment and the rest of the crack with the step function. In order to test the effectiveness of the method under mixed mode failure an interior angled crack example is used for the analysis. The plate is subjected to a uniform tensile stress of $\sigma_o$ at both ends. A crack of length 2$a$ is introduced in the center of the domain at an angle $\theta$. Stress intensity factors were calculated and compared with the exact solution. The exact solution of stress intensity factors for this problem is given by [Yau et al. 1980]:

\[
K_I = \sigma_o \sqrt{\pi a} \cos^2(\theta) \quad (7.6)
\]
\[
K_{II} = \sigma_o \sqrt{\pi a} \cos(\theta)\sin(\theta) \quad (7.7)
\]

For the analysis purpose, a square plate of width $W=10$in with a half crack length $a=0.5$in is used. It should be noted here that the exact values of stress intensity factors mentioned above is for an infinite plate, and as the problem at hand has the plate dimensions quite large as compare to the crack length used for analysis, the numerical solution can be compared with the
7.3 Modeling 2D Crack problems

exact solution. The domain is discretized into uniformly spaced grid of 41 X 41 quadrilateral elements. Furthermore only near tip enrichment is used for the analysis and not the modified enrichment scheme. Of course, the solution could be improved by extending the enrichment within certain region of influence around the crack tip. The domain for computing interaction integral is taken to be lying within radius $r_d = 2\sqrt{a_e}$. A uniaxial of stress of $\sigma_o = 1\,\text{psi}$ is applied at both ends. The problem geometry with boundary conditions is shown in the figure 7.26. The material properties used for the analysis were: young’s modulus $E=100\,\text{ksi}$ with poison’s ratio $\nu = 0.3$. The error in the computed mixed mode stress intensity factors $K_I$ and $K_{II}$ is calculated as:

$$ERROR = \frac{K_{num} - K_{exact}}{K_{exact}}$$

The analysis was performed for a range of center crack angles $\theta$. figures (7.27) below shows that the computed numerical results shows good agreement with the analytical results.

![Graphs showing $K_I$ and $K_{II}$ vs $\theta$ and error in $K_I$ and $K_{II}$](image)

(a) $K_I$ and $K_{II}$ vs $\theta$  
(b) Error in $K_I$ and $K_{II}$

**Figure 7.27**: Comparison of numerical $K_I$ and $K_{II}$ values with exact solutions for different crack angle $\theta$ in an infinite plate
7.4 Modeling voids using XFEM

In this section, we present a numerical example demonstrating the XFEM implementation for modeling arbitrary voids.

A finite plate with a square domain is considered. The plate is having traction free elliptical void under uniaxial tension $\sigma_o = 1$. In order to assess the accuracy and efficiency of the XFEM for modeling arbitrary discontinuities, we compare the finite element solution to that obtained by XFEM. The mesh size for the finite element solution was so chosen that further refining the mesh does not produce significant change in the solution, hence the FEM solution was also taken as the reference solution, with which we compare the accuracy of the XFEM solution.

An ellipse in general is represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a$ is the major principal axis and $b$ is the minor principal axis of an ellipse. It should be noted that for $b/a = 1$, the ellipse is a circle and for $b/a = 0$, the ellipse becomes a crack. In numerical model we consider a square domain of edge length $L$, with an elliptical hole at its center. We impose the exact tractions on the boundary of the square domain, with appropriate constraints added to remove rigid body modes. The material properties chosen for the analysis are: young’s modulus $E = 1000$ and Poisson’s ration $\nu = 0.3$. In numerical computations $L=2$

and plane strain conditions were assumed. An equi-spaced mesh of quadrilateral elements with 40x40 nodes were used. The analysis was performed with an elliptical void of $a=0.4$ and $b=0.2$
as major and minor principal axis. Figure 7.28 shows the finite element mesh and the extended finite element mesh used for the analysis. It is important to note here that, in the finite element mesh the hole is explicitly modeled by aligning the element edges with the boundary of an elliptical hole. However in case of extended finite element mesh, an elliptical hole is arbitrarily aligned with the mesh. To perform the XFEM analysis, the nodes whose nodal support is cut by the hole is enriched with the step function, while the nodes whose nodal support lies completely inside the hole is removed from the discrete system of equations. Figure 7.29 shows the enrichment scheme for XFEM analysis. The nodes with circles are enriched with step functions, while the nodes marked with red asterisks inside the hole were removed from the discrete system of equations.

Figure 7.29: Enrichment scheme for modeling voids

Comments :-

The finite element mesh consists of 1600 quadrilateral elements and 1680 nodes. The corresponding mesh for the XFEM implementation consists of 1521 quadrilateral elements having total nodes of 1600. The number of unknowns in FE computations are 3360 and that in the XFEM implementation are 3288. Figure 7.30 shows the contour plot of $\sigma_{yy}$ for the two approaches. It is seen that the agreement between the results obtained by the two methods is excellent.
7.5 Modeling Crack growth problems with XFEM

In the previous sections analysis of a body with a static crack was explained with some examples. In this section analysis results of bodies with evolving cracks is demonstrated. It is worthwhile to mention here that the standard finite element method requires the mesh to be aligned with the geometry of the body and crack. In case of evolving cracks this requirement becomes computationally expansive and burdensome, as the mesh is required to be updated at each step of crack increment. Using extended finite element method gives an elegant way of modeling discontinuities, where the discontinuities arbitrarily aligned with the mesh could be modeled. This alleviates the need of a conforming mesh and hence no mesh update is required, which in case of FEM might result in loss of accuracy as the data is transferred from one mesh to the other.

7.5.1 Edge crack in finite dimensional plate under uniaxial tension

This problem illustrates the crack growth using XFEM in a finite dimensional plate with center edge crack as shown in the figure 7.11. Further the same material properties are used for analysis as were used in section 7.3.1. A mesh with 24 X 48 nodes is used with modified enrichment scheme. The fixed area selected for the tip enrichment is \( R = 0.1 \). The domain for the computation of interaction integral is selected to be \( 3\sqrt{a_c} \). A crack increment length of \( \delta a = 0.04 \) is used for the crack propagation analysis. The crack is propagated through the domain till it
cut the whole body into two halves. Figure 7.32 shows the deformed shape at different crack increment steps. As the body was in mode I failure mode so it follows a straight path as was expected. Figure (7.31(a)) shows the variation of model I stress intensity factor with increase in crack length. Figure (7.31(b)) shows the error in numerical KI values with respect to the exact solution. where the error is calculated as

\[
ERROR = \frac{KI_{num} - KI_{exact}}{KI_{exact}}
\]

As the above equation 7.4 is only valid for the cases where \(a/w < 0.6\), so the error is computed only for crack increment up to 0.6 units. It can be seen from the graph that, numerically computed results are well in agreement with the exact solution and the error is below 1 percent.

![Graphs showing KI with crack length and error in KI](image)

(a) Variation of KI with crack length a  
(b) Error in KI

**Figure 7.31:** Numerical KI for edge crack growth problem

### 7.5.2 Interior crack in a finite dimensional plate under uniaxial tension

Two cases of interior crack in a finite dimensional plate is considered and discussed below. First example shows the crack evolution of a center interior crack of length 2\(a\) inclined at an angle \(\theta = 0\). The plate is loaded with uniaxial tension at the top edge, while the bottom edge is clamped. The material properties and the geometry of the domain is the same as in previous example. A modified or fixed area enrichment scheme was used for the analysis. Figure 7.33(a) shows the initial crack position and the enriched nodes for the crack growth analysis. As the body is subjected to a uniform tension perpendicular to the crack hence pure mode I failure is expected.
Figure 7.32: Deformed shape at different instants of crack growth in a finite dimensional plate with an initial edge crack

Figure 7.33: Center crack growth in a finite dimensional plate subjected to pure tension stress $\sigma_o$. 

7.5 Modeling Crack growth problems with XFEM

Comments :-

The obtained numerical results are an obvious outcome. The failure mode was a pure mode I and the crack progresses straight to the ends of the body. The crack path is shown in the figure
7.33(b).

7.5.3 Interior crack in an infinite plate

As a next example, in order to see the effectiveness and accuracy of the method in an alternative way. We consider an infinite plate with center crack of length 2a subjected to uniaxial tension stresses $\sigma_o$. The geometry of the body and boundary conditions used for modeling is shown in the figure7.26. An initial crack of length 2a is introduced with an angle of $\theta$ with the horizontal. For current investigation an initial crack is introduced at an angle of $0^\circ, 20^\circ, 30^\circ, 45^\circ, 60^\circ$ and $80^\circ$. The crack is propagated for four steps with a crack increment of 0.2in. Analysis was performed with a mesh of 41 X 41 nodes. Only near tip enrichment was used for crack modeling. Furthermore interaction integral is calculated within the domain of size $r_d = 3\sqrt{a_e}$. The fracture angle $\theta_{cr}$ is noted at the initiation of crack growth, for every orientation of the initial crack. It is to note here that, the crack propagation direction was determined using the maximum hoop stress criteria. The numerical results are then compared with the theoretical values of $\theta_{cr}$.

A theoretical result for this problem is given in Erdogan and Sih [1963], where the angle of maximum tangential stress for traction free crack surfaces is calculated from

$$KI \sin\theta_{cr} + KII \left(3\cos\theta_{cr} - 1\right) = 0$$

(7.8)

where the KI and KII for an angled crack in an infinite plate is given by

$$KI = \sigma_o \sqrt{\pi a} \cos^2 \theta$$

(7.9)

$$KII = \sigma_o \sqrt{\pi a} \cos \theta \sin \theta$$

(7.10)

The above equation 7.8 can now be written as

$$\sin\theta_{cr} + (3\cos\theta_{cr} - 1) \cot \gamma = 0$$

(7.11)

provided $\gamma = 0$, which is trivial case from the view point of fracture mechanics. where $\gamma = \frac{\pi}{2} - \theta$. This is demonstrated in the figure 7.34.

Comments :-
From the above equation 7.11 we can see that for the cases where $0 < \theta < \frac{\pi}{2}$, the fracture angle $\theta_{cr}$ is negative. Obtained results from the XFEM analysis shows good agreement with the theoretical values which shows the accuracy of the method in approximating accurately the field. The results are shown in the table (7.7). Figure (7.35(a)) shows the plot of both the exact solution and the numerical solution. Figure (7.35) shows the error in the numerical solution, which is less than 1 percent. Figure 7.36 shows the crack propagation for all seven orientation of an initial crack.
Figure 7.35: Comparison of crack propagation angle for different initial crack configurations
Figure 7.36: Center crack propagation in an infinite plate with different initial crack configurations
7.5 Modeling Crack growth problems with XFEM

7.5.4 Three point Bending test

To test the validity of the code, a standard three point bending test is performed. The geometry of the problem is shown in the figure 7.37. The analysis was performed to judge the efficiency and accuracy of the method in predicting the correct crack path. The analysis was performed using structured mesh of quadrilateral elements with 76 X 16 (1216) nodes. At each step the crack is advanced by an increment of $\delta a = 0.02$. A modified enrichment scheme is used for crack tip enrichment. The stress intensity factors are calculated using a domain form of interaction integral in a circular domain of radius $r_d = 3\sqrt{a_e}$ centered at the crack tip. Modified domain for interaction integral is used near the boundary of the body. The crack increment direction is determined using maximum hoop stress criteria. The analysis was initiated with an uncracked beam. As the load increases the crack appeared at the midspan of the beam, when the tensile strength of the material is reached. For the analysis, a beam of length $l = 500$cm and depth $b = 100$cm is used. The material properties used are as follows

$$E = 30,000 \text{ MPa}, \quad \nu = 0.1, \quad f_t = 1 \text{ MPa}$$
where $E$ is young’s modulus, $\nu$ is Poisson’s ratio and $f_t$ is the tensile strength of the material. The fracture energy of the material is taken to be $G_f = 20 \text{ Nm}^{-1}$. The node at (0,0) is restricted in both x and y directions while the node at (500,0) is allowed to move only in x direction. In order to numerically capture the snap back in the structural response, we followed an approach similar to the one proposed in Moes and Belytschko [2002]. The imposed tractions $P$ are considered to be depended linearly upon a loading factor $\lambda$, i.e $P = \lambda * \lambda_0$. The approach is then to find such a load factor $\lambda$, for which the mode I stress intensity factor, $K_I$ at the crack tip becomes equal to the critical stress intensity factor $K_{Ic}$. The XFEM analysis flowchart can then be given as

- **Step00**: Initialize with the given geometry of the body and crack (if a body contains an initial crack)
- **Step01**: Discretize the domain into finite elements
- **Step02**: Start with an initial guess of load factor, $\lambda$
- **Step03**: Perform the XFEM analysis
- **Step04**: Calculate the stress intensity factors using the domain form of interaction integral
- **Step05**: If the $K_I > K_{Ic}$ or $K_I < K_{Ic}$, update the load factor, (usually by means of secant method or newton methods) and go to step step03, Else if $K_I = K_{Ic}$ or within certain tolerance limit then find the crack growth direction and introduce a new crack segment and then go to step02.

![Figure 7.38: Load displacement curve for three point bending beam test](image_url)
7.5 Modeling Crack growth problems with XFEM

Comments :-

The crack is propagated throughout the whole depth of the beam section. As the failure mode in this case is a pure mode I failure, so the obtained straight propagation of the crack throughout the depth of the beam is an expected outcome. Figure 7.38 shows the obtained load-deflection points for the XFEM analysis of a three point bending test. It could be observed that the snap-back in the structural response is captured very well.

7.5.5 Shear crack propagation in Beams

In continuation to the previous problem, we shall now investigate the accuracy of XFEM in modeling curved cracks. Due to the ease that, XFEM does not require the mesh to be aligned with the geometry of the crack, it offers an elegant, computationally inexpensive and easy way of analyzing problems with curved crack propagation.

Let us consider the same beam model used for three point bending test in the previous example. In this case instead of placing the load at the middle of the beam, the load is placed at the distance L/3 from the end of the beam. The overall beam span is L=5cm and a shear span of 1.67cm, ultimately resulting in a shear span to depth ratio of 1.67. The geometry of the problem is shown in the figure 7.39(a). The analysis was carried out using three different mesh discretization in order to see the effect of mesh discretization on tracking crack path using XFEM.

Comments :-

Three mesh discretizations, used in the analysis, are mesh A: 720 nodes, mesh B: 1216 nodes and mesh C: 2121 nodes. Figure 7.40 shows the crack paths obtained with different mesh discretization. It is worthwhile to mention few things here, first it can be seen that XFEM was well able to predict the curved crack path without the need of mesh alignment with the crack. Further, the obtained crack path is qualitatively representing the expected crack pattern for shear cracks in beams. secondly, It can be seen from the figure that crack path was not fully captured when the mesh was coarse (mesh A), however the crack propagation pattern improved with the mesh refinement (mesh B) and after that further refinement of the mesh (mesh C) did not cause much change in the solution. The example clearly demonstrated that the crack growth pattern is also affected by the mesh discretization. One of the reasons for such behavior is, that the
stress intensity factors are affected by the mesh discretization and as the mesh becomes refine
the solution converges.
Another interesting feature that can also be observed from the crack paths is that, the crack
direction becomes unstable and shows oscillations near the end of the beam. This is due to
the fact that, in that zone the body is in compression and under compression loading the crack
direction tends to become unstable as the crack nears arrest. In this case it can be observed
that the crack path deviates from the smooth path. If we go into detail of the problem, so it
seems clear that this phenomena occur as the mode I stress intensity factor begins to decrease
and the ratio of mode II to mode I stress intensity factor begins to increase. As the crack
propagation criteria, specifically the maximum hoop stress criteria is based on determining the
stress intensity factors at the crack tip, and as this ratio(ratio of mode II to mode I) is essential
in determining the crack growth direction, so it directly affects the crack path. when this ratio
becomes too large, the crack path is susceptible to oscillation and becomes unstable as was
identified by Belytchko and Flemming [1999].
7.5 Modeling Crack growth problems with XFEM

![Graph showing crack propagation path with different meshes](image)

**Figure 7.40:** Effect of crack incremental length on crack propagation path

### 7.5.6 Peel Test

**Symmetrically loaded double cantilever beam (DCB)**

As a next example we consider a well known double cantilever beam problem. A peel test is carried out on the double cantilever beam to further investigate the factors which may affect the crack propagation in a body.

Let us consider a double cantilever beam (DCB) of length $l=5$ units and height $h=0.5$ units. The far end of the DCB is clamped both in horizontal and vertical directions. An initial crack of length $a=h$ is introduced into the beam. The geometry of the problem is shown in the figure 7.41. The beam is symmetrically loaded at the ends. The analysis was carried out with a structured uniform mesh of 91x10, 4-noded quadrilateral elements. The failure in this case is a pure mode I failure and the crack propagated straight all the way to the end of the double cantilever beam. Figure 7.42 shows the deformed shapes at different crack increment instants.

![Diagram of Double Cantilever Beam- symmetric crack opening](image)

**Figure 7.41:** Double Cantilever Beam- symmetric crack opening

**Un-Symmetrically loaded double cantilever beam (DCB)**
As a next test we alter the symmetric conditions of the double cantilever beam by incorporating a horizontal force component at one end of the beam. Further in order to get some intersecting and useful results, the crack propagation problem in an unsymmetrical DCB was studied with four different crack increment lengths, $\Delta a = 0.04, 0.06, 0.08, 0.1$, using two different domains for computing interaction integral i.e domain $1 = 2l_e$ and domain $2 = 3l_e$ where $l_e = \sqrt{a_e}$ and $a_e$ is the average area of the elements. The geometry of the body, crack depth, the mesh size and material properties remains the same as was considered in case of symmetrical double cantilever beam example.

**Comments :-**

Several important observations and conclusions could be made from the analysis. Figure 7.44 shows the crack propagation paths. The solid lines in the figure shows the crack paths obtained
using different crack increments and using domain 1 for the computation of interaction integral while the dotted lines show the corresponding crack paths, when domain 2 was used for the computation of interaction integral.

It can be seen from the figure that the crack paths are affected by the crack increments length. Using larger crack increment length resulted in wrong prediction of crack propagation path, while on the other hand using smaller crack increment length better predicted the crack path. This can be seen from the crack paths obtained using crack increments $\delta a = 0.06$ and $0.08$ which are nearly same. However using too small crack increment also failed to define the correct crack path. In order to understand this and to reach to some conclusion, let us formulate the problem in another way, and see the crack increments as a function of element size in the mesh. Then the ratios $\Delta a/\sqrt{a_e}$ for the four crack patterns are $0.72$, $1.08$, $1.4$, $1.8$ respectively. It can be observed that the crack path is well predicted when the ratio is between $1$ and $1.5$ times the element size. One of the reasons why using too small crack increment length fails to predict the correct path is that, when the crack increment length is too small as compare to element size then there will be number of crack segments inside an element, in that case a sequence of mapping is necessary to rotate each crack segment onto the crack model as explained earlier.

However, using level set functions can automatically eliminate this problem and the discontinuity in the enriching function will be aligned with the crack but it should be noted that using the simplified form of level set function, such as signed distance function itself incorporates a discontinuity, which results in wrong estimates of $r$ and $\theta$ values, thus affecting the solution and ultimately the crack propagation direction. More details on properties of level set functions can
be found in chapter 4. As we have used the simple signed distance function so possibly this is the source of error in computing the correct crack path.

It is well understood in the world of fracture mechanics that the contour integral is path independent but when it comes to curved cracks the J-integral is then no more path independent. The change in the crack pattern as clear from figure 7.44, from solid line to dotted lines, with the change in the domain for interaction integral is a manifestation of the same.

summarizing we can say that there are different factors which can effect the crack growth pattern naming the mesh refinement,crack increment length and the size of the domain for interaction integral.

7.5.7 Crack emanating from a void
Cracks emanating from voids includes a wider range of problems in various fields. Due to stress concentrations near a hole or near the corners of a polygonal void, cracks are likely to emanate at the voids under cyclic loading. Critical locations in a structure such as bolted joints and connections, openings in a wall or a panel, if subjected to cyclic loading may impair the structural integrity due to loss of strength and stiffness by crack formation. In this section a numerical study of crack emanating from a rectangular void in a rectangular plate subjected to shear loading is performed.

The geometry of the body and void is shown in the figure 7.45(a). The plate consist of length L=1unit and height H=2units. A rectangular void is present in the center of the domain. The panel is subjected to uniform shear stress $\tau_o$ at the top. Due to shear loading the stresses were maximum at the corners of the rectangular void. An initial cracks at $45^\circ$ were introduces at the corners where the tensile stresses exceeds the material strength. The obtained numerical results were then qualitatively compared with the shear crack propagation results of Erdogan and Sih [1963].

Comments :-
Figure7.45(b) shows the crack propagation path from a rectangular void subjected to shear loading. The example demonstrated the efficiency of the method in predicting the correct curved
7.5 Modeling Crack growth problems with XFEM

Figure 7.45: Shear crack propagation from a void in a plate subjected to shear stress $\tau_o$

crack path. The obtained shear crack path qualitatively is in good agreement with the experimental shear crack propagation path in a large plate subjected to uniform shear at infinity by Erdogan and Sih [1963]. Figure 7.46 shows some of the obtained results for an intermediate crack propagation step.

Figure 7.46: Crack emanating from a rectangular void
7.6 Multiple interacting cracks

XFEM offers an elegant way of modeling discontinuities. In this section we shall implement XFEM in modeling multiple cracks problems. The test examples will be carried out to analyze the efficiency of the XFEM in modeling interacting cracks and their effect on correct determination of crack propagation paths. The accuracy of the solution is judged by comparing numerically computed stress intensity factors form XFEM analysis with a reference solution given in Civelek and Erdogan [1982].

7.6.1 Interior multiple cracks in an infinite plate

The first problem investigated is the problem of interaction of two interior cracks present within an infinite plate subjected to uniform tension \( \sigma_o \). The cracks are perpendicular to the direction of loading and are apart by a distance of \( 2B \). The geometry of the body and crack is shown in the figure 7.47. An analytical solution for the determination of stress intensity factors were proposed in Civelek and Erdogan [1982], where the effect of interaction between multiple cracks

![Figure 7.47: Multiple cracks in an infinite plate under uniform tension stress \( \sigma_o \)](image-url)
7.6 Multiple interacting cracks

on crack propagation was discussed. It was shown that the cracks will move apart from each other as the distance between them decreases. Further it was shown that, the decrease in the distance between the cracks decreases the $K_I$(mode I stress intensity factor) and increases the $K_{II}$(mode II stress intensity factor). In the analysis below we shall investigate the effect of $B/H$ ratio on crack propagation using XFEM.

For the analysis, following geometric parameters were used, $V=10$, $H=1$. It was assumed that using $V/H = 10$, will simulate the infinite plate conditions. Uniform tension stress $\sigma_o=1$ was applied at the far ends as shown in the figure 7.47. The analysis was carried out with two different crack lengths $a=0.1$ and $0.2$.

Plain strain conditions were assumed for the analysis with $\nu = 0.3$ and young’s modulus $E = 1000$ksi. A mesh consisting of $24 \times 240$, 4 nodded quadrilateral elements was used for the analysis. The analysis was performed by enriching the tip elements with 4 near-tip enrichment functions and the rest of the crack with a step function. A domain with radius $r_d = 3\sqrt{a_e}$ was used for the computation of stress intensity factors using the domain form of interaction integral.

Table 7.8 shows the results of the analysis performed with different crack lengths and separation distances. The stress intensity factors computed are normalized to stress intensity factors of a single crack in an infinite plate like 7.47 as

$$k_I = \frac{K_I}{\sigma_o \sqrt{\pi a}} \quad (7.12)$$

where $k_I$ is the normalized stress intensity factor for mode I and $K_I$ is the interacting cracks mode I stress intensity factor. The error in the computed result with respect to the reference solution is computed as

$$ERROR = \frac{|k_{IXFEM} - k_{IReff}|}{k_{IReff}}$$

As the mode II stress intensity factors were too low as compare to mode I, so the efficiency of the method is judged by only the mode I stress intensity factors. Table 7.8 also gives the error in the computed results. Figure(7.48) shows the plots of numerical and reference solution for $K_I$ and $K_{II}$.

Comments :-
7.6 Multiple interacting cracks

Table 7.8: Comparison of XFEM results with Reference solution

| a  | B/H | $k_{I_{XFEM}}$ | $k_{I_{Ref}}$ | Error = $\frac{|k_{I_{XFEM}} - k_{I_{Ref}}|}{k_{I_{Ref}}}$ |
|----|-----|----------------|--------------|-----------------------------------------------|
| 0.2| 0.2 | 0.971532       | 0.9749       | 0.003455                                      |
| 0.2| 0.3 | 1.038017       | 1.0437       | 0.005445                                      |
| 0.2| 0.4 | 1.07763        | 1.0839       | 0.005785                                      |
| 0.2| 1.0 | 1.1016         | 1.1096       | 0.00721                                       |
| 0.1| 0.2 | 0.964854       | 0.966        | 0.001186                                      |
| 0.1| 0.5 | 1.018378       | 1.0219       | 0.003446                                      |
| 0.1| 1   | 1.019805       | 1.0247       | 0.004777                                      |

The obtained results show good agreement with the reference solution. The computed error in the results is about 0.5 percent. It is clear from the figure 7.48 that mode I stress intensity factor decreases as the cracks come closer to each other, further the mode II stress intensity factor becomes more significant at smaller B/H. This has a great effect on the crack propagation direction in case of interacting cracks. This issue will be discussed in detail in the next section. Another thing that one can note from figure 7.48 by comparing the $k_I$ curves for two different crack lengths is, that the stress intensity factors are also affected by the crack length. For a smaller crack of length 0.1H, the change in separation distance is not much significant as compare to the cracks of greater length, where we can see large changes in stress intensity factors with changes in crack separation distance. In short to say that interaction has smaller effect on the overall crack propagation behavior for smaller crack depths, but as the crack length increases the interaction becomes stronger.

7.6.2 Multiple edge cracks in an infinite plate

Let us consider an infinite strip containing two edge cracks separated apart by a distance 2B. The plate is subjected to a uniform tension stress of $\sigma_o=1$. The cracks are perpendicular to the direction of loading with length $a=0.2$. The numerical model, geometric and material properties are the same as in the previous example. The geometry of the body is shown in the figure(??) below. The analysis was carried out to see the efficiency of the XFEM in predicting the interacting/coupled behavior of cracks and crack propagation in a body with various cracks. The
7.6 Multiple interacting cracks

![Graph showing stress intensity factors vs B/H ratio]

**Figure 7.48:** Comparison of numerical results with the reference solution of multiple interior cracks in an infinite plate

Numerical results obtained were compared with the reference solution of Civelek and Erdogan [1982].

**Comments :-**

Figure(7.50(a)) shows the plot of stress intensity factors for different crack separation distances. The dotted line shows the normalized stress intensity factors for a single edge crack. The results re-affirmed our previous conclusions. It can be seen from the figure that as the separation distance between the crack becomes smaller, mode I stress intensity factor starts decreasing. At the same time the mode II stress intensity factor becomes more dominant. This affects the crack propagation and the crack start moving away from each other. This can also be seen from figure7.50(b), where the variation of crack propagation angle with respect to B/H ratio is shown. Further it can be inferred form the figures that as the distance 2B between the cracks or B/H increases, the stress intensity factors approaches to values for single edge crack, the interaction effect vanishes, the mode II stress intensity factor becomes zero, consequently the crack propagation angle becomes 0 (meaning crack will propagate straight) and hence both the cracks will behave like an independent edge cracks. Hence it can be summarized that enriching the domain using PoU in XFEM also well able to capture the interaction between multiple cracks.
7.6 Multiple interacting cracks

7.6.3 Three point bending test on an infinite plate with multiple cracks

In this section we shall examine the interaction between multiple cracks in an infinite strip subjected to three point bending test. The geometry of the problem domain is shown in the figure 7.51(a). Three edge cracks are incorporated into the beam. The effect of interaction of each crack with the other is analyzed by varying the separation distance between the cracks. A numerical model consists of depth $H=1$ and length $L=10$. The domain was meshed with $24 \times 240$ 4-noded quadrilateral elements. The material properties used are Young’s modulus $E=1000\text{Ksi}$ and $\nu = 0.3$. An initial cracks of lengths $a_1=a_2=0.2$ were introduced. Stress intensity factors for the two modes, mode I and mode II, were computed for each $B/H$ value. The results were qualitatively compared with the results of Civelek and Erdogan [1982]. However knowing the fact that central crack will propagate straight with mode II stress intensity factor being zero, puts another check on our numerically obtained results.

Comments :-

Figure 7.52(a) shows the variation of normalized stress intensity factors with varying $B/H$ ratios. The dashed line shows the normalized stress intensity factor values for a single edge.
7.6 Multiple interacting cracks

![Graphs](image)

(a) $K_I$ and $K_{II}$ vs B/H  
(b) Probable crack growth angle vs B/H

**Figure 7.50:** Effect of B/H on crack propagation

Crack in a beam subjected to three point bending test. Figure 7.52(b) shows variation of crack propagation angle of outer cracks with changing B/H ratios. The obtained results are qualitatively in good agreement with the results of Civelek and Erdogan [1982]. This shows that the method (XFEM) is capturing well the effects of interaction between the cracks. It can be seen from the figures 7.52(a) and 7.52(b), that the mode II stress intensity factor for central crack ($k_{II_{c2}}$) is zero, meaning the central crack will propagate straight for all crack separation values. Further it can be seen that at smaller B/H values the stress intensity factor for the middle crack is smaller than the side cracks, hence most of stresses will be given to other part of the beam resulting in greater stress intensity factors for the outer cracks. with the increasing B/H values the mode I stress intensity factor approaches towards the single edge crack value.

Now coming to the outer cracks, as can be seen from the graph 7.52(a) that the mode II stress intensity factor is negative for smaller B/H values. This means that when the separation between the crack is small, there will be greater interaction between the cracks. In this case, the outer cracks will propagate away from the middle crack. This can be seen in figures 7.51(b) and 7.53(a).

When the cracks are far from each other most of the stresses will be taken by the central crack. The central crack actually relaxes the stress states in rest of the domain, thus resulting in smaller stress intensity factors for the outer cracks as can be seen in figures 7.52(a) and also from the stress plot in figure 7.51(c). Further as shear component dominates at the far end so the outer cracks will move towards the center as shown in figure 7.51(c) and 7.53(b).
7.6 Multiple interacting cracks

(a) Three point bending test model

(b) stress component \( \sigma_{yy} \) for cracks separated by \( B=0.2H \)

(c) stress component \( \sigma_{yy} \) for cracks separated by \( B=4H \)

**Figure 7.51:** Geometry of the problem and stress plots for three point bending beam test with initial multiple cracks

(a) \( K_I \) and \( K_{II} \) vs \( B/H \)

(b) crack propagation angle vs \( B/H \)

**Figure 7.52:** Effect of interaction between cracks (\( B/H \)) on crack propagation
7.6 Multiple interacting cracks

Figure 7.53: Zoom at cracked zones
Chapter 8

Conclusions and Future work

8.1 Summary and conclusions

XFEM is a partition of unity based method which is able to incorporate functions, typically non-polynomials into the standard finite element approximating space. The method relies on an enhancement of the approximating space with enrichment functions. Additionally it is noteworthy that any kind of function describing the general behavior of the solution can be used as an enrichment function. It may be a polynomial or a non-polynomial functions, discontinuous functions, singular functions, trigonometric functions or any priori knowledge on the characteristic behavior of the solution.

In this work the extended finite element methodology was studied. The document shows the implementation of XFEM methodology to some of the standard problems to set guidelines for more complicated problems and future work. Detail comments on the numerical tests that were performed are given after every numerical example in chapter 7. In the passage below summary of the analysis results is presented.

Level set method

A notion of level set method for geometric modeling was presented and its coupling with XFEM was studied. A simple form of level set function usually employed for modeling and tracking discontinuities (i.e signed distance function) was analyzed and discussed in detail. It was seen during the analysis that the construction of level set functions requires a mesh, like finite difference, specially for tracking moving discontinuities where the level set update is necessary.
8.1 Summary and conclusions

However in order to define very curved geometries a refined mesh for the construction of level set function is necessary to resolve the geometry of the discontinuity.

Level set method when couples with XFEM becomes a very handy tool. For crack growth problems the evaluation of enrichment function also becomes very convenient using the level set functions.

It is also suggested in the literature that, the level set function can also be used for selection of cut elements and tip elements, however it was found in our implementation that this may not give correct results in many instances. In our work we used level set function for geometric modeling, however for the selection of enriched nodes and elements, we used explicit criteria where we explicitly searched for the element whose edges are cut by the crack.

Furthermore using level set functions, computations of crack tip polar coordinates is a straight forward, but using the signed distance function which is more commonly used, may not be useful specially for curved crack paths. This factor consequently results in erroneous solution to the analysis problem.

**Modified/fixed area enrichment**

Modeling cracks with a step function and using near tip enrichment function for crack tip regions, gives very good results even for the coarse meshes. Using the modified or fixed area near tip enrichment scheme improves the rate of convergence. However determining the size of enrichment domain has no fixed rules. This can be set by running few initial analysis to check what could be the optimum domain for getting results with minimum error. Using the modified enrichment scheme helps in getting improved convergence rates however it also affects the selection of domain for interaction integral. In case a modified enrichment scheme is used the domain for interaction integral should be chosen greater then the domain of enrichment in order to get stable accurate results.

**Shifted enrichment**

Using the shifted enrichment within XFEM not only helps in making post processing easier by making standard part of XFEM displacements as equal to the nodal displacements but also
8.1 Summary and conclusions

helps in applying the Dirichlet boundary conditions as the enriched degree of freedom or more precisely enrichment is zero at the node.

**Modified domain for interaction integral**

A modified domain for the computation of interaction integral has been developed. It is seen that, near the boundaries of the body the contour integral does not surround the point of singularity, hence unable to integrate properly the integrand quantity and does not give the correct measure of energy at the crack tip. However using the modified form of domain, optimal results can be obtained.

**Crack propagation path**

Factors affecting the crack propagation paths were deeply studied by simulating crack propagation using different useful numerical examples. Tracking correct crack path is imperative for determining the true failure strength of the material. In this regard mainly three factors affects the crack path (1) length of crack incremental segment (2) mesh refinement (3) domain for interaction integral. It can be concluded from the examples presented above that using crack incremental length equal to 1 -1.5 times the element size gives good results. furthermore using a finer mesh resolve the field near the singularity and hence the parameters of intersect(stress intensity factors) can be evaluated more accurate as is obvious. Consequently it helps in determining the correct crack path. In case of curved cracks the J-integral is no more path independent and hence affects the solution. Further it is also observed during the analysis that crack path tries to deviate from original path and shows oscillations as the crack approaches the arrest under compression. Hence it can be said that , the crack path seems to become unstable as it approaches arrest.

**Modeling multiple cracks**

Thanks to the partition of unity property which enables us to enhance or extend the standard finite element space by pasting together special functions that best describes the field. This enables to model flaws arbitrarily aligned with the finite element mesh. In case of crack propagation, this has a greater implications and hence a material with several hundreds of arbitrary discontinuities could be modeled with ease and accuracy. One of the major aspect of our work was, to extend the XFEM to model multiple crack growth problems. In this context numerical
studies were performed and the numerical results were compared with the available analytical solutions in the literature. The methodology was successful in modeling multiple crack growth problems. The approach was fully able to capture the interacting effects of multiple cracks in a body.

8.2 Future work

Extended finite element method has a lots of potentials and presents a wider area of its implementation for the solution of myriad of problems. In order to predict not only the failure load but also the post-peak behavior correctly, robust and stable computational algorithms, that are capable of dealing with the highly non-linear set of governing equations are an essential requirement. Performing numerical study either to verify some experimental test results or to develop some numerical models presents another motivation to develop a numerical tool, where such analysis could be done with accuracy and computational efficiency. Extended finite element method in this sense is quite appealing as the discontinuity can be aligned arbitrarily with the mesh, within the element interior.

In XFEM the numerical integration in the elements cut by the crack is usually done by partitioning the element into sub domains. This is straight forward, however in any case it requires partitioning in some sense which can be burdensome and hence clashes with the qualities of the XFEM methodology.

Blending element effect although seems to be limited locally to a small domain but in reality it affects the whole solution and degrades the convergence of the solution. Chessa et al. [2003] and Fries [2008] had developed methods for dealing with blending element effect, however they in turn increases the degrees of freedom, means more computation and hence impairs the qualities of XFEM over FEM. Efficient methods needs to be developed in the future to handle such issues.

Much of the work has been done in 2D crack growth modeling, however 3d crack propagation and determining the correct crack path is still a question and needs an extensive research in the area.


Although level set method has been coupled with the XFEM, but this is done usually the simplest form of level set function, which gives erroneous results in many circumstances. It is therefore required to develop and use other forms of level set functions which are computationally efficient, accurate and best suited with the framework of extended finite element method in order to get full benefit of the level set method for tracking discontinuities.

Due to the fact that XFEM can model arbitrary discontinuities independent of the mesh, modeling composite material using the methodology of extended finite element method presents another challenging field for future research.
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